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Research Article

Analytic Solution of a Class of Fractional Differential Equations

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We consider the analytic solution of a class of fractional differential equations with variable coefficients by operatorial methods. We obtain three theorems which extend the Garra's results to the general case.

1. Introduction

Recently, Garra [1] studied the analytic solution of a class of fractional differential equations with variable coefficients by using operatorial methods.

The proof of his main results is strongly based on operatorial properties of Caputo fractional derivative and the following theorem by Miller and Ross [2].

Theorem M.-R. (Theorem 1.1 in [1]). *Let $f(t)$ be any function of the form $t^\lambda\eta(t)$ or $t^\lambda(\ln t)\eta(t)$, where $\lambda > -1$, and*

$$\eta(t) = \sum_{n=0}^{\infty} a_n t^n \tag{1}$$

has a radius of convergence $R > 0$ and $0 < X < R$. Then

$$D_t^\alpha D_t^\beta f(t) = D_t^{\alpha+\beta} f(t) \tag{2}$$

holds for all $t \in (0, X)$, provided

(1) *$\beta < \lambda + 1$ and α are arbitrary, or*

(2) *$\beta \geq \lambda + 1$ and α are arbitrary and $a_k = 0$ for $k = 0, 1, \ldots, m-1$, where $m = \lceil\beta\rceil$ (ceiling of the number).*

Garra's main results are as follows.

Theorem Garra (Theorem 1.2 in [1]). *Consider the following boundary value problem (BVP):*

$$\begin{aligned} D_{L_x} f(x,t) &= D_t^\alpha f(x,t), \\ f(0,t) &= g(t), \end{aligned} \tag{3}$$

in the half plane $t > 0$, with analytic boundary condition $g(t)$ such that the conditions of Theorem 1.1 (Theorem M.-R.) are satisfied. The operatorial solution of BVP (3) is given by

$$f(x,t) = \sum_{k=0}^{\infty} \frac{x^k D_t^{\alpha k}}{(k!)^2} g(t), \tag{4}$$

where D_{L_x} is the Laguerre derivative and D_t^α denotes Caputo fractional derivative. On the basis of the previous result, Garra proved (Example 1 in [1]) that if $g(t) = t^m$, $m \in \mathbb{N}$, then the analytic solution of the BVP (3) is given by

$$f(x,t) = \sum_{k=0}^{\infty} \frac{x^k D_t^{\alpha k} t^m}{(k!)^2} = \sum_{r=0}^{[m/\alpha]} \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha r)} \cdot \frac{x^r t^{m-\alpha r}}{(r!)^2}, \tag{5}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

Motivated by the results, in the present note, we extend Garra's results to the general case.

2. Preliminaries

Definition 1 (see [3]). For every positive integer n, the operator $D_{nL_x} := Dx, \ldots, DxDxD$ (containing $n+1$ ordinary derivatives) is called the n-order Laguerre derivatives and the nL-exponential function is defined by

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}. \tag{6}$$

In [4], the following result is proved.

Lemma 2. *Let a be an arbitrary real or complex number. The function $e_n(ax)$ is an eigenfunction of the operator D_{nL_x}; that is,*

$$D_{nL_x} e_n = a e_n(ax). \tag{7}$$

For $n = 0$, we have $D_{0L} := D$. Thus, (7) leads to the classical property of the exponential function $De^{ax} = ae^{ax}$.

Similarly, the spectral properties can be obtained [3] by using the general Laguerre derivatives

$$D_{1L} + mD = D(xD + m) \tag{8}$$

(here m is a real or complex constant) or more generally the operator

$$D(xD+m)^n = \sum_{k=0}^{n} \binom{n}{k} D_{kL} m^{n-k} \tag{9}$$

($n \in \mathbb{N}$) and the corresponding eigenfunctions

$$\sum_{k=0}^{\infty} \frac{x^k}{k!\,(k+m)!} \tag{10}$$

or

$$\sum_{k=0}^{\infty} \frac{x^k}{k!((k+m)!)^n}. \tag{11}$$

Throughout this paper, we use the Caputo fractional derivatives as in [1].

Definition 3 (see [5]). The Caputo derivative of fractional order α of function $f(t)$ is defined as

$$D^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1<\alpha<m \\ \dfrac{d^m f}{dt^m}, & \alpha = m, \end{cases} \tag{12}$$

where $m := \lceil \alpha \rceil$.

Lemma 4 (see [5]). *Let $f(t) = t^{\lambda}$ for some $\lambda \geq 0$. Then*

$$D^{\alpha} f(t) = \begin{cases} 0, & \text{if } \lambda \in 0,1,\dots,m-1, \\ \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} t^{\lambda-\alpha}, & \text{if } \lambda \in \mathbb{N},\ \lambda \geq m \text{ or} \\ & \lambda \notin \mathbb{N},\ \lambda > m-1, \end{cases} \tag{13}$$

where $m =: \lceil \alpha \rceil$.

Lemma 5. *Let $\lambda > 0$. Let $\alpha > 0$, $\beta > 0$, and $\alpha + \beta \leq \lambda$. Then*

$$D^{\alpha+\beta} t^{\lambda} = D^{\alpha} D^{\beta} t^{\lambda} \quad (t > 0). \tag{14}$$

Proof. This follows immediately from Lemma 4. □

Remark 6. In general, for $\alpha, \beta > 0$, $D^{\alpha}D^{\beta} = D^{\alpha+\beta}$ is not true. For example,

$$D^{0.6} D^{0.6} t = \frac{\Gamma(2)}{\Gamma(0.8)} t^{-0.2}, \tag{15}$$

but

$$D^{0.6+0.6} t = D^{1.2} t = 0. \tag{16}$$

Lemma 7. *Suppose $\alpha > 0$, and let $g(t) = \sum_{k=0}^{\infty} a_k t^k$ have a radius of convergence $R > 0$ and $0 < X < R$. If $\lceil \alpha k \rceil = \lceil \alpha(k+1) \rceil - 1$ for some $k \in \mathbb{N}$, then*

$$D^{(k+1)\alpha} g(t) = D^{\alpha} D^{k\alpha} g(t). \tag{17}$$

Proof. Let $\lceil \alpha(k+1) \rceil = m$. Then $m - 1 < \alpha(k+1) \leq 0$.

By lemmas 4 and 5, we have

$$\begin{aligned} D^{(k+1)\alpha} g(t) &= D^{(k+1)\alpha} \left(\sum_{l=0}^{\infty} a_l t^l \right) \\ &= D^{(k+1)\alpha} \left(\sum_{l=m}^{\infty} a_l t^l \right). \end{aligned} \tag{18}$$

Since $\lceil \alpha k \rceil = \lceil \alpha(k+1) \rceil - 1 = m - 1$, together with Definition 3, we have

$$D^{\alpha} D^{\alpha k} \left(\sum_{l=0}^{m-1} a_l t^l \right) = 0, \tag{19}$$

and therefore (18) implies that

$$D^{(k+1)\alpha} g(t) = D^{\alpha} D^{\alpha k} \left(\sum_{l=0}^{\infty} a_l t^l \right) = D^{\alpha} D^{\alpha k} g(t). \tag{20}$$

This completes the proof. □

3. Main Results

We first study the following BVP in the plane $t > 0$:

$$\begin{aligned} D_{nL_x} f(x,t) &= D_t^{\alpha} f(x,t), \\ f(0,t) &= g(t). \end{aligned} \tag{21}$$

Theorem 8. *Let $g(t) = t^{\lambda}$ with $\lambda > 0$ and assume that λ/α is a positive integer. Then the operatorial form solution of BVP (21) is*

$$\begin{aligned} f(x,t) &= \sum_{k=0}^{\lambda/\alpha} \frac{x^k D^{\alpha k} t^{\lambda}}{(k!)^{n+1}} \\ &= \sum_{k=0}^{\lambda/\alpha} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha k)} \cdot \frac{x^k}{(k!)^{n+1}} t^{\lambda-\alpha k}. \end{aligned} \tag{22}$$

Proof. Let $\lambda/\alpha = m$. By Lemmas 4 and 5, we have

$$\begin{aligned} D_{nL_x} f(x,t) &= \sum_{k=1}^{m} \frac{k^{n+1} D_t^{\alpha k} t^{\lambda}}{(k!)^{n+1}} = \sum_{k=1}^{m} \frac{x^{k-1} D_t^{\alpha k} t^{\lambda}}{[(k-1)!]^{n+1}} \\ &= \sum_{k=0}^{m-1} \frac{D_t^{\alpha(k+1)} t^{\lambda}}{(k!)^{n+1}} = D_t^{\alpha} \left(\sum_{k=0}^{m-1} \frac{x^k D^{\alpha k} t^{\lambda}}{(k!)^{n+1}} \right) \\ &= D_t^{\alpha} \left(\sum_{k=0}^{m} \frac{x^k D^{\alpha k} t^{\lambda}}{(k!)^{n+1}} \right) = D_t^{\alpha} f(x,t). \end{aligned} \tag{23}$$

In the fifth previous equality, we use the fact that

$$D_t^{\alpha} \left(\frac{x^m D^{\alpha m} t^{\lambda}}{(k!)^{n+1}} \right) = D_t^{\alpha} \left(\frac{x^m D^{\lambda} t^{\lambda}}{(k!)^{n+1}} \right) = 0. \tag{24}$$

□

Remark 9. We point out that the result of Example 1 in [1] is incorrect. A counterexample is as follows. Let $g(t) = t$ and $\alpha = 0.6$. By Lemma 5, we have

$$\begin{aligned} & D_{L_x} \left(\sum_{k=0}^{[1/0.6]} \frac{x^k D_t^{\alpha k}}{(k!)^2} t \right) \\ &= D_x x D_x \left(t + \frac{x D_t^{0.6}}{(1!)^2} t \right) = D_t^{0.6}(t). \end{aligned} \tag{25}$$

On the other hand, we have

$$\begin{aligned} & D_t^{0.6} \left(\sum_{k=0}^{[1/0.6]} \frac{x^k D_t^{\alpha k}}{(k!)^2} t \right) \\ &= D_t^{0.6} \left(t + \frac{x D_t^{0.6}}{(1!)^2} t \right) = D_t^{0.6}(t) + x D_t^{0.6} \left(D_t^{0.6} t \right). \end{aligned} \tag{26}$$

Note that by Remark 6, we have

$$\begin{aligned} D_t^{0.6+0.6} t &= 0, \\ D_t^{0.6} D_t^{0.6} t &= \frac{1}{\Gamma(0.8)} t^{-0.2} \neq 0. \end{aligned} \tag{27}$$

Hence,

$$D_{L_x} \left(\sum_{k=0}^{[1/0.6]} \frac{x^k D_t^{\alpha k}}{(k!)^2} t \right) \neq D_t^{0.6} \left(\sum_{k=0}^{[1/0.6]} \frac{x^k D_t^{\alpha k}}{(k!)^2} t \right). \tag{28}$$

Theorem 10. *Let $g(t)$ satisfy the assumptions of Lemma 7. If $[\alpha k] = \lceil \alpha(k+1) \rceil - 1$, $(k = 1, 2, \ldots)$, then the operatorial form solution of BVP (21) is given by*

$$f(x,t) = e_n \left(x D_t^{\alpha} \right) g(t) = \sum_{k=0}^{\infty} \frac{x^k D_t^{\alpha k}}{(k!)^{n+1}} g(t). \tag{29}$$

Proof. By Lemmas 2 and 7, we have

$$\begin{aligned} D_{nL_x} f(x,t) &= D_{nL_x} \left(e_n \left(x D_t^{\alpha} \right) g(t) \right) \\ &= D_t^{\alpha} e_n \left(x D_t^{\alpha} \right) g(t) = D_t^{\alpha} f(x,t). \end{aligned} \tag{30}$$

This completes the proof. □

The following generalization of the Theorem 10 can be proved similarly.

Theorem 11. *Let m be a real or complex constant and $n \in \mathbb{N}$. Consider the following BVP:*

$$\begin{aligned} D_x \left(x D_x + m \right)^n f(x,t) &= D_t^{\alpha} f(x,t), \\ f(0,t) &= g(t), \end{aligned} \tag{31}$$

in the half plane $t > 0$, with analytic boundary condition $g(t)$ such that the conditions of Lemma 7 are satisfied. If $[\alpha k] = \lceil \alpha(k+1) \rceil - 1$, $(k = 1, 2, \ldots)$, then the operatorial solution of BVP (31) is given by

$$f(x,t) = \sum_{k=0}^{\infty} \frac{x^k D_t^{\alpha k}}{k!((k+m)!)^n} g(t). \tag{32}$$

Proof. Using spectral properties of Laguerre derivative, together with Lemma 7, we have

$$\begin{aligned} D_x \left(x D_x + m \right)^n f(x,t) &= D_x \left(x D_x + m \right)^n \\ &\quad \times \left(\sum_{k=0}^{\infty} \frac{x^k D_t^{\alpha k}}{k!((k+m)!)^n} g(t) \right) \\ &= \sum_{k=1}^{\infty} \frac{k(k+m)^n x^{k-1} D_t^{\alpha k}}{k!((k+m)!)^n} g(t) \\ &= \sum_{k=0}^{\infty} \frac{x^k D^{\alpha(k+1)}}{k!((k+m)!)^{n+1}} g(t) \\ &= D^{\alpha} \left(\sum_{k=0}^{\infty} \frac{x^k D^{\alpha k}}{k!((k+m)!)^n} g(t) \right) \\ &= D^{\alpha} f(x,t). \end{aligned} \tag{33}$$

This completes the proof. □

Remark 12. Conditions $[\alpha k] = \lceil \alpha(k+1) \rceil - 1$ $(k = 1, 2, \ldots)$ are very harsh. However, if we remove them, Theorems 10 and 11 no longer remain valid. It should be noted that the main result in [1] (Theorem 1.2 in [1]) is incorrect in general case. A counterexample is as follows. Let $g(t) = e^t$ and $\alpha = 0.6$. One has

$$\begin{aligned} D_{L_x} \left(\sum_{k=0}^{\infty} \frac{x^k D_t^{\alpha k} e^t}{(k!)^2} \right) &= D_x x D_x \left(\frac{x}{(1!)^2} D_t^{0.6} e^t + \frac{x^2}{(2!)^2} D_t^{1.2} e^t \right. \\ &\qquad \left. + \frac{x^3}{(3!)^2} D_t^{1.8} e^t + \cdots \right) \\ &= D_t^{0.6} e^t + \frac{x}{(1!)^2} D_t^{1.2} e^t \\ &\quad + \frac{x^2}{(2!)^2} D_t^{1.8} e^t + \cdots. \end{aligned} \tag{34}$$

On the other hand, we have

$$D_t^{\alpha}\left(\sum_{k=0}^{\infty}\frac{x^k D_t^{\alpha k}e^t}{(k!)^2}\right) = D_t^{0.6}\left(e^t + \frac{x}{(1!)^2}D_t^{0.6}e^t + \frac{x^2}{(2!)^2}D_t^{1.2}e^t + \cdots\right). \quad (35)$$

Set $x = 1$ in (34) and (35). By Definition 3 and Lemma 4, we deduce that all terms in (34) are positive and cannot contain negative exponent of variable t. For example,

$$\begin{aligned} D_t^{0.6}e^t &= D_t^{0.6}\left(1 + t + \frac{t^2}{2!} + \cdots\right) = D_t^{0.6}\left(t + \frac{t^2}{2!} + \cdots\right) \\ &= \frac{1}{\Gamma(1.4)}t^{0.4} + \frac{1}{\Gamma(2.4)}t^{1.4} + \cdots, \\ D_t^{1.8}e^t &= D_t^{1.8}\left(1 + t + \frac{t^2}{2!} + \cdots\right) = D_t^{1.8}\left(\frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \\ &= \frac{1}{\Gamma(1.2)}t^{0.2} + \frac{1}{\Gamma(2.2)}t^{1.2} + \cdots, \\ &\text{and so forth.} \end{aligned} \quad (36)$$

Similarly, all terms in (35) are also positive, except that some terms contain negative exponent of variable t. For example,

$$\begin{aligned} D_t^{0.6}D_t^{0.6}e^t &= D_t^{0.6}D_t^{0.6}\left(1 + t + \frac{t^2}{2!} + \cdots\right) \\ &= D_t^{0.6}D_t^{0.6}\left(t + \frac{t^2}{2!} + \cdots\right) \\ &= \frac{1}{\Gamma(0.8)}t^{-0.2} + \cdots, \\ D_t^{0.6}D_t^{6.6}e^t &= D_t^{0.6}D_t^{6.6}\left(1 + t + \frac{t^2}{2!} + \cdots\right) \\ &= D_t^{0.6}D_t^{6.6}\left(t + \frac{t^2}{2!} + \cdots\right) \\ &= \frac{1}{\Gamma(0.8)}t^{-0.2} + \cdots, \\ &\text{and so forth.} \end{aligned} \quad (37)$$

Thus, we conclude that

$$D_{L_x}\left(\sum_{k=0}^{\infty}\frac{x^k D_t^{0.6k}e^t}{(k!)^2}\right) \neq D_t^{0.6}\left(\sum_{k=0}^{\infty}\frac{x^k D_t^{0.6k}e^t}{(k!)^2}\right). \quad (38)$$

4. Conclusion and Discussion

In this paper, we point out that Garra's results are incorrect and give some necessary counterexamples. In addition, we established three theorems (Theorems 8, 10, and 11) which correct and extend the corresponding results of [1].

Different from integer-order derivative, there are many kinds of definitions for fractional derivatives, including Riemann-Liouville, Caputo, Grunwald-Letnikov, Weyl, Jumarie, Hadamard, Davison and Essex, Riesz, Erdelyi-Kober, and Coimbra (see [1, 6–8]). These definitions are generally not equivalent to each other. Every derivative has its own serviceable range. In other words, all these fractional derivatives definitions have their own advantages and disadvantages. For example, the Caputo derivative is very useful when dealing with real-world problem, since it allows traditional initial and boundary conditions to be included in the formulation of the problem and the Laplace transform of Caputo fractional derivative is a natural generalization of the corresponding well-known Laplace transform of integer-order derivative. So, the Caputo fractional-order system is often used in modelling and analysis. However, the functions that are not differentiable do not have fractional derivative, which reduces the field of application of Caputo derivative (see [1, 8, 9]).

When solving fractional-order systems, the law of exponents (semigroup property) is the most important. Unlike integer-order derivative, for $\alpha > 0$ and $\beta > 0$, α derivative of the β derivative of a function is, in general, not equal to the $\alpha+\beta$ derivative of such function. About the semigroup property of the fractional derivatives, under suitable assumptions of fractional order, there have existed some studies, but only a few studies provide valuable judgment methods (see [1, 9]).

Fortunately, if we define Ω as the class of all functions f which are infinitely differentiable everywhere and are such that f and all its derivatives are of order t^{-N} for all N, $N = 1, 2, \ldots$, then, for all functions of class Ω, Weyl fractional derivatives possess the semigroup property [1]. This has brought us great convenience for studying Weyl fractional differential equations. We will considered this topic in a forthcoming paper.

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