Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 102820, 44 pages
http://dx.doi.org/10.1155/2013/102820

Research Article

Hybrid and Relaxed Mann Iterations for General Systems of Variational Inequalities and Nonexpansive Mappings

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Received 4 August 2013; Accepted 10 August 2013

Academic Editor: Jen-Chih Yao

We introduce hybrid and relaxed Mann iteration methods for a general system of variational inequalities with solutions being also common solutions of a countable family of variational inequalities and common fixed points of a countable family of nonexpansive mappings in real smooth and uniformly convex Banach spaces. Here, the hybrid and relaxed Mann iteration methods are based on Korpelevich's extragradient method, viscosity approximation method, and Mann iteration method. Under suitable assumptions, we derive some strong convergence theorems for hybrid and relaxed Mann iteration algorithms not only in the setting of uniformly convex and 2-uniformly smooth Banach space but also in a uniformly convex Banach space having a uniformly Gateaux differentiable norm. The results presented in this paper improve, extend, supplement, and develop the corresponding results announced in the earlier and very recent literature.

1. Introduction

Let X be a real Banach space whose dual space is denoted by X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in X, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Let C be a nonempty closed convex subset of X. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\to$ to indicate the strong convergence. A mapping $A : C \to X$ is said to be

(i) accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \ge 0; \tag{2}$$

(ii) α-strongly accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \ge \alpha \|x - y\|^2, \tag{3}$$

for some $\alpha \in (0, 1)$;

(iii) β-inverse strongly accretive if, for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \ge \beta \|Ax - Ay\|^2, \tag{4}$$

for some $\beta > 0$;

(iv) λ-strictly pseudocontractive [1] (see also [2]) if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \le \|x - y\|^2 - \lambda \|x - y - (Ax - Ay)\|^2 \tag{5}$$

for some $\lambda \in (0, 1)$.

It is worth emphasizing that the definition of the inverse strongly accretive mapping is based on that of the inverse strongly monotone mapping, which was studied by so many authors; see, for example, [3–5]. Let $U = \{x \in X : \|x\| = 1\}$ denote the unite sphere of X. A Banach space X is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \Longrightarrow \frac{\|x + y\|}{2} \leq 1 - \delta. \tag{6}$$

It is known that a uniformly convex Banach space is reflexive and strict convex. A Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{7}$$

exists for all $x, y \in U$; in this case, X is also said to have a Gateaux differentiable norm. X is said to have a uniformly, Gateaux differentiable norm if, for each $y \in U$, the limit is attained uniformly for $x \in U$. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of X is said to be the Frechet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$. In the meantime, we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x, y \in X, \right.$$
$$\left. \|x\| = 1, \|y\| = \tau \right\}. \tag{8}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then, a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. As pointed out in [6], no Banach space is q-uniformly smooth for $q > 2$. In addition, it is also known that J is single valued if and only if X is smooth, whereas if X is uniformly smooth, then the mapping J is norm-to-norm uniformly continuous on bounded subsets of X. If X has a uniformly Gateaux differentiable norm, then the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X.

Recently, Yao et al. [7] combined the viscosity approximation method and Mann iteration method and gave the following hybrid viscosity approximation method.

Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X, $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f : C \to C$ a contraction with coefficient $\rho \in (0, 1)$. For an arbitrary $x_0 \in C$, define $\{x_n\}$ in the following way:

$$y_n = \alpha_n x_n + (1 - \alpha_n) T x_n,$$
$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) y_n, \quad \forall n \geq 0, \tag{YCY}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$.

They proved under certain control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a fixed point of T. Subsequently, under the following control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$:

(i) $1 \leq \beta_n \leq 1 - \rho$, for all $n \geq n_0$ for some integer $n_0 \geq 1$,

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$,

(iv) $\lim_{n \to \infty} (\beta_{n+1}/(1 - (1 - \beta_{n+1})\alpha_{n+1}) - \beta_n/(1 - (1 - \beta_n)\alpha_n)) = 0$.

Ceng and Yao [8] proved that

$$x_n \longrightarrow q \Longleftrightarrow \beta_n (f(x_n) - x_n) \longrightarrow 0, \tag{9}$$

where $q \in \mathrm{Fix}(T)$ solves the variational inequality problem (VIP):

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{10}$$

Such a result includes [7, Theorem 1] as a special case.

Let C be a nonempty closed convex subset of a real Banach space X and $f \in \Xi_C$ with a contractive coefficient $\rho \in (0, 1)$, where Ξ_C is the set of all contractive self-mappings on C. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=0}^{\infty}$ a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 0$, define a self-mapping W_n on C as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n) I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k) I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1}) I, \\
&\vdots \\
U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1 - \lambda_1) I, \\
W_n &= U_{n,0} = \lambda_0 T_0 U_{n,1} + (1 - \lambda_0) I.
\end{aligned} \tag{CY}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_0$, and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_0$; see [9].

In 2008, Ceng and Yao [10] introduced and analyzed the following relaxed viscosity approximation method for finding a common fixed point of an infinite family of nonexpansive mappings in a strictly convex and reflexive Banach space with a uniformly Gateaux differentiable norm.

Theorem 1 (see [10]). *Let X be a strictly convex and reflexive Banach space with a uniformly Gateaux differentiable norm, C a nonempty closed convex subset of X, $\{T_n\}_{n=0}^{\infty}$ a sequence of nonexpansive self-mappings on C such that the common fixed point set $F := \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$, and $f \in \Xi_C$ with a contractive* *coefficient* $\rho \in (1/2, 1)$. *For any given* $x_0 \in C$, *let* $\{x_n\}_{n=0}^{\infty}$ *be the iterative sequence defined by*

$$y_n = (1 - \gamma_n) x_n + \gamma_n W_n x_n,$$
$$x_{n+1} = (1 - \alpha_n - \beta_n) x_n + \alpha_n f(y_n) + \beta_n W_n y_n, \quad \forall n \geq 0, \tag{11}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are two sequences in* $(0, 1)$ *with* $\alpha_n + \beta_n \leq 1$ $(n \geq 0)$, $\{\gamma_n\}_{n=0}^{\infty}$ *is a sequence in* $[0, 1]$, *and* W_n *is the* W*-mapping generated by* (CY). *Assume that*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(ii) $\lim_{n\to\infty}|\gamma_n - \gamma_{n-1}| = 0$ *and* $\limsup_{n\to\infty}\gamma_n < 1$.

Then, there hold the following:

(i) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;

(ii) *the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to some* $p \in F$ *which is the unique solution of the variational inequality problem (VIP)*

$$\langle (I - f) q, J(q - p) \rangle \leq 0, \quad \forall f \in \Xi_C, \; p \in F, \tag{12}$$

provided $\lim_{n\to\infty}\gamma_n = 0$ *and* $\beta_n \equiv \beta$ *for some fixed* $\beta \in (0, 1)$.

On the other hand, Cai and Bu [11] considered the following general system of variational inequalities (GSVI) in a real smooth Banach space X, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C,$$
$$\langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*) \rangle \geq 0, \quad \forall x \in C, \tag{13}$$

where C is a nonempty, closed, and convex subset of X, $B_1, B_2 : C \to X$ are two nonlinear mappings, and μ_1 and μ_2 are two positive constants. Here, the set of solutions of GSVI (13) is denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In particular, if $X = H$, a real Hilbert space, then GSVI (13) reduces to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,$$
$$\langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \tag{14}$$

in which μ_1 and μ_2 are two positive constants. The set of solutions of problem (14) is still denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In particular, if $B_1 = B_2 = A$, then problem (14) reduces to the new system of variational inequalities (NSVI), introduced and studied by Verma [12]. Further, if $x^* = y^*$ additionally, then the NSVI reduces to the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{15}$$

The solution set of the VIP (15) is denoted by $\mathrm{VI}(C, A)$. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems. It is now well known that the variational inequalities are equivalent to the fixed point problems, the origin of which can be traced back to Lions and Stampacchia [13]. This alternative formulation has been used to suggest and analyze projection iterative method for solving variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous.

Recently, Ceng et al. [14] transformed problem (14) into a fixed point problem in the following way.

Lemma 2 (see [14]). *For given* $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ *is a solution of problem* (14) *if and only if* $\overline{x}$ *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C, \tag{16}$$

where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$ *and* P_C *is the projection of* H *onto* C.

In particular, if the mapping $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$.

In 1976, Korpelevič [15] proposed an iterative algorithm for solving the VIP (15) in Euclidean space $\mathbf{R}^n$:

$$y_n = P_C(x_n - \tau A x_n),$$
$$x_{n+1} = P_C(x_n - \tau A y_n), \quad n \geq 0 \tag{17}$$

with $\tau > 0$ a given number, which is known as the extragradient method (see also [16]). The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see, for example, [3, 11, 13, 17–33] and references therein, to name but a few.

In particular, whenever X is still a real smooth Banach space, $B_1 = B_2 = A$ and $x^* = y^*$, then GSVI (13) reduces to the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \tag{18}$$

which was considered by Aoyama et al. [34]. Note that VIP (18) is connected with the fixed point problem for nonlinear mapping (see, e.g., [35]), the problem of finding a zero point of a nonlinear operator (see, e.g., [36]), and so on. It is clear that VIP (18) extends VIP (15) from Hilbert spaces to Banach spaces.

In order to find a solution of VIP (18), Aoyama et al. [34] introduced the following Mann-type iterative scheme for an accretive operator A:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \Pi_C(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{19}$$

where Π_C is a sunny nonexpansive retraction from X onto C. Then, they proved a weak convergence theorem. For the related work, see [37] and the references therein.

Let C be a nonempty convex subset of a real Banach space X. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of

C into itself and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i = 1, \ldots, N$. Define a mapping $K : C \to C$ as follows:

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1-\lambda_1) I, \\ U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2) U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1-\lambda_3) U_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1}) U_{N-2}, \\ K = U_N &= \lambda_N T_N U_{N-1} + (1-\lambda_N) U_{N-1}. \end{aligned} \quad (20)$$

Such a mapping K is called the K-mapping generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$.

Very recently, Kangtunyakarn [38] introduced and analyzed an iterative algorithm by the modification of Mann's iteration process for finding a common element of the set of solutions of a finite family of variational inequalities and the set of fixed points of an η-strictly pseudocontractive mapping and a nonexpansive mapping in uniformly convex and 2-uniformly smooth Banach spaces.

Theorem 3 (see [38]). *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $A_i : C \to X$ be an α_i-inverse-strongly accretive mapping for each $i = 1, \ldots, N$. Define the mapping $G_i : C \to C$ by $G_i = \Pi_C(I - \lambda_i A_i)$ for $i = 1, \ldots, N$, where $\lambda_i \in (0, \alpha_i/\kappa^2)$ and κ is the 2-uniformly smooth constant of X. Let $B : C \to C$ be the K-mapping generated by $G_1, \ldots, G_N$ and $\rho_1, \ldots, \rho_N$, where $\rho_i \in (0,1)$, for all $i = 1, \ldots, N-1$, and $\rho_N \in (0,1]$. Let $f : C \to C$ a contraction with coefficient $\rho \in (0,1)$. Let $V : C \to C$ be an η-strictly pseudocontractive mapping and $S : C \to C$ be a nonexpansive mapping such that $F = \mathrm{Fix}(S) \cap \mathrm{Fix}(V) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(C, A_i)) \neq \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n S((1-\alpha) I + \alpha V) x_n, \quad \forall n \ge 0, \quad (21)$$

where $\alpha \in (0, \eta/\kappa^2)$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $[0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c,d]$ *for some* $c, d \in (0,1)$;

(iii) $\sum_{n=1}^{\infty} (|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then, $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q-p) \rangle \le 0, \quad \forall p \in F. \quad (22)$$

Beyond doubt, it is an interesting and valuable problem of constructing some algorithms with strong convergence for solving GSVI (13) which contains VIP (18) as a special case. Very recently, Cai and Bu [11] constructed an iterative algorithm for solving GSVI (13) and a common fixed point problem of a countable family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space. They proved the strong convergence of the proposed algorithm by virtue of the following inequality in a 2-uniformly smooth Banach space X.

Lemma 4 (see [39]). *Let X be a 2-uniformly smooth Banach space. Then,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x)\rangle + 2\|\kappa y\|^2, \quad \forall x, y \in X, \quad (23)$$

where κ is the 2-uniformly smooth constant of X and J is the normalized duality mapping from X into X^*.

Define the mapping $G : C \to C$ as follows:

$$G(x) := \Pi_C (I - \mu_1 B_1) \Pi_C (I - \mu_2 B_2) x, \quad \forall x \in C. \quad (24)$$

The fixed point set of G is denoted by Ω. Then, their strong convergence theorem on the proposed method is stated as follows.

Theorem 5 (see [11]). *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be β_i-inverse-strongly accretive with $0 < \mu_i < \beta_i/\kappa^2$ for $i = 1, 2$. Let f be a contraction of C into itself with coefficient $\delta \in (0,1)$. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping G defined by (24). For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1-\beta_n) S_n y_n, \\ y_n &= \alpha_n f(x_n) + (1-\alpha_n) z_n, \\ z_n &= \Pi_C (u_n - \mu_1 B_1 u_n), \\ u_n &= \Pi_C (x_n - \mu_2 B_2 x_n), \quad \forall n \ge 1. \end{aligned} \quad (25)$$

Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ for any bounded subset D of C and let T be a mapping of C into X defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Then, $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q-p) \rangle \le 0, \quad \forall p \in F. \quad (26)$$

It is easy to see that the iterative scheme in Theorem 5 is essentially equivalent to the following two-step iterative scheme:

$$y_n = \alpha_n f(x_n) + (1-\alpha_n) G x_n,$$
$$x_{n+1} = \beta_n x_n + (1-\beta_n) T_n y_n, \quad \forall n \geq 1. \tag{27}$$

For the convenience of implementing the argument techniques in [14], the authors of [11] have used the following inequality in a real smooth and uniform convex Banach space X.

Proposition 6 (see [40]). *Let X be a real smooth and uniform convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbf{R}$, $g(0) = 0$ such that*

$$g(\|x-y\|) \leq \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2, \quad \forall x, y \in B_r, \tag{28}$$

where $B_r = \{x \in X : \|x\| \leq r\}$.

Let C be a nonempty closed convex subset of a real smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and $f : C \to C$ a contraction with coefficient $\rho \in (0,1)$. Motivated and inspired by the research going on this area, we consider and introduce hybrid and relaxed Mann iteration methods for finding solutions of the GSVI (13) which are also common solutions of a countable family of variational inequalities and common fixed points of a countable family of nonexpansive mappings in X. Here, the hybrid and relaxed Mann iteration methods are based on Korpelevich's extragradient method, viscosity approximation method, and Mann iteration method. Under suitable assumptions, we derive some strong convergence theorems for hybrid and relaxed Mann iteration algorithms not only in the setting of uniformly convex and 2-uniformly smooth Banach space but also in a uniformly convex Banach space having a uniformly Gateaux differentiable norm. The results presented in this paper improve, extend, supplement, and develop the corresponding results announced in the earlier and very recent literature; see, for example, [8, 10, 11, 14, 33, 38].

2. Preliminaries

We list some lemmas that will be used in the sequel.

Lemma 7 (see [41]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n) s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \geq 0, \tag{29}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) *$\{\alpha_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;*

(ii) *$\limsup_{n\to\infty} \beta_n \leq 0$;*

(iii) *$\gamma_n \geq 0$, for all $n \geq 0$, and $\sum_{n=0}^{\infty} \gamma_n < \infty$.*

Then, $\limsup_{n\to\infty} s_n = 0$.

The following lemma is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

Lemma 8 (see [42]). *Let X be a real Banach space X. Then, for all $x, y \in X$*

(i) *$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$ for all $j(x+y) \in J(x+y)$;*

(ii) *$\|x+y\|^2 \geq \|x\|^2 + 2\langle y, j(x)\rangle$ for all $j(x) \in J(x)$.*

Let D be a subset of C and let Π be a mapping of C into D. Then, Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x), \tag{30}$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$ where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 9 (see [43]). *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let D be a nonempty subset of C. Let Π be a retraction of C onto D. Then, the following are equivalent:*

(i) *Π is sunny and nonexpansive;*

(ii) *$\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle$, for all $x, y \in C$;*

(iii) *$\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0$, for all $x \in C$, $y \in D$.*

It is well known that if $X = H$ a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C = P_C$. If C is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space X and if $T : C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

Lemma 10. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let $B_1, B_2 : C \to X$ be nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (13) if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

Proof. We can rewrite GSVI (13) as

$$\langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*)\rangle \geq 0, \quad \forall x \in C,$$
$$\langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*)\rangle \geq 0, \quad \forall x \in C, \tag{31}$$

which is obviously equivalent to

$$x^* = \Pi_C(y^* - \mu_1 B_1 y^*),$$
$$y^* = \Pi_C(x^* - \mu_2 B_2 x^*), \tag{32}$$

because of Lemma 9. This completes the proof. □

In terms of Lemma 10, we observe that

$$x^* = \Pi_C \left[\Pi_C \left(x^* - \mu_2 B_2 x^*\right) - \mu_1 B_1 \Pi_C \left(x^* - \mu_2 B_2 x^*\right)\right], \quad (33)$$

which implies that x^* is a fixed point of the mapping G. Throughout this paper, the set of fixed points of the mapping G is denoted by Ω.

Lemma 11 (see [44]). *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0, \infty] \to [0, \infty]$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta g\left(\|x - y\|\right) \quad (34)$$

for all $x, y, z \in B_r$, and all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 12 (see [45]). *Let C be a nonempty closed convex subset of a Banach space X. Let $S_0, S_1, \ldots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1} x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0$.*

Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. As previous, let Ξ_C be the set of all contractions on C. For $t \in (0, 1)$ and $f \in \Xi_C$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) + (1 - t)Tx$ on C; that is,

$$x_t = tf\left(x_t\right) + (1 - t)\, T x_t. \quad (35)$$

Lemma 13 (see [35, 46]). *Let X be a uniformly smooth Banach space, or a reflexive and strictly convex Banach space with a uniformly Gateaux differentiable norm. Let C be a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f \in \Xi_C$. Then, the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1 - t)Tx_t$ converges strongly to a point in $\mathrm{Fix}(T)$. If we define a mapping $Q : \Xi_C \to \mathrm{Fix}(T)$ by $Q(f) := s - \lim_{t\to 0} x_t$, for all $f \in \Xi_C$, then $Q(f)$ solves the VIP:*

$$\left\langle (I - f)\, Q(f), J\left(Q(f) - p\right)\right\rangle \leq 0, \quad \forall f \in \Xi_C,\ p \in \mathrm{Fix}(T). \quad (36)$$

Lemma 14 (see [47]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then, a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ for $x \in C$ is defined well; nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

Lemma 15 (see [39]). *Given a number $r > 0$, A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\left\|\lambda x + (1 - \lambda)\, y\right\|^2 \leq \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda (1 - \lambda)\, g\left(\|x - y\|\right) \quad (37)$$

for all $\lambda \in [0, 1]$ and $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

Lemma 16 (see [48, Lemma 3.2]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 0$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

Using Lemma 16, one can define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,0} x \quad (38)$$

for every $x \in C$. Such a W is called the W-mapping generated by the sequences $\{T_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$. Throughout this paper, we always assume that $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$.

Lemma 17 (see [48]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$. Then, $\mathrm{Fix}(W) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$.*

Let μ be a continuous linear functional on l^{∞} and $s = (a_0, a_1, \ldots) \in l^{\infty}$. One writes $\mu_n(a_n)$ instead of $\mu(s)$. μ is called a Banach limit if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \ldots) \in l^{\infty}$. If μ is a Banach limit, then, there hold the following:

(i) *for all $n \geq 0$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$;*

(ii) *$\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r;*

(iii) *$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \ldots) \in l^{\infty}$.*

Lemma 18 (see [49]). *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_n\} \in l^{\infty}$ satisfy the condition $\mu_n(a_n) \leq a$ for all Banach limit μ. If $\limsup_{n\to\infty}(a_{n+r} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

In particular, if $r = 1$ in Lemma 18, then we immediately obtain the following corollary.

Corollary 19 (see [50]). *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_n\} \in l^{\infty}$ satisfy the condition $\mu_n(a_n) \leq a$ for all Banach limit μ. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then, $\limsup_{n\to\infty} a_n \leq a$.*

Lemma 20 (see [51]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence of nonnegative numbers in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$ for all integers*

$n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Lemma 21 (see [34]). *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and $A : C \to X$ an accretive mapping. Then for all $\lambda > 0$,*

$$\mathrm{VI}(C, A) = \mathrm{Fix}(\Pi_C(I - \lambda A)). \tag{39}$$

Lemma 22 (see [11]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $B_i : C \to X$ be $\widehat{\beta}_i$-inverse-strongly accretive. Then, one has*

$$\begin{aligned}&\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\|^2\\ &\quad\leq \|x - y\|^2 + 2\mu_i(\mu_i\kappa^2 - \widehat{\beta}_i)\|B_i x - B_i y\|^2,\\ &\quad\forall x, y \in C,\end{aligned} \tag{40}$$

for $i = 1, 2$ where $\mu_i > 0$. In particular, if $0 < \mu_i \leq \widehat{\beta}_i/\kappa^2$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Lemma 23 (see [11]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be $\widehat{\beta}_i$-inverse-strongly accretive for $i = 1, 2$. Let $G : C \to C$ be the mapping defined by*

$$\begin{aligned}Gx = \Pi_C[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)],\\ \forall x \in C.\end{aligned} \tag{41}$$

If $0 < \mu_i \leq \widehat{\beta}_i/\kappa^2$ for $i = 1, 2$, then $G : C \to C$ is nonexpansive.

3. Hybrid Mann Iterations and Their Convergence Criteria

In this section, we introduce our hybrid Mann iteration algorithms in real smooth and uniformly convex Banach spaces and present their convergence criteria.

Theorem 24. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$ and $A_i : C \to E$ an $\widehat{\alpha}_i$-inverse strongly accretive mapping for each $i = 0, 1, \ldots$. Define a mapping $G_i : C \to C$ by $\Pi_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 0, 1, \ldots$, where $\lambda_i \in (0, \widehat{\alpha}_i/\kappa^2]$, κ is the 2-uniformly smooth constant of X. Let $B_n : C \to C$ be the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$ and $\rho_n, \rho_{n-1}, \ldots, \rho_0$. Let the mapping $B_i : C \to X$ be $\widehat{\beta}_i$-inverse strongly accretive for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = (\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)) \cap \Omega \cap (\bigcap_{i=0}^{\infty}\mathrm{VI}(C, A_i)) \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i < \widehat{\beta}_i/\kappa^2$ for $i = 1, 2$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}y_n &= \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n G x_n,\\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad \forall n \geq 0,\end{aligned} \tag{42}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $[0, 1]$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) *$\sum_{n=0}^{\infty}\alpha_n = \infty$ and $0 \leq \alpha_n \leq 1 - \rho$, for all $n \geq n_0$ for some integer $n_0 \geq 0$;*

(ii) *$\liminf_{n\to\infty}\gamma_n > 0$ and $\liminf_{n\to\infty}\delta_n > 0$;*

(iii) *$\lim_{n\to\infty}(|\alpha_{n+1}/(1 - (1 - \alpha_{n+1})\beta_{n+1}) - \alpha_n/(1 - (1 - \alpha_n)\beta_n)| + |\delta_{n+1}/(1 - \beta_{n+1}) - \delta_n/(1 - \beta_n)|) = 0$;*

(iv) *$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.*

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x - S_n x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty}S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)$. Then, there hold the following:

(I) *$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;*

(II) *$x_n \to q \Leftrightarrow \alpha_n(f(x_n) - x_n) \to 0$ provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0, 1)$, where $q \in F$ solves the following VIP:*

$$\langle q - f(q), J(q - p)\rangle \leq 0, \quad \forall p \in F. \tag{43}$$

Proof. First of all, since $0 < \lambda_i < \widehat{\alpha}_i/\kappa^2$ for $i = 0, 1, \ldots$, it is easy to see that G_i is a nonexpansive mapping for each $i = 0, 1, \ldots$. Since $B_n : C \to C$ is the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$ and $\rho_n, \rho_{n-1}, \ldots, \rho_0$, by Lemma 16 we know that, for each $x \in C$ and $k \geq 0$, the limit $\lim_{n\to\infty}U_{n,k}x$ exists. Moreover, one can define a mapping $B : C \to C$ as follows:

$$Bx = \lim_{n\to\infty}B_n x = \lim_{n\to\infty}U_{n,0}x \tag{44}$$

for every $x \in C$. That is, such a B is the W-mapping generated by the sequences $\{G_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$. According to Lemma 17, we know that $\mathrm{Fix}(B) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(G_i)$. From Lemma 15 and the definition of G_i, we have $\mathrm{Fix}(G_i) = \mathrm{VI}(C, A_i)$ for each $i = 0, 1, \ldots$. Hence, we have

$$\mathrm{Fix}(B) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(G_i) = \bigcap_{i=0}^{\infty}\mathrm{VI}(C, A_i). \tag{45}$$

Next, let us show that the sequence $\{x_n\}$ is bounded. Indeed, take a fixed $p \in F$ arbitrarily. Then, we get $p = Gp$, $p = B_n p$, and $p = S_n p$ for all $n \geq 0$. By Lemma 23 we know that G is nonexpansive. Then, from (42), we have

$$\begin{aligned}\|y_n - p\| &\leq \beta_n\|x_n - p\| + \gamma_n\|B_n x_n - p\| + \delta_n\|S_n G x_n - p\|\\ &\leq \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| + \delta_n\|G x_n - p\|\\ &\leq \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| + \delta_n\|x_n - p\|\\ &= \|x_n - p\|,\end{aligned} \tag{46}$$

and hence

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n \|f(x_n)-p\| + (1-\alpha_n)\|y_n-p\| \\ &\le \alpha_n (\|f(x_n)-f(p)\| + \|f(p)-p\|) \\ &\quad + (1-\alpha_n)\|y_n-p\| \\ &\le \alpha_n (\rho\|x_n-p\| + \|f(p)-p\|) \\ &\quad + (1-\alpha_n)\|x_n-p\| \\ &= (1-\alpha_n(1-\rho))\|x_n-p\| \\ &\quad + \alpha_n(1-\rho)\frac{\|f(p)-p\|}{1-\rho} \\ &\le \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-\rho}\right\}.\end{aligned} \tag{47}$$

By induction, we obtain

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\rho}\right\}, \quad \forall n \ge 0. \tag{48}$$

Thus, $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{Gx_n\}$ and $\{f(x_n)\}$.

Let us show that

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = 0. \tag{49}$$

As a matter of fact, put $\sigma_n = (1-\alpha_n)\beta_n$, for all $n \ge 0$. Then, it follows from (i) and (iv) that

$$\beta_n \ge \sigma_n = (1-\alpha_n)\beta_n \ge (1-(1-\rho))\beta_n = \rho\beta_n, \quad \forall n \ge n_0, \tag{50}$$

and hence

$$0 < \liminf_{n\to\infty} \sigma_n \le \limsup_{n\to\infty} \sigma_n < 1. \tag{51}$$

Define

$$x_{n+1} = \sigma_n x_n + (1-\sigma_n) z_n. \tag{52}$$

Observe that

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+2}-\sigma_{n+1}x_{n+1}}{1-\sigma_{n+1}} - \frac{x_{n+1}-\sigma_n x_n}{1-\sigma_n} \\ &= \frac{\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1}) y_{n+1} - \sigma_{n+1}x_{n+1}}{1-\sigma_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + (1-\alpha_n) y_n - \sigma_n x_n}{1-\sigma_n} \\ &= \left(\frac{\alpha_{n+1} f(x_{n+1})}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\sigma_n}\right) \\ &\quad - \frac{(1-\alpha_n)[\beta_n x_n + \gamma_n B_n x_n + \delta_n S_n G x_n] - \sigma_n x_n}{1-\sigma_n} \\ &\quad + (1-\alpha_{n+1})[\beta_{n+1}x_{n+1} + \gamma_{n+1}B_{n+1}x_{n+1} \\ &\qquad + \delta_{n+1}S_{n+1}Gx_{n+1}] \\ &\quad - \sigma_{n+1}x_{n+1} \times (1-\sigma_{n+1})^{-1} \\ &= \left(\frac{\alpha_{n+1} f(x_{n+1})}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\sigma_n}\right) + \frac{1-\alpha_{n+1}}{1-\sigma_{n+1}} \\ &\quad \times (\gamma_{n+1}B_{n+1}x_{n+1} + \delta_{n+1}S_{n+1}Gx_{n+1}) \\ &\quad - \frac{1-\alpha_n}{1-\sigma_n}(\gamma_n B_n x_n + \delta_n S_n G x_n) \\ &= \left(\frac{\alpha_{n+1} f(x_{n+1})}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\sigma_n}\right) \\ &\quad + \frac{(1-\alpha_{n+1})(1-\beta_{n+1})}{1-\sigma_{n+1}} \\ &\quad \times \left[\frac{\gamma_{n+1}B_{n+1}x_{n+1} + \delta_{n+1}S_{n+1}Gx_{n+1}}{1-\beta_{n+1}} \right. \\ &\qquad \left. - \frac{\gamma_n B_n x_n + \delta_n S_n G x_n}{1-\beta_n}\right] \\ &\quad + \left[\frac{(1-\alpha_{n+1})(1-\beta_{n+1})}{1-\sigma_{n+1}} - \frac{(1-\alpha_n)(1-\beta_n)}{1-\sigma_n}\right] \\ &\quad \times \frac{\gamma_n B_n x_n + \delta_n S_n G x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\sigma_{n+1}}(f(x_{n+1}) - f(x_n)) \\ &\quad + \left(\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right) f(x_n) \\ &\quad + \frac{(1-\alpha_{n+1})(1-\beta_{n+1})}{1-\sigma_{n+1}} \\ &\quad \times \left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(B_{n+1}x_{n+1} - B_n x_n) \right. \\ &\qquad + \left(\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\gamma_n}{\gamma_n+\delta_n}\right) B_n x_n \\ &\qquad + \frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(S_{n+1}Gx_{n+1} - S_n G x_n) \\ &\qquad \left. + \left(\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right) S_n G x_n\right] \\ &\quad - \left(\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right)\frac{\gamma_n B_n x_n + \delta_n S_n G x_n}{\gamma_n+\delta_n} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_{n+1}}{1-\sigma_{n+1}} \left(f\left(x_{n+1}\right) - f\left(x_n\right)\right) \\
&\quad + \left(\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right) \\
&\quad \times \left(f\left(x_n\right) - \frac{\gamma_n B_n x_n + \delta_n S_n G x_n}{\gamma_n + \delta_n}\right) \\
&\quad + \frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}} \\
&\quad \times \left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}} \left(B_{n+1} x_{n+1} - B_n x_n\right)\right. \\
&\qquad + \left(\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\gamma_n}{\gamma_n+\delta_n}\right) B_n x_n \\
&\qquad + \frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} \left(S_{n+1} G x_{n+1} - S_n G x_n\right) \\
&\qquad \left. + \left(\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right) S_n G x_n \right],
\end{aligned} \tag{53}
$$

and hence

$$
\begin{aligned}
&\left\|z_{n+1} - z_n\right\| \\
&\quad \leq \frac{\alpha_{n+1}}{1-\sigma_{n+1}} \left\|f\left(x_{n+1}\right) - f\left(x_n\right)\right\| \\
&\qquad + \left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right| \left\|f\left(x_n\right) - \frac{\gamma_n B_n x_n + \delta_n S_n G x_n}{\gamma_n + \delta_n}\right\| \\
&\qquad + \frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}} \\
&\qquad \times \left\|\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}} \left(B_{n+1} x_{n+1} - B_n x_n\right)\right. \\
&\qquad\quad + \left(\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\gamma_n}{\gamma_n+\delta_n}\right) B_n x_n \\
&\qquad\quad + \frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} \left(S_{n+1} G x_{n+1} - S_n G x_n\right) \\
&\qquad\quad \left. + \left(\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right) S_n G x_n \right\| \\
&\quad \leq \frac{\rho \alpha_{n+1}}{1-\sigma_{n+1}} \left\|x_{n+1} - x_n\right\| + \left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right| \\
&\qquad \times \left(\left\|f\left(x_n\right)\right\| + \left\|B_n x_n\right\| + \left\|S_n G x_n\right\|\right) \\
&\qquad + \frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}} \\
&\qquad \times \left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}} \left\|B_{n+1} x_{n+1} - B_n x_n\right\|\right. \\
&\qquad\quad + \left|\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\gamma_n}{\gamma_n+\delta_n}\right| \left\|B_n x_n\right\| \\
&\qquad\quad + \frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} \left\|S_{n+1} G x_{n+1} - S_n G x_n\right\| \\
&\qquad\quad \left. + \left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}} - \frac{\delta_n}{\gamma_n+\delta_n}\right| \left\|S_n G x_n\right\| \right].
\end{aligned} \tag{54}
$$

On the other hand, we note that, for all $n \geq 0$,

$$
\begin{aligned}
&\left\|S_{n+1} G x_{n+1} - S_n G x_n\right\| \\
&\quad \leq \left\|S_{n+1} G x_{n+1} - S_{n+1} G x_n\right\| + \left\|S_{n+1} G x_n - S_n G x_n\right\| \\
&\quad \leq \left\|G x_{n+1} - G x_n\right\| + \left\|S_{n+1} G x_n - S_n G x_n\right\| \\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \left\|S_{n+1} G x_n - S_n G x_n\right\|.
\end{aligned} \tag{55}
$$

Furthermore, by (CY), since G_i and $U_{n,i}$ are nonexpansive, we deduce that for each $n \geq 0$

$$
\begin{aligned}
&\left\|B_{n+1} x_{n+1} - B_n x_n\right\| \\
&\quad \leq \left\|B_{n+1} x_{n+1} - B_{n+1} x_n\right\| + \left\|B_{n+1} x_n - B_n x_n\right\| \\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \left\|B_{n+1} x_n - B_n x_n\right\| \\
&\quad = \left\|x_{n+1} - x_n\right\| + \left\|\lambda_0 G_0 U_{n+1,1} x_n - \lambda_0 G_0 U_{n,1} x_n\right\| \\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \lambda_0 \left\|U_{n+1,1} x_n - U_{n,1} x_n\right\| \\
&\quad = \left\|x_{n+1} - x_n\right\| + \lambda_0 \left\|\lambda_1 G_1 U_{n+1,2} x_n - \lambda_1 G_1 U_{n,2} x_n\right\| \\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \lambda_0 \lambda_1 \left\|U_{n+1,2} x_n - U_{n,2} x_n\right\| \\
&\qquad \vdots \\
&\quad \leq \left\|x_{n+1} - x_n\right\| + \left(\prod_{i=0}^{n} \lambda_i\right) \left\|U_{n+1,n+1} x_n - U_{n,n+1} x_n\right\| \\
&\quad \leq \left\|x_{n+1} - x_n\right\| + M_0 \prod_{i=0}^{n} \lambda_i,
\end{aligned} \tag{56}
$$

for some constant $M_0 > 0$. Utilizing (54)–(56), we have

$$
\begin{aligned}
&\left\|z_{n+1} - z_n\right\| \\
&\quad \leq \frac{\rho \alpha_{n+1}}{1-\sigma_{n+1}} \left\|x_{n+1} - x_n\right\| + \left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}} - \frac{\alpha_n}{1-\sigma_n}\right| \\
&\qquad \times \left(\left\|f\left(x_n\right)\right\| + \left\|B_n x_n\right\| + \left\|S_n G x_n\right\|\right) \\
&\qquad + \frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}}
\end{aligned}
$$

$$\times\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\left(\|x_{n+1}-x_n\|+M_0\prod_{i=0}^{n}\lambda_i\right)\right.$$
$$+\left|\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right|\|B_nx_n\|$$
$$+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\left(\|x_{n+1}-x_n\|+\|S_{n+1}Gx_n-S_nGx_n\|\right)$$
$$\left.+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|\|S_nGx_n\|\right]$$
$$=\frac{1-\sigma_{n+1}-\alpha_{n+1}(1-\rho)}{1-\sigma_{n+1}}\|x_{n+1}-x_n\|$$
$$+\left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right|\left(\|f(x_n)\|+\|B_nx_n\|+\|S_nGx_n\|\right)$$
$$+\frac{1-\sigma_{n+1}-\alpha_{n+1}}{1-\sigma_{n+1}}$$
$$\times\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}M_0\prod_{i=0}^{n}\lambda_i\right.$$
$$+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\|S_{n+1}Gx_n-S_nGx_n\|$$
$$\left.+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|\left(\|B_nx_n\|+\|S_nGx_n\|\right)\right]$$
$$\le\|x_{n+1}-x_n\|+\left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right|M$$
$$+M\prod_{i=0}^{n}\lambda_i+\|S_{n+1}Gx_n-S_nGx_n\|$$
$$+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|M$$
$$=\|x_{n+1}-x_n\|$$
$$+M\left(\left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right|\right.$$
$$\left.+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|+\prod_{i=0}^{n}\lambda_i\right)$$
$$+\|S_{n+1}Gx_n-S_nGx_n\|, \tag{57}$$

which hence yields

$$\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|$$
$$\le M\left(\left|\frac{\alpha_{n+1}}{1-\sigma_{n+1}}-\frac{\alpha_n}{1-\sigma_n}\right|+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|\right.$$
$$\left.+\prod_{i=0}^{n}\lambda_i\right)+\|S_{n+1}Gx_n-S_nGx_n\|, \tag{58}$$

where $\sup_{n\ge0}\{\|f(x_n)\|+\|B_nx_n\|+\|S_nGx_n\|+M_0\}\le M$ for some $M>0$. So, from (58), condition (iii), and the assumption on $\{S_n\}$, it follows that (noting that $0<\lambda_i\le b<1$, for all $i\ge0$)

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\right)\le 0. \tag{59}$$

Consequently, by Lemma 20, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{60}$$

It follows from (51) and (52) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\sigma_n)\|z_n-x_n\|=0. \tag{61}$$

From (42), we have

$$x_{n+1}-x_n=\alpha_n(f(x_n)-x_n)+(1-\alpha_n)(y_n-x_n), \tag{62}$$

which hence implies that

$$\begin{aligned}\rho\|y_n-x_n\|&=(1-(1-\rho))\|y_n-x_n\|\\&\le(1-\alpha_n)\|y_n-x_n\|\\&=\|x_{n+1}-x_n-\alpha_n(f(x_n)-x_n)\|\\&\le\|x_{n+1}-x_n\|+\|\alpha_n(f(x_n)-x_n)\|.\end{aligned} \tag{63}$$

Since $x_{n+1}-x_n\to0$ and $\alpha_n(f(x_n)-x_n)\to0$, we get

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{64}$$

Next, we show that $\|x_n-Gx_n\|\to0$ as $n\to\infty$.

Indeed, for simplicity, put $q=\Pi_C(p-\mu_2B_2p)$, $u_n=\Pi_C(x_n-\mu_2B_2x_n)$ and $v_n=\Pi_C(u_n-\mu_1B_1u_n)$. Then, $v_n=Gx_n$ for all $n\ge0$. From Lemma 22, we have

$$\begin{aligned}\|u_n-q\|^2&=\|\Pi_C(x_n-\mu_2B_2x_n)-\Pi_C(p-\mu_2B_2p)\|^2\\&\le\|x_n-p-\mu_2(B_2x_n-B_2p)\|^2\\&\le\|x_n-p\|^2-2\mu_2(\widehat{\beta}_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2,\end{aligned} \tag{65}$$

$$\begin{aligned}\|v_n-p\|^2&=\|\Pi_C(u_n-\mu_1B_1u_n)-\Pi_C(q-\mu_1B_1q)\|^2\\&\le\|u_n-q-\mu_1(B_1u_n-B_1q)\|^2\\&\le\|u_n-q\|^2-2\mu_1(\widehat{\beta}_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2.\end{aligned} \tag{66}$$

Substituting (65) for (66), we obtain

$$\begin{aligned}\|v_n-p\|^2\le{}&\|x_n-p\|^2-2\mu_2(\widehat{\beta}_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\\&-2\mu_1(\widehat{\beta}_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2.\end{aligned} \tag{67}$$

From (42) and (67), we have

$$\begin{aligned}\|y_n - p\|^2 &\le \beta_n\|x_n - p\|^2 + \gamma_n\|B_n x_n - p\|^2 + \delta_n\|S_n G x_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|v_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\ &\quad + \delta_n\left[\|x_n - p\|^2 - 2\mu_2\left(\widehat{\beta}_2 - \kappa^2\mu_2\right)\right. \\ &\qquad \times \|B_2 x_n - B_2 p\|^2 \\ &\qquad \left. - 2\mu_1\left(\widehat{\beta}_1 - \kappa^2\mu_1\right)\|B_1 u_n - B_1 q\|^2\right] \\ &= \|x_n - p\|^2 \\ &\quad - 2\delta_n\left[\mu_2\left(\widehat{\beta}_2 - \kappa^2\mu_2\right)\|B_2 x_n - B_2 p\|^2\right. \\ &\qquad \left. + 2\mu_1\left(\widehat{\beta}_1 - \kappa^2\mu_1\right)\|B_1 u_n - B_1 q\|^2\right],\end{aligned} \tag{68}$$

which hence implies that

$$\begin{aligned}&2\delta_n\left[\mu_2\left(\widehat{\beta}_2 - \kappa^2\mu_2\right)\|B_2 x_n - B_2 p\|^2\right. \\ &\qquad \left. + \mu_1\left(\widehat{\beta}_1 - \kappa^2\mu_1\right)\|B_1 u_n - B_1 q\|^2\right] \\ &\quad \le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad \le \left(\|x_n - p\| + \|y_n - p\|\right)\|x_n - y_n\|.\end{aligned} \tag{69}$$

Since $0 < \mu_i < \widehat{\beta}_i/\kappa^2$ for $i = 1, 2$, and $\{x_n\}$, $\{y_n\}$ are bounded, we obtain from (64), (69), and condition (ii) that

$$\lim_{n\to\infty}\|B_2 x_n - B_2 p\| = 0, \qquad \lim_{n\to\infty}\|B_1 u_n - B_1 q\| = 0. \tag{70}$$

Utilizing Proposition 6 and Lemma 9, we have

$$\begin{aligned}\|u_n - q\|^2 &= \|\Pi_C(x_n - \mu_2 B_2 x_n) - \Pi_C(p - \mu_2 B_2 p)\|^2 \\ &\le \langle x_n - \mu_2 B_2 x_n - (p - \mu_2 B_2 p), J(u_n - q)\rangle \\ &= \langle x_n - p, J(u_n - q)\rangle + \mu_2\langle B_2 p - B_2 x_n, J(u_n - q)\rangle \\ &\le \frac{1}{2}\left[\|x_n - p\|^2 + \|u_n - q\|^2 - g_1\left(\|x_n - u_n - (p - q)\|\right)\right] \\ &\quad + \mu_2\|B_2 p - B_2 x_n\|\|u_n - q\|,\end{aligned} \tag{71}$$

which implies that

$$\begin{aligned}\|u_n - q\|^2 &\le \|x_n - p\|^2 - g_1\left(\|x_n - u_n - (p - q)\|\right) \\ &\quad + 2\mu_2\|B_2 p - B_2 x_n\|\|u_n - q\|.\end{aligned} \tag{72}$$

In the same way, we derive

$$\begin{aligned}\|v_n - p\|^2 &= \|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(q - \mu_1 B_1 q)\|^2 \\ &\le \langle u_n - \mu_1 B_1 u_n - (q - \mu_1 B_1 q), J(v_n - p)\rangle \\ &= \langle u_n - q, J(v_n - p)\rangle + \mu_1\langle B_1 q - B_1 u_n, J(v_n - p)\rangle \\ &\le \frac{1}{2}\left[\|u_n - q\|^2 + \|v_n - p\|^2\right. \\ &\qquad \left. - g_2\left(\|u_n - v_n + (p - q)\|\right)\right] \\ &\quad + \mu_1\|B_1 q - B_1 u_n\|\|v_n - p\|,\end{aligned} \tag{73}$$

which implies that

$$\begin{aligned}\|v_n - p\|^2 &\le \|u_n - q\|^2 - g_2\left(\|u_n - v_n + (p - q)\|\right) \\ &\quad + 2\mu_1\|B_1 q - B_1 u_n\|\|v_n - p\|.\end{aligned} \tag{74}$$

Substituting (72) for (74), we get

$$\begin{aligned}\|v_n - p\|^2 &\le \|x_n - p\|^2 - g_1\left(\|x_n - u_n - (p - q)\|\right) \\ &\quad - g_2\left(\|u_n - v_n + (p - q)\|\right) \\ &\quad + 2\mu_2\|B_2 p - B_2 x_n\|\|u_n - q\| \\ &\quad + 2\mu_1\|B_1 q - B_1 u_n\|\|v_n - p\|.\end{aligned} \tag{75}$$

By Lemma 8(i), we have from (68) and (75)

$$\begin{aligned}\|y_n - p\|^2 &\le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|v_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\ &\quad + \delta_n\left[\|x_n - p\|^2 - g_1\left(\|x_n - u_n - (p - q)\|\right)\right. \\ &\qquad - g_2\left(\|u_n - v_n + (p - q)\|\right) + 2\mu_2\|B_2 p - B_2 x_n\| \\ &\qquad \left. \times \|u_n - q\| + 2\mu_1\|B_1 q - B_1 u_n\|\|v_n - p\|\right] \\ &\le \|x_n - p\|^2 \\ &\quad - \delta_n\left[g_1\left(\|x_n - u_n - (p - q)\|\right)\right. \\ &\qquad \left. + g_2\left(\|u_n - v_n + (p - q)\|\right)\right] \\ &\quad + 2\mu_2\|B_2 p - B_2 x_n\|\|u_n - q\| \\ &\quad + 2\mu_1\|B_1 q - B_1 u_n\|\|v_n - p\|,\end{aligned} \tag{76}$$

which hence leads to

$$\begin{aligned}
&\delta_n [g_1 (\|x_n - u_n - (p - q)\|) + g_2 (\|u_n - v_n + (p - q)\|)] \\
&\le \|x_n - p\|^2 - \|y_n - p\|^2 + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\| \\
&\le (\|x_n - p\| + \|y_n - p\|) \|x_n - y_n\| \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\| .
\end{aligned} \tag{77}$$

From (70), (77), condition (ii), and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} g_1 (\|x_n - u_n - (p - q)\|) = 0, \\
&\lim_{n\to\infty} g_2 (\|u_n - v_n + (p - q)\|) = 0.
\end{aligned} \tag{78}$$

Utilizing the properties of g_1 and g_2, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \|x_n - u_n - (p - q)\| = 0, \\
&\lim_{n\to\infty} \|u_n - v_n + (p - q)\| = 0.
\end{aligned} \tag{79}$$

From (79), we get

$$\begin{aligned}
\|x_n - v_n\| &\le \|x_n - u_n - (p - q)\| \\
&\quad + \|u_n - v_n + (p - q)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{80}$$

That is,

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{81}$$

Next, let us show that

$$\lim_{n\to\infty} \|B_n x_n - x_n\| = 0, \qquad \lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{82}$$

Indeed, utilizing Lemma 15 and (42), we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&= \left\|\delta_n (S_n Gx_n - p) + (\beta_n + \gamma_n) \left(\frac{\beta_n x_n + \gamma_n B_n x_n}{\beta_n + \gamma_n} - p\right)\right\|^2 \\
&\le \delta_n \|S_n Gx_n - p\|^2 + (\beta_n + \gamma_n) \\
&\quad \times \left\|\frac{\beta_n x_n + \gamma_n B_n x_n}{\beta_n + \gamma_n} - p\right\|^2 \\
&= \delta_n \|S_n Gx_n - p\|^2 + (\beta_n + \gamma_n) \\
&\quad \times \left\|\frac{\beta_n}{\beta_n + \gamma_n} (x_n - p) + \frac{\gamma_n}{\beta_n + \gamma_n} (B_n x_n - p)\right\|^2 \\
&\le \delta_n \|Gx_n - p\|^2 + (\beta_n + \gamma_n) \\
&\quad \times \left[\frac{\beta_n}{\beta_n + \gamma_n} \|x_n - p\|^2 + \frac{\gamma_n}{\beta_n + \gamma_n} \|B_n x_n - p\|^2 \right. \\
&\qquad \left. - \frac{\beta_n \gamma_n}{(\beta_n + \gamma_n)^2} g_3 (\|x_n - B_n x_n\|)\right] \\
&\le \delta_n \|x_n - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 \\
&\quad - \frac{\beta_n \gamma_n}{\beta_n + \gamma_n} g_3 (\|x_n - B_n x_n\|) \\
&= \|x_n - p\|^2 - \frac{\beta_n \gamma_n}{\beta_n + \gamma_n} g_3 (\|x_n - B_n x_n\|) ,
\end{aligned} \tag{83}$$

which immediately implies that

$$\begin{aligned}
&\beta_n \gamma_n g_3 (\|x_n - B_n x_n\|) \\
&\qquad \le \frac{\beta_n \gamma_n}{\beta_n + \gamma_n} g_3 (\|x_n - B_n x_n\|) \\
&\qquad \le \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\qquad \le (\|x_n - p\| + \|y_n - p\|) \|x_n - y_n\| .
\end{aligned} \tag{84}$$

So, from (64), the boundedness of $\{x_n\}$, $\{y_n\}$, and conditions (ii), (iv), it follows that

$$\lim_{n\to\infty} g_3 (\|x_n - B_n x_n\|) = 0. \tag{85}$$

From the properties of g_3, we have

$$\lim_{n\to\infty} \|x_n - B_n x_n\| = 0. \tag{86}$$

Taking into account that

$$y_n - x_n = \gamma_n (B_n x_n - x_n) + \delta_n (S_n Gx_n - x_n) , \tag{87}$$

we have

$$\begin{aligned}
&\delta_n \|S_n Gx_n - x_n\| \\
&\qquad = \|y_n - x_n - \gamma_n (B_n x_n - x_n)\| \\
&\qquad \le \|y_n - x_n\| + \gamma_n \|B_n x_n - x_n\| \\
&\qquad \le \|y_n - x_n\| + \|B_n x_n - x_n\| .
\end{aligned} \tag{88}$$

From (64), (86), and condition (ii), it follows that

$$\lim_{n\to\infty} \|S_n Gx_n - x_n\| = 0, \qquad \lim_{n\to\infty} \|S_n Gx_n - B_n x_n\| = 0. \tag{89}$$

Note that

$$\begin{aligned} &\|x_n - Sx_n\| \\ &\quad \le \|x_n - S_nGx_n\| + \|S_nGx_n - S_nx_n\| \\ &\qquad + \|S_nx_n - Sx_n\| \\ &\quad \le \|x_n - S_nGx_n\| + \|Gx_n - x_n\| \\ &\qquad + \|S_nx_n - Sx_n\|. \end{aligned} \tag{90}$$

So, in terms of (81), (89), and Lemma 12, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{91}$$

Suppose that $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$ such that $\beta + \gamma_n + \delta_n = 1$ for all $n \ge 0$. Define a mapping $Vx = (1 - \theta_1 - \theta_2)Sx + \theta_1 Bx + \theta_2 Gx$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then, by Lemmas 14 and 17, we have that $\mathrm{Fix}(V) = \mathrm{Fix}(S) \cap \mathrm{Fix}(B) \cap \mathrm{Fix}(G) = F$. For each $k \ge 1$, let $\{p_k\}$ be a unique element of C such that

$$p_k = \frac{1}{k} f(p_k) + \left(1 - \frac{1}{k}\right) Vp_k. \tag{92}$$

From Lemma 13, we conclude that $p_k \to q \in \mathrm{Fix}(V) = F$ as $k \to \infty$. Observe that for every n, k

$$\begin{aligned} &\|y_n - Bp_k\| \\ &\quad = \|\beta(x_n - Bp_k) + \gamma_n(B_nx_n - Bp_k) + \delta_n(S_nGx_n - Bp_k)\| \\ &\quad \le \beta\|x_n - Bp_k\| + \gamma_n\|B_nx_n - Bp_k\| \\ &\qquad + \delta_n(\|S_nGx_n - B_nx_n\| + \|B_nx_n - Bp_k\|) \\ &\quad = \beta\|x_n - Bp_k\| + (1-\beta)\|B_nx_n - Bp_k\| \\ &\qquad + \delta_n\|S_nGx_n - B_nx_n\|, \end{aligned} \tag{93}$$

and hence

$$\begin{aligned} &\|x_{n+1} - Bp_k\| \\ &\quad \le \alpha_n\|f(x_n) - Bp_k\| + (1-\alpha_n)\|y_n - Bp_k\| \\ &\quad \le \alpha_n(\|f(x_n) - x_n\| + \|x_n - Bp_k\|) \\ &\qquad + (1-\alpha_n)\|y_n - Bp_k\| \\ &\quad \le \alpha_n\|f(x_n) - x_n\| + \alpha_n\|x_n - Bp_k\| + (1-\alpha_n) \\ &\qquad \times [\beta\|x_n - Bp_k\| \\ &\qquad\quad + (1-\beta)\|B_nx_n - Bp_k\| + \delta_n\|S_nGx_n - B_nx_n\|] \\ &\quad \le \alpha_n\|f(x_n) - x_n\| + \alpha_n\|x_n - Bp_k\| + (1-\alpha_n) \\ &\qquad \times [\beta\|x_n - Bp_k\| \\ &\qquad\quad + (1-\beta)(\|B_nx_n - B_np_k\| + \|B_np_k - Bp_k\|) \\ &\qquad\quad + \delta_n\|S_nGx_n - B_nx_n\|] \\ &\quad \le \alpha_n\|f(x_n) - x_n\| + \alpha_n\|x_n - Bp_k\| \\ &\qquad + (1-\alpha_n)[\beta\|x_n - Bp_k\| \\ &\qquad\quad + (1-\beta)(\|x_n - p_k\| + \|B_np_k - Bp_k\|) \\ &\qquad\quad + \delta_n\|S_nGx_n - B_nx_n\|] \\ &\quad = \alpha_n\|f(x_n) - x_n\| + (\beta + \alpha_n(1-\beta)) \\ &\qquad \times \|x_n - Bp_k\| + (1-\alpha_n)(1-\beta)\|x_n - p_k\| \\ &\qquad + (1-\alpha_n)[(1-\beta)\|B_np_k - Bp_k\| \\ &\qquad\quad + \delta_n\|S_nGx_n - B_nx_n\|] \\ &\quad \le \alpha_n\|f(x_n) - x_n\| + (\beta + \alpha_n(1-\beta)) \\ &\qquad \times (\|x_n - x_{n+1}\| + \|x_{n+1} - Bp_k\|) \\ &\qquad + (1-\alpha_n)(1-\beta)(\|x_n - x_{n+1}\| + \|x_{n+1} - p_k\|) \\ &\qquad + (1-\alpha_n)[(1-\beta)\|B_np_k - Bp_k\| \\ &\qquad\quad + \delta_n\|S_nGx_n - B_nx_n\|] \\ &\quad = \alpha_n\|f(x_n) - x_n\| + (\beta + \alpha_n(1-\beta)) \\ &\qquad \times \|x_{n+1} - Bp_k\| + (1-\alpha_n)(1-\beta)\|x_{n+1} - p_k\| \\ &\qquad + (1-\alpha_n)[(1-\beta)\|B_np_k - Bp_k\| \\ &\qquad\quad + \delta_n\|S_nGx_n - B_nx_n\|] + \|x_n - x_{n+1}\|. \end{aligned} \tag{94}$$

So, it immediately follows from $0 \le \alpha_n \le 1 - \rho$, for all $n \ge n_0$, that

$$\begin{aligned} &\|x_{n+1} - Bp_k\| \\ &\quad \le \|x_{n+1} - p_k\| + \|B_np_k - Bp_k\| + \frac{1}{(1-\alpha_n)(1-\beta)} \\ &\qquad \times (\|\alpha_n(x_n - f(x_n))\| + \|x_n - x_{n+1}\|) \\ &\qquad + \frac{\delta_n}{1-\beta}\|S_nGx_n - B_nx_n\| \\ &\quad \le \|x_{n+1} - p_k\| + \|B_np_k - Bp_k\| \\ &\qquad + \|S_nGx_n - B_nx_n\| + \frac{1}{\rho(1-\beta)} \\ &\qquad \times (\|\alpha_n(x_n - f(x_n))\| + \|x_n - x_{n+1}\|) \\ &\quad = \|x_{n+1} - p_k\| + \theta_n, \quad \forall n \ge n_0, \end{aligned} \tag{95}$$

where $\theta_n = \|B_n p_k - Bp_k\| + \|S_n G x_n - B_n x_n\| + (1/\rho(1-\beta))(\|\alpha_n(x_n - f(x_n))\| + \|x_n - x_{n+1}\|)$. Since $\lim_{n\to\infty}\|B_n p_k - Bp_k\| = \lim_{n\to\infty}\|S_n G x_n - B_n x_n\| = \lim_{n\to\infty}\|\alpha_n(x_n - f(x_n))\| = \lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$, we know that $\theta_n \to 0$ as $n \to \infty$.

From (95), we obtain

$$\begin{aligned}\|x_{n+1} - Bp_k\|^2 &\le \|x_{n+1} - p_k\|^2 \\ &\quad + \theta_n (2\|x_{n+1} - p_k\| + \theta_n), \quad \forall n \ge n_0.\end{aligned} \tag{96}$$

For any Banach limit μ, from (96), we derive

$$\begin{aligned}\mu_n\|x_n - Bp_k\|^2 &= \mu_n\|x_{n+1} - Bp_k\|^2 \\ &\le \mu_n\|x_{n+1} - p_k\|^2 = \mu_n\|x_n - p_k\|^2.\end{aligned} \tag{97}$$

In addition, note that

$$\begin{aligned}&\|x_n - Gp_k\|^2 \\ &\quad \le \|x_n - Gx_n + Gx_n - Gp_k\|^2 \\ &\quad \le (\|x_n - Gx_n\| + \|x_n - p_k\|)^2 \\ &\quad = \|x_n - p_k\|^2 + \|x_n - Gx_n\| \\ &\qquad \times (2\|x_n - p_k\| + \|x_n - Gx_n\|), \\ &\|x_n - Sp_k\|^2 \\ &\quad \le \|x_n - Sx_n + Sx_n - Sp_k\|^2 \\ &\quad \le (\|x_n - Sx_n\| + \|x_n - p_k\|)^2 \\ &\quad = \|x_n - p_k\|^2 + \|x_n - Sx_n\| \\ &\qquad \times (2\|x_n - p_k\| + \|x_n - Sx_n\|).\end{aligned} \tag{98}$$

It is easy to see from (81) and (91) that

$$\begin{aligned}\mu_n\|x_n - Gp_k\|^2 &\le \mu_n\|x_n - p_k\|^2, \\ \mu_n\|x_n - Sp_k\|^2 &\le \mu_n\|x_n - p_k\|^2.\end{aligned} \tag{99}$$

Utilizing (97) and (99), we deduce that

$$\begin{aligned}&\mu_n\|x_n - Vp_k\|^2 \\ &\quad = \mu_n \|(1-\theta_1-\theta_2)(x_n - Sp_k) \\ &\qquad + \theta_1 (x_n - Bp_k) + \theta_2 (x_n - Gp_k)\|^2 \\ &\quad \le (1-\theta_1-\theta_2)\mu_n\|x_n - Sp_k\|^2 \\ &\qquad + \theta_1\mu_n\|x_n - Bp_k\|^2 + \theta_2\mu_n\|x_n - Gp_k\|^2 \\ &\quad \le \mu_n\|x_n - p_k\|^2.\end{aligned} \tag{100}$$

Also, observe that

$$x_n - p_k = \frac{1}{k}(x_n - f(p_k)) + \left(1 - \frac{1}{k}\right)(x_n - Vp_k); \tag{101}$$

that is,

$$\left(1 - \frac{1}{k}\right)(x_n - Vp_k) = x_n - p_k - \frac{1}{k}(x_n - f(p_k)). \tag{102}$$

It follows from Lemma 8 (ii) and (102) that

$$\begin{aligned}&\left(1 - \frac{1}{k}\right)^2\|x_n - Vp_k\|^2 \\ &\quad \ge \|x_n - p_k\|^2 - \frac{2}{k}\langle x_n - p_k + p_k - f(p_k), J(x_n - p_k)\rangle \\ &\quad = \left(1 - \frac{2}{k}\right)\|x_n - p_k\|^2 + \frac{2}{k}\langle f(p_k) - p_k, J(x_n - p_k)\rangle.\end{aligned} \tag{103}$$

So by (100) and (103), we have

$$\begin{aligned}&\left(1 - \frac{1}{k}\right)^2\mu_n\|x_n - p_k\|^2 \\ &\quad \ge \left(1 - \frac{2}{k}\right)\mu_n\|x_n - p_k\|^2 \\ &\qquad + \frac{2}{k}\mu_n\langle f(p_k) - p_k, J(x_n - p_k)\rangle,\end{aligned} \tag{104}$$

and hence

$$\frac{1}{k^2}\mu_n\|x_n - p_k\|^2 \ge \frac{2}{k}\mu_n\langle f(p_k) - p_k, J(x_n - p_k)\rangle. \tag{105}$$

This implies that

$$\frac{1}{2k}\mu_n\|x_n - p_k\|^2 \ge \mu_n\langle f(p_k) - p_k, J(x_n - p_k)\rangle. \tag{106}$$

Since $p_k \to q \in \mathrm{Fix}(V) = F$ as $k \to \infty$, by the uniform Frechet differentiability of the norm of X we have

$$\mu_n\langle f(q) - q, J(x_n - q)\rangle \le 0. \tag{107}$$

On the other hand, from (49) and the norm-to-norm uniform continuity of J on bounded subsets of X, it follows that

$$\lim_{n\to\infty}|\langle f(q) - q, J(x_{n+1} - q)\rangle - \langle f(q) - q, J(x_n - q)\rangle| = 0. \tag{108}$$

So, utilizing Lemma 18 we deduce from (107) and (108) that

$$\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \le 0, \tag{109}$$

which together with (49) and the norm-to-norm uniform continuity of J on bounded subsets of X, implies that

$$\limsup_{n\to\infty}\langle f(q) - q, J(x_{n+1} - q)\rangle \le 0. \tag{110}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. Utilizing Lemma 8 (i), from (42) and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned} &\|y_n - q\|^2 \\ &\quad\leq \beta_n\|x_n - q\|^2 + \gamma_n\|B_nx_n - q\|^2 \\ &\qquad + \delta_n\|S_nGx_n - q\|^2 \\ &\quad\leq \beta_n\|x_n - q\|^2 + \gamma_n\|x_n - q\|^2 \\ &\qquad + \delta_n\|x_n - q\|^2 \\ &\quad= \|x_n - q\|^2, \end{aligned} \tag{111}$$

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad= \|\alpha_n(f(x_n) - f(q)) + (1-\alpha_n)(y_n - q) \\ &\qquad + \alpha_n(f(q) - q)\|^2 \\ &\quad\leq \|\alpha_n(f(x_n) - f(q)) + (1-\alpha_n)(y_n - q)\|^2 \\ &\qquad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\quad\leq \alpha_n\|f(x_n) - f(q)\|^2 + (1-\alpha_n)\|y_n - q\|^2 \\ &\qquad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\quad\leq \alpha_n\rho\|x_n - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 \\ &\qquad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\quad= (1-\alpha_n(1-\rho))\|x_n - q\|^2 \\ &\qquad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\quad= (1-\alpha_n(1-\rho))\|x_n - q\|^2 \\ &\qquad + \alpha_n(1-\rho)\frac{2\langle f(q) - q, J(x_{n+1} - q)\rangle}{1-\rho}. \end{aligned} \tag{112}$$

Applying Lemma 7 to (112), we obtain that $x_n \to q$ as $n \to \infty$.

Conversely, if $x_n \to q \in F$ as $n \to \infty$, then from (42) it follows that

$$\begin{aligned} &\|y_n - q\| \\ &\quad\leq \beta_n\|x_n - q\| + \gamma_n\|B_nx_n - q\| \\ &\qquad + \delta_n\|S_nGx_n - q\| \\ &\quad\leq \beta_n\|x_n - q\| + \gamma_n\|x_n - q\| + \delta_n\|x_n - q\| \\ &\quad= \|x_n - q\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{113}$$

that is, $y_n \to q$. Again from (42) we obtain that

$$\begin{aligned} &\|\alpha_n(f(x_n) - x_n)\| \\ &\quad= \|x_{n+1} - x_n - (1-\alpha_n)(y_n - x_n)\| \\ &\quad\leq \|x_{n+1} - x_n\| + (1-\alpha_n)\|y_n - x_n\| \\ &\quad\leq \|x_{n+1} - q\| + \|x_n - q\| \\ &\qquad + (1-\alpha_n)(\|y_n - q\| + \|x_n - q\|) \\ &\quad\leq \|x_{n+1} - q\| + 2\|x_n - q\| + \|y_n - q\|. \end{aligned} \tag{114}$$

Since $x_n \to q$ and $y_n \to q$, we get $\alpha_n(f(x_n) - x_n) \to 0$. This completes the proof. □

Corollary 25. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $\{\rho_n\}_{n=0}^\infty$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0,1)$ and $A_i : C \to E$ an $\widehat{\alpha}_i$-inverse strongly accretive mapping for each $i = 0, 1, \ldots$. Define a mapping $G_i : C \to C$ by $\Pi_C(I - \lambda_iA_i)x = G_ix$ for all $x \in C$ and $i = 0, 1, \ldots$, where $\lambda_i \in (0, \widehat{\alpha}_i/\kappa^2]$ and κ is the 2-uniformly smooth constant of X. Let $B_n : C \to C$ be the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$ and $\rho_n, \rho_{n-1}, \ldots, \rho_0$. Let $V : C \to C$ be an α-strictly pseudocontractive mapping. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=0}^\infty$ be a countable family of nonexpansive mappings of C into itself such that $F = (\bigcap_{i=0}^\infty \mathrm{Fix}(S_i)) \cap \mathrm{Fix}(V) \cap (\bigcap_{i=0}^\infty \mathrm{VI}(C, A_i)) \neq \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} y_n &= \beta_nx_n + \gamma_nB_nx_n + \delta_nS_n((1-l)I + lV)x_n, \\ x_{n+1} &= \alpha_nf(x_n) + (1-\alpha_n)y_n, \quad \forall n \geq 0, \end{aligned} \tag{115}$$

where $0 < l < \alpha/\kappa^2$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $[0,1]$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) *$\sum_{n=0}^\infty \alpha_n = \infty$ and $0 \leq \alpha_n \leq 1 - \rho$, for all $n \geq n_0$ for some integer $n_0 \geq 0$;*

(ii) *$\liminf_{n\to\infty}\gamma_n > 0$ and $\liminf_{n\to\infty}\delta_n > 0$;*

(iii) *$\lim_{n\to\infty}(|\alpha_{n+1}/(1-(1-\alpha_{n+1})\beta_{n+1}) - \alpha_n/(1-(1-\alpha_n)\beta_n)| + |\delta_{n+1}/(1-\beta_{n+1}) - \delta_n/(1-\beta_n)|) = 0$;*

(iv) *$0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.*

Assume that $\sum_{n=0}^\infty \sup_{x\in D}\|S_{n+1}x - S_nx\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty}S_nx$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^\infty \mathrm{Fix}(S_i)$. Then, there hold the following:

(I) *$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;*

(II) *$x_n \to q \Leftrightarrow \alpha_n(f(x_n) - x_n) \to 0$ provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$, where $q \in F$ solves the following VIP*

$$\langle q - f(q), J(q - p)\rangle \leq 0, \quad \forall p \in F. \tag{116}$$

Proof. In Theorem 24, we put $B_1 = I - V$, $B_2 = 0$, and $\mu_1 = l$, where $0 < l < \alpha/\kappa^2$. Then, GSVI (13) is equivalent to the VIP of finding $x^* \in C$ such that

$$\langle B_1x^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{117}$$

In this case, $B_1 : C \to X$ is α-inverse strongly accretive. It is not hard to see that $\mathrm{Fix}(V) = \mathrm{VI}(C, B_1)$. As a matter of fact, we have, for $l > 0$,

$$\begin{aligned} u \in \mathrm{VI}(C, B_1) & \\ &\Longleftrightarrow \langle B_1 u, J(y-u)\rangle \geq 0 \quad \forall y \in C \\ &\Longleftrightarrow \langle u - lB_1 u - u, J(u-y)\rangle \geq 0 \quad \forall y \in C \\ &\Longleftrightarrow u = \Pi_C (u - lB_1 u) \\ &\Longleftrightarrow u = \Pi_C (u - lu + lVu) \\ &\Longleftrightarrow \langle u - lu + lVu - u, J(u-y)\rangle \geq 0 \quad \forall y \in C \\ &\Longleftrightarrow \langle u - Vu, J(u-y)\rangle \leq 0 \quad \forall y \in C \\ &\Longleftrightarrow u = Vu \\ &\Longleftrightarrow u \in \mathrm{Fix}(V). \end{aligned} \tag{118}$$

Accordingly, we know that $F = (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)) \cap \Omega \cap (\bigcap_{i=0}^{\infty} \mathrm{VI}(C, A_i)) = (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)) \cap \mathrm{Fix}(V) \cap (\bigcap_{i=0}^{\infty} \mathrm{VI}(C, A_i))$, and

$$\begin{aligned} \Pi_C (I - \mu_1 B_1) \Pi_C (I - \mu_2 B_2) x_n &= \Pi_C (I - \mu_1 B_1) x_n \\ &= \Pi_C ((1-l) x_n + lVx_n) \\ &= ((1-l) I + lV) x_n. \end{aligned} \tag{119}$$

So, the scheme (42) reduces to (115). Therefore, the desired result follows from Theorem 24. □

Here, we prove the following important lemmas which will be used in the sequel.

Lemma 26. *Let C be a nonempty closed convex subset of a smooth Banach space X and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Then, for $\mu_i \in (0, 1]$ one has*

$$\begin{aligned} &\|(I - \mu_i B_i) x - (I - \mu_i B_i) y\| \\ &\quad \leq \left\{ \sqrt{\frac{1-\alpha_i}{\lambda_i}} + (1-\mu_i)\left(1 + \frac{1}{\lambda_i}\right)\right\} \|x - y\|, \\ &\qquad \forall x, y \in C, \end{aligned} \tag{120}$$

for $i = 1, 2$. In particular, if $1 - (\lambda_i/(1+\lambda_i))(1 - \sqrt{(1-\alpha_i)/\lambda_i}) \leq \mu_i \leq 1$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Proof. Taking into account the λ_i-strict pseudocontractivity of B_i, we derive for every $x, y \in C$

$$\begin{aligned} \lambda_i \|(I - B_i) x - (I - B_i) y\|^2 &\leq \langle (I - B_i) x - (I - B_i) y, J(x-y)\rangle \\ &\leq \|(I - B_i) x - (I - B_i) y\| \|x - y\|, \end{aligned} \tag{121}$$

which implies that

$$\|(I - B_i) x - (I - B_i) y\| \leq \frac{1}{\lambda_i} \|x - y\|. \tag{122}$$

Hence,

$$\begin{aligned} \|B_i x - B_i y\| &\leq \|(I - B_i) x - (I - B_i) y\| + \|x - y\| \\ &\leq \left(1 + \frac{1}{\lambda_i}\right) \|x - y\|. \end{aligned} \tag{123}$$

Utilizing the α_i-strong accretivity and λ_i-strict pseudocontractivity of B_i, we get

$$\begin{aligned} \lambda_i \|(I - B_i) x - (I - B_i) y\|^2 &\leq \|x - y\|^2 - \langle B_i x - B_i y, J(x-y)\rangle \\ &\leq (1-\alpha_i) \|x - y\|^2. \end{aligned} \tag{124}$$

So, we have

$$\|(I - B_i) x - (I - B_i) y\| \leq \sqrt{\frac{1-\alpha_i}{\lambda_i}} \|x - y\|. \tag{125}$$

Therefore, for $\mu_i \in (0, 1]$ we have

$$\begin{aligned} &\|(I - \mu_i B_i) x - (I - \mu_i B_i) y\| \\ &\quad \leq \|(I - B_i) x - (I - B_i) y\| + (1-\mu_i) \|B_i x - B_i y\| \\ &\quad \leq \sqrt{\frac{1-\alpha_i}{\lambda_i}} \|x - y\| + (1-\mu_i)\left(1 + \frac{1}{\lambda_i}\right) \|x - y\| \\ &\quad = \left\{ \sqrt{\frac{1-\alpha_i}{\lambda_i}} + (1-\mu_i)\left(1 + \frac{1}{\lambda_i}\right)\right\} \|x - y\|. \end{aligned} \tag{126}$$

Since $1 - (\lambda_i/(1+\lambda_i))(1 - \sqrt{(1-\alpha_i)/\lambda_i}) \leq \mu_i \leq 1$, it follows immediately that

$$\sqrt{\frac{1-\alpha_i}{\lambda_i}} + (1-\mu_i)\left(1 + \frac{1}{\lambda_i}\right) \leq 1. \tag{127}$$

This implies that $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$. □

Lemma 27. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Let $G : C \to C$ be the mapping defined by*

$$G(x) = \Pi_C [\Pi_C (x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C (x - \mu_2 B_2 x)], \quad \forall x \in C. \tag{128}$$

If $1 - (\lambda_i/(1+\lambda_i))(1 - \sqrt{(1-\alpha_i)/\lambda_i}) \leq \mu_i \leq 1$, then $G : C \to C$ is nonexpansive.

Proof. According to Lemma 26, we know that $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$. Hence, for all $x, y \in C$, we have

$$\begin{aligned}
&\|G(x) - G(y)\| \\
&= \|\Pi_C[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)] \\
&\quad - \Pi_C[\Pi_C(y - \mu_2 B_2 y) - \mu_1 B_1 \Pi_C(y - \mu_2 B_2 y)]\| \\
&= \|\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x \\
&\quad - \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y\| \\
&\leq \|(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x \\
&\quad - (I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y\| \\
&\leq \|\Pi_C(I - \mu_2 B_2)x - \Pi_C(I - \mu_2 B_2)y\| \\
&\leq \|(I - \mu_2 B_2)x - (I - \mu_2 B_2)y\| \\
&\leq \|x - y\|.
\end{aligned} \tag{129}$$

This shows that $G : C \to C$ is nonexpansive. This completes the proof. □

Theorem 28. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gateaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$ and $A_i : C \to X$ be ξ_i-strictly pseudocontractive and $\widehat{\alpha}_i$-strongly accretive with $\xi_i + \widehat{\alpha}_i \geq 1$ for each $i = 0, 1, \ldots$. Define a mapping $G_i : C \to C$ by $\Pi_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 0, 1, \ldots$, where $1 - (\xi_i/(1 + \xi_i))(1 - \sqrt{(1 - \widehat{\alpha}_i)/\xi_i}) \leq \lambda_i \leq 1$ for all $i = 0, 1, \ldots$. Let $B_n : C \to C$ be the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$ and $\rho_n, \rho_{n-1}, \ldots, \rho_0$. Let the mapping $B_i : C \to X$ ζ_i-strictly pseudocontractive and $\widehat{\beta}_i$-strongly accretive with $\zeta_i + \widehat{\beta}_i \geq 1$ for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)) \cap \Omega \cap (\bigcap_{i=0}^{\infty} \mathrm{VI}(C, A_i)) \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $1 - (\zeta_i/(1 + \zeta_i))(1 - \sqrt{(1 - \widehat{\beta}_i)/\zeta_i}) \leq \mu_i \leq 1$ for $i = 1, 2$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n G x_n, \\
x_{n+1} &= \alpha_n f(x_n) + \sigma_n G x_n + (1 - \alpha_n - \sigma_n) y_n, \quad \forall n \geq 0,
\end{aligned} \tag{130}$$

where $\{\sigma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $[0, 1]$ such that $\beta_n + \gamma_n + \delta_n = 1$ and $\alpha_n + \sigma_n \leq 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $0 \leq \alpha_n + \sigma_n \leq 1 - \rho$, *for all* $n \geq n_0$ *for some integer* $n_0 \geq 0$;

(ii) $\liminf_{n\to\infty} \sigma_n > 0$, $\liminf_{n\to\infty} \gamma_n > 0$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iii) $\lim_{n\to\infty}(|\alpha_{n+1}/(1-(1-\alpha_{n+1}-\sigma_{n+1})\beta_{n+1}) - (\alpha_n/(1-(1-\alpha_n-\sigma_n)\beta_n))| + |\sigma_{n+1}/(1-(1-\alpha_{n+1}-\sigma_{n+1})\beta_{n+1}) - (\sigma_n/(1-(1-\alpha_n-\sigma_n)\beta_n))| + |\delta_{n+1}/(1-\beta_{n+1}) - \delta_n/(1-\beta_n)|) = 0$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Assume that $\sum_{n=0}^{\infty} \sup_{x \in D} \|S_{n+1}x - S_n x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. Then there hold the following:

(I) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(II) $x_n \to q \Leftrightarrow \alpha_n(f(x_n) - x_n) \to 0$ *provided* $\beta_n \equiv \beta$ *for some fixed* $\beta \in (0, 1)$, *where* $q \in F$ *solves the following VIP*

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall p \in F. \tag{131}$$

Proof. First of all, take a fixed $p \in F$ arbitrarily. Then we obtain $p = Gp$, $p = B_n p$ and $S_n p = p$ for all $n \geq 0$. By Lemma 27, we get from (130)

$$\begin{aligned}
&\|y_n - p\| \\
&\leq \beta_n \|x_n - p\| + \gamma_n \|B_n x_n - p\| + \delta_n \|S_n G x_n - p\| \\
&\leq \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|x_n - p\| \\
&= \|x_n - p\|,
\end{aligned} \tag{132}$$

and hence

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\leq \alpha_n \|f(x_n) - p\| + \sigma_n \|Gx_n - p\| \\
&\quad + (1 - \alpha_n - \sigma_n) \|y_n - p\| \\
&\leq \alpha_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) \\
&\quad + \sigma_n \|x_n - p\| + (1 - \alpha_n - \sigma_n) \|x_n - p\| \\
&\leq \alpha_n (\rho \|x_n - p\| + \|f(p) - p\|) \\
&\quad + \sigma_n \|x_n - p\| + (1 - \alpha_n - \sigma_n) \|x_n - p\| \\
&= (1 - \alpha_n(1 - \rho)) \|x_n - p\| + \alpha_n \|f(p) - p\| \\
&= (1 - \alpha_n(1 - \rho)) \|x_n - p\| + \alpha_n (1 - \rho) \frac{\|f(p) - p\|}{1 - \rho} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \rho}\right\}.
\end{aligned} \tag{133}$$

By induction, we have

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \rho}\right\}, \quad \forall n \geq 0, \tag{134}$$

which implies that $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{Gx_n\}$, and $\{f(x_n)\}$.

Let us show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{135}$$

As a matter of fact, put $\theta_n=(1-\alpha_n-\sigma_n)\beta_n$, for all $n\geq 0$. Then, it follows from (i) and (iv) that

$$\beta_n \geq \theta_n=(1-\alpha_n-\sigma_n)\beta_n \geq (1-(1-\rho))\beta_n=\rho\beta_n, \quad \forall n\geq n_0, \tag{136}$$

and hence

$$0<\liminf_{n\to\infty}\theta_n\leq\limsup_{n\to\infty}\theta_n<1. \tag{137}$$

Define

$$x_{n+1}=\theta_n x_n+(1-\theta_n)z_n. \tag{138}$$

Observe that

$$\begin{aligned}
&z_{n+1}-z_n\\
&=\frac{x_{n+2}-\theta_{n+1}x_{n+1}}{1-\theta_{n+1}}-\frac{x_{n+1}-\theta_n x_n}{1-\theta_n}\\
&=(\alpha_{n+1}f(x_{n+1})+\sigma_{n+1}Gx_{n+1}\\
&\qquad+(1-\alpha_{n+1}-\sigma_{n+1})y_{n+1}-\theta_{n+1}x_{n+1})\\
&\quad\times(1-\theta_{n+1})^{-1}\\
&\quad-\frac{\alpha_n f(x_n)+\sigma_n Gx_n+(1-\alpha_n-\sigma_n)y_n-\theta_n x_n}{1-\theta_n}\\
&=\left(\frac{\alpha_{n+1}f(x_{n+1}+\sigma_{n+1}Gx_{n+1})}{1-\theta_{n+1}}-\frac{\alpha_n f(x_n)+\sigma_n Gx_n}{1-\theta_n}\right)\\
&\quad-\frac{(1-\alpha_n-\sigma_n)[\beta_n x_n+\gamma_n B_n x_n+\delta_n S_n Gx_n]-\theta_n x_n}{1-\theta_n}\\
&\quad+(1-\alpha_{n+1}-\sigma_{n+1})\\
&\quad\times[\beta_{n+1}x_{n+1}+\gamma_{n+1}B_{n+1}x_{n+1}+\delta_{n+1}S_{n+1}Gx_{n+1}]\\
&\quad-\theta_{n+1}x_{n+1}\times(1-\theta_{n+1})^{-1}\\
&=\left(\frac{\alpha_{n+1}f(x_{n+1})+\sigma_{n+1}Gx_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n f(x_n)+\sigma_n Gx_n}{1-\theta_n}\right)\\
&\quad+\frac{1-\alpha_{n+1}-\sigma_{n+1}}{1-\theta_{n+1}}(\gamma_{n+1}B_{n+1}x_{n+1}+\delta_{n+1}S_{n+1}Gx_{n+1})\\
&\quad-\frac{1-\alpha_n-\sigma_n}{1-\theta_n}(\gamma_n B_n x_n+\delta_n S_n Gx_n)\\
&=\left(\frac{\alpha_{n+1}f(x_{n+1})+\sigma_{n+1}Gx_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n f(x_n)+\sigma_n Gx_n}{1-\theta_n}\right)\\
&\quad+\frac{(1-\alpha_{n+1}-\sigma_{n+1})(1-\beta_{n+1})}{1-\theta_{n+1}}\\
&\quad\times\left[\frac{\gamma_{n+1}B_{n+1}x_{n+1}+\delta_{n+1}S_{n+1}Gx_{n+1}}{1-\beta_{n+1}}\right.\\
&\qquad\left.-\frac{\gamma_n B_n x_n+\delta_n S_n Gx_n}{1-\beta_n}\right]\\
&\quad+\left[\frac{(1-\alpha_{n+1}-\sigma_{n+1})(1-\beta_{n+1})}{1-\theta_{n+1}}\right.\\
&\qquad\left.-\frac{(1-\alpha_n-\sigma_n)(1-\beta_n)}{1-\theta_n}\right]\\
&\quad\times\frac{\gamma_n B_n x_n+\delta_n S_n Gx_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}}{1-\theta_{n+1}}(f(x_{n+1})-f(x_n))\\
&\quad+\frac{\sigma_{n+1}}{1-\theta_{n+1}}(Gx_{n+1}-Gx_n)\\
&\quad+\left(\frac{\alpha_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n}{1-\theta_n}\right)f(x_n)\\
&\quad+\left(\frac{\sigma_{n+1}}{1-\theta_{n+1}}-\frac{\sigma_n}{1-\theta_n}\right)Gx_n\\
&\quad+\frac{(1-\alpha_{n+1}-\sigma_{n+1})(1-\beta_{n+1})}{1-\theta_{n+1}}\\
&\quad\times\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(B_{n+1}x_{n+1}-B_n x_n)\right.\\
&\qquad+\left(\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right)B_n x_n\\
&\qquad+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}(S_{n+1}Gx_{n+1}-S_n Gx_n)\\
&\qquad\left.+\left(\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right)S_n Gx_n\right]\\
&\quad-\left(\frac{\alpha_{n+1}+\sigma_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n+\sigma_n}{1-\theta_n}\right)\frac{\gamma_n B_n x_n+\delta_n S_n Gx_n}{\gamma_n+\delta_n}\\
&=\frac{\alpha_{n+1}}{1-\theta_{n+1}}(f(x_{n+1})-f(x_n))\\
&\quad+\frac{\sigma_{n+1}}{1-\theta_{n+1}}(Gx_{n+1}-Gx_n)\\
&\quad+\left(\frac{\alpha_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n}{1-\theta_n}\right)\left(f(x_n)-\frac{\gamma_n B_n x_n+\delta_n S_n Gx_n}{\gamma_n+\delta_n}\right)\\
&\quad+\left(\frac{\sigma_{n+1}}{1-\theta_{n+1}}-\frac{\sigma_n}{1-\theta_n}\right)\left(Gx_n-\frac{\gamma_n B_n x_n+\delta_n S_n Gx_n}{\gamma_n+\delta_n}\right)\\
&\quad+\frac{1-\alpha_{n+1}-\sigma_{n+1}-\theta_{n+1}}{1-\theta_{n+1}}
\end{aligned}$$

$$\times\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\left(B_{n+1}x_{n+1}-B_nx_n\right)\right.$$
$$+\left(\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right)B_nx_n$$
$$+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\left(S_{n+1}Gx_{n+1}-S_nGx_n\right)$$
$$\left.+\left(\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right)S_nGx_n\right], \tag{139}$$

and hence

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \frac{\alpha_{n+1}}{1-\theta_{n+1}}\|f(x_{n+1})-f(x_n)\| \\ &+\frac{\sigma_{n+1}}{1-\theta_{n+1}}\|Gx_{n+1}-Gx_n\| \\ &+\left|\frac{\alpha_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n}{1-\theta_n}\right|\left\|f(x_n)-\frac{\gamma_nB_nx_n+\delta_nS_nGx_n}{\gamma_n+\delta_n}\right\| \\ &+\left|\frac{\sigma_{n+1}}{1-\theta_{n+1}}-\frac{\sigma_n}{1-\theta_n}\right|\left\|Gx_n-\frac{\gamma_nB_nx_n+\delta_nS_nGx_n}{\gamma_n+\delta_n}\right\| \\ &+\frac{1-\alpha_{n+1}-\sigma_{n+1}-\theta_{n+1}}{1-\theta_{n+1}} \\ &\times\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\|B_{n+1}x_{n+1}-B_nx_n\|\right. \\ &\quad+\left|\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right|\|B_nx_n\| \\ &\quad+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\|S_{n+1}Gx_{n+1}-S_nGx_n\| \\ &\quad\left.+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|\|S_nGx_n\|\right].\end{aligned} \tag{140}$$

On the other hand, repeating the same arguments as those of (55) and (56) in the proof of Theorem 24, we can get

$$\|S_{n+1}Gx_{n+1}-S_nGx_n\|\le\|x_{n+1}-x_n\|+\|S_{n+1}Gx_n-S_nGx_n\|,$$
$$\|B_{n+1}x_{n+1}-B_nx_n\|\le\|x_{n+1}-x_n\|+M_0\prod_{i=0}^{n}\lambda_i, \tag{141}$$

for some constant $M_0>0$. Utilizing (140)-(141), we have

$$\begin{aligned}\|z_{n+1}-z_n\| &\le\frac{\alpha_{n+1}}{1-\theta_{n+1}}\rho\|x_{n+1}-x_n\| \\ &+\frac{\sigma_{n+1}}{1-\theta_{n+1}}\|x_{n+1}-x_n\| \\ &+\left|\frac{\alpha_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n}{1-\theta_n}\right| \\ &\times\left(\|f(x_n)\|+\|B_nx_n\|+\|S_nGx_n\|\right) \\ &+\left|\frac{\sigma_{n+1}}{1-\theta_{n+1}}-\frac{\sigma_n}{1-\theta_n}\right| \\ &\times\left(\|Gx_n\|+\|B_nx_n\|+\|S_nGx_n\|\right) \\ &+\frac{1-\alpha_{n+1}-\sigma_{n+1}-\theta_{n+1}}{1-\theta_{n+1}} \\ &\times\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\left(\|x_{n+1}-x_n\|+M_0\prod_{i=0}^{n}\lambda_i\right)\right. \\ &\quad+\left|\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\gamma_n}{\gamma_n+\delta_n}\right|\|B_nx_n\| \\ &\quad+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\left(\|x_{n+1}-x_n\|+\|S_{n+1}Gx_n-S_nGx_n\|\right) \\ &\quad\left.+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|\|S_nGx_n\|\right] \\ &=\frac{1-\alpha_{n+1}(1-\rho)-\theta_{n+1}}{1-\theta_{n+1}}\|x_{n+1}-x_n\| \\ &+\frac{1-\alpha_{n+1}-\sigma_{n+1}-\theta_{n+1}}{1-\theta_{n+1}} \\ &\times\left[\frac{\gamma_{n+1}}{\gamma_{n+1}+\delta_{n+1}}M_0\prod_{i=0}^{n}\lambda_i+\left|\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}-\frac{\delta_n}{\gamma_n+\delta_n}\right|\right. \\ &\quad\times\left(\|B_nx_n\|+\|S_nGx_n\|\right) \\ &\quad\left.+\frac{\delta_{n+1}}{\gamma_{n+1}+\delta_{n+1}}\|S_{n+1}Gx_n-S_nGx_n\|\right] \\ &+\left|\frac{\alpha_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n}{1-\theta_n}\right| \\ &\times\left(\|f(x_n)\|+\|B_nx_n\|+\|S_nGx_n\|\right) \\ &+\left|\frac{\sigma_{n+1}}{1-\theta_{n+1}}-\frac{\sigma_n}{1-\theta_n}\right| \\ &\times\left(\|Gx_n\|+\|B_nx_n\|+\|S_nGx_n\|\right) \\ &\le\|x_{n+1}-x_n\| \\ &+M\left(\prod_{i=0}^{n}\lambda_i+\left|\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n}\right|\right. \\ &\quad\left.+\left|\frac{\alpha_{n+1}}{1-\theta_{n+1}}-\frac{\alpha_n}{1-\theta_n}\right|+\left|\frac{\sigma_{n+1}}{1-\theta_{n+1}}-\frac{\sigma_n}{1-\theta_n}\right|\right) \\ &+\|S_{n+1}Gx_n-S_nGx_n\|,\end{aligned} \tag{142}$$

where $\sup_{n\geq 0}\{\|f(x_n)\| + \|Gx_n\| + \|B_n x_n\| + \|S_n G x_n\| + M_0\} \leq M$ for some $M > 0$. So, from (142), condition (iii), and the assumption on $\{S_n\}$ it follows that (noting that $0 < \lambda_i \leq b < 1$, for all $i \geq 0$)

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{143}$$

Consequently, by Lemma 20, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{144}$$

It follows from (137) and (138) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\theta_n) \|z_n - x_n\| = 0. \tag{145}$$

Next, we show that $\|x_n - Gx_n\| \to 0$ as $n \to \infty$. Indeed, in terms of Lemma 11, from (130), we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad \leq \beta_n \|x_n - p\|^2 + \gamma_n \|B_n x_n - p\|^2 \\
&\qquad + \delta_n \|S_n G x_n - p\|^2 \\
&\quad \leq \beta_n \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 + \delta_n \|x_n - p\|^2 \\
&\quad = \|x_n - p\|^2, \\
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n (x_n - p) + \sigma_n (Gx_n - p) \\
&\quad + (1-\alpha_n-\sigma_n)(y_n - p) \\
&\quad + \alpha_n (f(x_n) - x_n)\|^2 \\
&\leq [\|\alpha_n (x_n - p) + \sigma_n (Gx_n - p) + (1-\alpha_n-\sigma_n)(y_n - p)\| \\
&\quad + \|\alpha_n (f(x_n) - x_n)\|]^2 \\
&= \|\alpha_n (x_n - p) + \sigma_n (Gx_n - p) + (1-\alpha_n-\sigma_n)(y_n - p)\|^2 \\
&\quad + \|\alpha_n (f(x_n) - x_n)\| \\
&\quad \times [2\|\alpha_n (x_n - p) + \sigma_n (Gx_n - p) \\
&\qquad + (1-\alpha_n-\sigma_n)(y_n - p)\| + \|\alpha_n (f(x_n) - x_n)\|] \\
&= \Bigg\|\alpha_n (x_n - p) + (1-\alpha_n) \\
&\quad \times \left[\frac{\sigma_n}{1-\alpha_n}(Gx_n - p) + \frac{1-\alpha_n-\sigma_n}{1-\alpha_n}(y_n - p)\right]\Bigg\|^2 \\
&\quad + \|\alpha_n (f(x_n) - x_n)\| \\
&\quad \times [2\|\alpha_n (x_n - p) + \sigma_n (Gx_n - p) \\
&\qquad + (1-\alpha_n-\sigma_n)(y_n - p)\| \\
&\qquad + \|\alpha_n (f(x_n) - x_n)\|] \\
&\leq \alpha_n \|x_n - p\|^2 \\
&\quad + (1-\alpha_n)\left\|\frac{\sigma_n}{1-\alpha_n}(Gx_n - p) + \frac{1-\alpha_n-\sigma_n}{1-\alpha_n}(y_n - p)\right\|^2 \\
&\quad + \|\alpha_n (f(x_n) - x_n)\| \\
&\quad \times [2(\alpha_n \|x_n - p\| + \sigma_n \|Gx_n - p\| \\
&\qquad + (1-\alpha_n-\sigma_n)\|y_n - p\|) + \|\alpha_n (f(x_n) - x_n)\|] \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \\
&\quad \times \left[\frac{\sigma_n}{1-\alpha_n}\|Gx_n - p\|^2 + \frac{1-\alpha_n-\sigma_n}{1-\alpha_n}\|y_n - p\|^2 \right. \\
&\qquad \left. - \frac{\sigma_n(1-\alpha_n-\sigma_n)}{(1-\alpha_n)^2} g(\|Gx_n - y_n\|)\right] \\
&\quad + \|\alpha_n (f(x_n) - x_n)\| \\
&\quad \times [2(\alpha_n \|x_n - p\| + \sigma_n \|x_n - p\| + (1-\alpha_n-\sigma_n)\|x_n - p\|) \\
&\qquad + \|\alpha_n (f(x_n) - x_n)\|] \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \\
&\quad \times \left[\frac{\sigma_n}{1-\alpha_n}\|x_n - p\|^2 + \frac{1-\alpha_n-\sigma_n}{1-\alpha_n}\|x_n - p\|^2 \right. \\
&\qquad \left. - \frac{\sigma_n(1-\alpha_n-\sigma_n)}{(1-\alpha_n)^2} g(\|Gx_n - y_n\|)\right] \\
&\quad + \|\alpha_n (f(x_n) - x_n)\| (2\|x_n - p\| + \|\alpha_n (f(x_n) - x_n)\|) \\
&= \|x_n - p\|^2 - \frac{\sigma_n(1-\alpha_n-\sigma_n)}{1-\alpha_n} g(\|Gx_n - y_n\|) \\
&\quad + \|\alpha_n (f(x_n) - x_n)\| (2\|x_n - p\| + \|\alpha_n (f(x_n) - x_n)\|).
\end{aligned} \tag{146}$$

Then, it immediately follows from $0 \leq \alpha_n + \sigma_n \leq 1-\rho$, for all $n \geq n_0$ that

$$\begin{aligned}
&\rho\sigma_n g(\|Gx_n - y_n\|) \\
&\quad \leq \frac{\sigma_n(1-\alpha_n-\sigma_n)}{1-\alpha_n} g(\|Gx_n - y_n\|) \\
&\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad + \|\alpha_n (f(x_n) - x_n)\| (2\|x_n - p\| + \|\alpha_n (f(x_n) - x_n)\|)
\end{aligned}$$

$$\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| + \|\alpha_n (f(x_n) - x_n)\| \times (2\|x_n - p\| + \|\alpha_n (f(x_n) - x_n)\|), \tag{147}$$

for all $n \geq n_0$. Since $\|\alpha_n(f(x_n) - x_n)\| \to 0$ and $\{x_n\}$ is bounded, we deduce from (145) and condition (ii) that

$$\lim_{n\to\infty} g(\|Gx_n - y_n\|) = 0. \tag{148}$$

Utilizing the properties of g, we have

$$\lim_{n\to\infty} \|Gx_n - y_n\| = 0. \tag{149}$$

Also, from (130) we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n (f(x_n) - x_n) + \sigma_n (Gx_n - x_n) + (1 - \alpha_n - \sigma_n)(y_n - x_n) \\ &= \alpha_n (f(x_n) - x_n) + \sigma_n (Gx_n - y_n + y_n - x_n) + (1 - \alpha_n - \sigma_n)(y_n - x_n) \\ &= \alpha_n (f(x_n) - x_n) + \sigma_n (Gx_n - y_n) + (1 - \alpha_n)(y_n - x_n), \end{aligned} \tag{150}$$

which hence leads to

$$\begin{aligned} \rho \|y_n - x_n\| &\leq (1 - \alpha_n - \sigma_n) \|y_n - x_n\| \\ &\leq (1 - \alpha_n) \|y_n - x_n\| \\ &= \|x_{n+1} - x_n - \alpha_n (f(x_n) - x_n) - \sigma_n (Gx_n - y_n)\| \\ &\leq \|x_{n+1} - x_n\| + \|\alpha_n (f(x_n) - x_n)\| + \sigma_n \|Gx_n - y_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\alpha_n (f(x_n) - x_n)\| + \|Gx_n - y_n\|. \end{aligned} \tag{151}$$

So, it is easy to see from (145), (149), and $\|\alpha_n(f(x_n) - x_n)\| \to 0$ that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{152}$$

We note that

$$\|Gx_n - x_n\| \leq \|Gx_n - y_n\| + \|y_n - x_n\|. \tag{153}$$

Therefore, from (149) and (152) it follows that

$$\lim_{n\to\infty} \|Gx_n - x_n\| = 0. \tag{154}$$

Repeating the same arguments as those of (86), (89), and (91) in the proof of Theorem 24, we can obtain

$$\lim_{n\to\infty} \|x_n - B_n x_n\| = \lim_{n\to\infty} \|S_n G x_n - B_n x_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{155}$$

Suppose that $\beta_n \equiv \beta$ for some fixed $\beta \in (0, 1)$ such that $\beta + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Define a mapping $Vx = (1 - \theta_1 - \theta_2)Sx + \theta_1 Bx + \theta_2 Gx$, where $\theta_1, \theta_2 \in (0, 1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then, by Lemmas 14 and 17, we have that $\mathrm{Fix}(V) = \mathrm{Fix}(S) \cap \mathrm{Fix}(B) \cap \mathrm{Fix}(G) = F$. For each $k \geq 1$, let $\{p_k\}$ be a unique element of C such that

$$p_k = \frac{1}{k} f(p_k) + \left(1 - \frac{1}{k}\right) V p_k. \tag{156}$$

From Lemma 13, we conclude that $p_k \to q \in \mathrm{Fix}(V) = F$ as $k \to \infty$. Observe that for every n, k

$$\begin{aligned} \|y_n - Bp_k\| &\leq \beta \|x_n - Bp_k\| + \gamma_n \|B_n x_n - Bp_k\| + \delta_n (\|S_n G x_n - B_n x_n\| + \|B_n x_n - Bp_k\|) \\ &= \beta \|x_n - Bp_k\| + (1 - \beta) \|B_n x_n - Bp_k\| + \delta_n \|S_n G x_n - B_n x_n\|, \end{aligned} \tag{157}$$

and hence

$$\begin{aligned} \|x_{n+1} - Bp_k\| &\leq \alpha_n \|f(x_n) - Bp_k\| + \sigma_n \|Gx_n - Bp_k\| + (1 - \alpha_n - \sigma_n) \|y_n - Bp_k\| \\ &\leq \alpha_n (\|f(x_n) - x_n\| + \|x_n - Bp_k\|) + \sigma_n (\|Gx_n - x_n\| + \|x_n - Bp_k\|) + (1 - \alpha_n - \sigma_n) \\ &\quad \times [\beta \|x_n - Bp_k\| + (1 - \beta) \|B_n x_n - Bp_k\| + \delta_n \|S_n G x_n - B_n x_n\|] \\ &\leq \alpha_n (\|f(x_n) - x_n\| + \|x_n - Bp_k\|) + \sigma_n (\|Gx_n - x_n\| + \|x_n - Bp_k\|) + (1 - \alpha_n - \sigma_n) \\ &\quad \times [\beta \|x_n - Bp_k\| + (1 - \beta)(\|B_n x_n - B_n p_k\| + \|B_n p_k - Bp_k\|) + \delta_n \|S_n G x_n - B_n x_n\|] \\ &\leq \alpha_n (\|f(x_n) - x_n\| + \|x_n - Bp_k\|) + \sigma_n (\|Gx_n - x_n\| + \|x_n - Bp_k\|) + (1 - \alpha_n - \sigma_n) \\ &\quad \times [\beta \|x_n - Bp_k\| + (1 - \beta)(\|x_n - p_k\| + \|B_n p_k - Bp_k\|) + \delta_n \|S_n G x_n - B_n x_n\|] \\ &= \alpha_n \|f(x_n) - x_n\| + \sigma_n \|Gx_n - x_n\| + [\beta + (\alpha_n + \sigma_n)(1 - \beta)] \|x_n - Bp_k\| \\ &\quad + (1 - \alpha_n - \sigma_n)(1 - \beta) \|x_n - p_k\| + (1 - \alpha_n - \sigma_n) \\ &\quad \times [(1 - \beta) \|B_n p_k - Bp_k\| + \delta_n \|S_n G x_n - B_n x_n\|] \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|f(x_n) - x_n\| + \sigma_n \|Gx_n - x_n\| \\
&\quad + [\beta + (\alpha_n + \sigma_n)(1-\beta)] \\
&\quad \times (\|x_n - x_{n+1}\| + \|x_{n+1} - Bp_k\|) + (1-\alpha_n-\sigma_n)(1-\beta) \\
&\quad \times (\|x_n - x_{n+1}\| + \|x_{n+1} - p_k\|) + (1-\alpha_n-\sigma_n) \\
&\quad \times [(1-\beta)\|B_n p_k - Bp_k\| + \delta_n \|S_n G x_n - B_n x_n\|] \\
&= \alpha_n \|f(x_n) - x_n\| + \sigma_n \|Gx_n - x_n\| \\
&\quad + [\beta + (\alpha_n + \sigma_n)(1-\beta)] \|x_{n+1} - Bp_k\| \\
&\quad + (1-\alpha_n-\sigma_n)(1-\beta)\|x_{n+1} - p_k\| + (1-\alpha_n-\sigma_n) \\
&\quad \times [(1-\beta)\|B_n p_k - Bp_k\| + \delta_n \|S_n G x_n - B_n x_n\|] \\
&\quad + \|x_n - x_{n+1}\|.
\end{aligned} \tag{158}$$

So, it immediately follows from $0 \leq \alpha_n \leq 1-\rho$, for all $n \geq n_0$ that

$$\begin{aligned}
&\|x_{n+1} - Bp_k\| \\
&\quad \leq \|x_{n+1} - p_k\| + \|B_n p_k - Bp_k\| \\
&\qquad + \frac{1}{(1-\alpha_n-\sigma_n)(1-\beta)} \\
&\qquad \times (\|\alpha_n (x_n - f(x_n))\| + \sigma_n \|Gx_n - x_n\| \\
&\qquad\quad + \|x_n - x_{n+1}\|) + \frac{\delta_n}{1-\beta}\|S_n G x_n - B_n x_n\| \\
&\quad \leq \|x_{n+1} - p_k\| + \|B_n p_k - Bp_k\| \\
&\qquad + \|S_n G x_n - B_n x_n\| + \frac{1}{\rho(1-\beta)} \\
&\qquad \times (\|\alpha_n (x_n - f(x_n))\| \\
&\qquad\quad + \|Gx_n - x_n\| + \|x_n - x_{n+1}\|) \\
&\quad = \|x_{n+1} - p_k\| + \tau_n, \quad \forall n \geq n_0,
\end{aligned} \tag{159}$$

where $\theta_n = \|B_n p_k - Bp_k\| + \|S_n G x_n - B_n x_n\| + 1/(\rho(1-\beta))(\|\alpha_n(x_n - f(x_n))\| + \|Gx_n - x_n\| + \|x_n - x_{n+1}\|)$. Since $\lim_{n\to\infty}\|B_n p_k - Bp_k\| = \lim_{n\to\infty}\|S_n G x_n - B_n x_n\| = \lim_{n\to\infty}\|\alpha_n(x_n - f(x_n))\| = \lim_{n\to\infty}\|Gx_n - x_n\| = \lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$, we know that $\tau_n \to 0$ as $n \to \infty$.

From (159), we obtain

$$\|x_{n+1} - Bp_k\|^2 \leq \|x_{n+1} - p_k\|^2 + \tau_n (2\|x_{n+1} - p_k\| + \tau_n), \quad \forall n \geq n_0. \tag{160}$$

For any Banach limit μ, from (160) we derive

$$\begin{aligned}
\mu_n \|x_n - Bp_k\|^2 &= \mu_n \|x_{n+1} - Bp_k\|^2 \\
&\leq \mu_n \|x_{n+1} - p_k\|^2 = \mu_n \|x_n - p_k\|^2.
\end{aligned} \tag{161}$$

Repeating the same arguments as those of (99), in the proof of Theorem 24, we can get

$$\begin{aligned}
\mu_n \|x_n - Gp_k\|^2 &\leq \mu_n \|x_n - p_k\|^2, \\
\mu_n \|x_n - Sp_k\|^2 &\leq \mu_n \|x_n - p_k\|^2.
\end{aligned} \tag{162}$$

Utilizing (161) and (162), we deduce that

$$\begin{aligned}
&\mu_n \|x_n - Vp_k\|^2 \\
&\quad \leq (1-\theta_1-\theta_2)\mu_n \|x_n - Sp_k\|^2 \\
&\qquad + \theta_1 \mu_n \|x_n - Bp_k\|^2 + \theta_2 \mu_n \|x_n - Gp_k\|^2 \\
&\quad \leq \mu_n \|x_n - p_k\|^2.
\end{aligned} \tag{163}$$

Also, observe that

$$\left(1 - \frac{1}{k}\right)(x_n - Vp_k) = x_n - p_k - \frac{1}{k}(x_n - f(p_k)). \tag{164}$$

Repeating the same arguments as those of (106) in the proof of Theorem 24, we can get

$$\frac{1}{2k}\mu_n \|x_n - p_k\|^2 \geq \mu_n \langle f(p_k) - p_k, J(x_n - p_k)\rangle. \tag{165}$$

Since $p_k \to q \in \mathrm{Fix}(V) = F$ as $k \to \infty$, by the uniform Gateaux differentiability of the norm of X, we have

$$\mu_n \langle f(q) - q, J(x_n - q)\rangle \leq 0. \tag{166}$$

On the other hand, from (135) and the norm-to-weak* uniform continuity of J on bounded subsets of X, it follows that

$$\lim_{n\to\infty} |\langle f(q) - q, J(x_{n+1} - q)\rangle - \langle f(q) - q, J(x_n - q)\rangle| = 0. \tag{167}$$

So, utilizing Lemma 18, we deduce from (166) and (167) that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \leq 0, \tag{168}$$

which, together with (135) and the norm-to-norm uniform continuity of J on bounded subsets of X, implies that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_{n+1} - q)\rangle \leq 0. \tag{169}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. Utilizing Lemma 8 (i), from (130) and the convexity of $\|\cdot\|^2$, we get

$$\|y_n - q\|^2 \le \beta_n\|x_n - q\|^2 + \gamma_n\|B_n x_n - q\|^2 + \delta_n\|S_n G x_n - q\|^2 \le \|x_n - q\|^2, \tag{170}$$

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n (f(x_n) - f(q)) + \sigma_n (Gx_n - q) \\
&\quad + (1-\alpha_n-\sigma_n)(y_n - q) + \alpha_n (f(q) - q)\|^2 \\
&\le \|\alpha_n (f(x_n) - f(q)) + \sigma_n (Gx_n - q) \\
&\quad + (1-\alpha_n-\sigma_n)(y_n - q)\|^2 \\
&\quad + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\le \alpha_n\|f(x_n) - f(q)\|^2 + \sigma_n\|Gx_n - q\|^2 \\
&\quad + (1-\alpha_n-\sigma_n)\|y_n - q\|^2 \\
&\quad + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\le \alpha_n\rho\|x_n - q\|^2 + \sigma_n\|x_n - q\|^2 \\
&\quad + (1-\alpha_n-\sigma_n)\|x_n - q\|^2 \\
&\quad + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q)\rangle \\
&= (1-\alpha_n(1-\rho))\|x_n - q\|^2 \\
&\quad + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q)\rangle \\
&= (1-\alpha_n(1-\rho))\|x_n - q\|^2 \\
&\quad + \alpha_n(1-\rho)\frac{2\langle f(q) - q, J(x_{n+1} - q)\rangle}{1-\rho}.
\end{aligned} \tag{171}$$

Applying Lemma 7 to (171), we obtain that $x_n \to q$ as $n \to \infty$.

Conversely, if $x_n \to q \in F$ as $n \to \infty$, then from (130) it follows that

$$\|y_n - q\| \le \beta_n\|x_n - q\| + \gamma_n\|B_n x_n - q\| + \delta_n\|S_n G x_n - q\| \le \|x_n - q\| \longrightarrow 0 \tag{172}$$

as $n \to \infty$; that is, $y_n \to q$. Again from (130) we obtain that

$$\begin{aligned}
\|\alpha_n (f(x_n) - x_n)\| &= \|x_{n+1} - x_n - \sigma_n (Gx_n - x_n) - (1-\alpha_n-\sigma_n)(y_n - x_n)\| \\
&\le \|x_{n+1} - x_n\| + \sigma_n\|Gx_n - x_n\| + (1-\alpha_n-\sigma_n)\|y_n - x_n\| \\
&\le \|x_{n+1} - q\| + \|x_n - q\| + \sigma_n(\|Gx_n - q\| + \|x_n - q\|) \\
&\quad + (1-\alpha_n-\sigma_n)(\|y_n - q\| + \|x_n - q\|) \\
&\le \|x_{n+1} - q\| + \|x_n - q\| + \sigma_n(\|x_n - q\| + \|x_n - q\|) \\
&\quad + (1-\alpha_n-\sigma_n)(\|y_n - q\| + \|x_n - q\|) \\
&\le \|x_{n+1} - q\| + 3\|x_n - q\| + \|y_n - q\|.
\end{aligned} \tag{173}$$

Since $x_n \to q$ and $y_n \to q$, we get $\alpha_n(f(x_n) - x_n) \to 0$. This completes the proof. □

Corollary 29. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gateaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0,1)$ and $A_i : C \to X$ ξ_i-strictly pseudocontractive and $\widehat{\alpha}_i$-strongly accretive with $\xi_i + \widehat{\alpha}_i \ge 1$ for each $i = 0, 1, \ldots$. Define a mapping $G_i : C \to C$ by $\Pi_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 0, 1, \ldots$, where $1-(\xi_i/(1+\xi_i))(1-\sqrt{(1-\widehat{\alpha}_i)/\xi_i}) \le \lambda_i \le 1$ for all $i = 0, 1, \ldots$. Let $B_n : C \to C$ be the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$ and $\rho_n, \rho_{n-1}, \ldots, \rho_0$. Let $V : C \to C$ be a self-mapping such that $I - V : C \to X$ is λ-strictly pseudocontractive and α-strongly accretive with $\alpha + \lambda \ge 1$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)) \cap \mathrm{Fix}(V) \cap (\bigcap_{i=0}^{\infty} \mathrm{VI}(C, A_i)) \ne \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n((1-l)I + lV)x_n, \\
x_{n+1} &= \alpha_n f(x_n) + \sigma_n((1-l)I + lV)x_n + (1-\alpha_n-\sigma_n)y_n, \\
&\qquad \forall n \ge 0,
\end{aligned} \tag{174}$$

where $1 - (\lambda/(1+\lambda))(1 - \sqrt{(1-\alpha)/\lambda}) \le l \le 1$ and $\{\sigma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $[0,1]$ such that $\beta_n + \gamma_n + \delta_n = 1$ and $\alpha_n + \sigma_n \le 1$ for all $n \ge 0$. Suppose that the following conditions hold:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $0 \le \alpha_n + \sigma_n \le 1 - \rho$, *for all* $n \ge n_0$ *for some integer* $n_0 \ge 0$;

(ii) $\liminf_{n\to\infty}\sigma_n > 0$, $\liminf_{n\to\infty}\gamma_n > 0$ *and* $\liminf_{n\to\infty}\delta_n > 0$;

(iii) $\lim_{n\to\infty}(|\alpha_{n+1}/(1-(1-\alpha_{n+1}-\sigma_{n+1})\beta_{n+1}) - \alpha_n/(1-(1-\alpha_n-\sigma_n)\beta_n)| + |\sigma_{n+1}/(1-(1-\alpha_{n+1}-\sigma_{n+1})\beta_{n+1}) - \sigma_n/(1-(1-\alpha_n-\sigma_n)\beta_n)| + |\delta_{n+1}/(1-\beta_{n+1}) - \delta_n/(1-\beta_n)|) = 0$;

(iv) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Assume that $\sum_{n=0}^{\infty} \sup_{x\in D}\|S_{n+1}x - S_n x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. Then there hold the following:

(I) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;

(II) $x_n \to q \Leftrightarrow \alpha_n(f(x_n) - x_n) \to 0$ *provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$, where $q \in F$ solves the following VIP*

$$\langle q - f(q), J(q-p)\rangle \le 0, \quad \forall p \in F. \tag{175}$$

Proof. In Theorem 28, we put $B_1 = I - V$, $B_2 = 0$, and $\mu_1 = l$, where $1 - (\lambda/(1+\lambda))(1 - \sqrt{(1-\alpha)/\lambda}) \leq l \leq 1$. Then, GSVI (13) is equivalent to the VIP of finding $x^* \in C$ such that

$$\langle B_1 x^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{176}$$

In this case, $B_1 : C \to X$ is λ-strictly pseudocontractive and α-strongly accretive. Repeating the same arguments as those in the proof of Corollary 25, we can infer that $\text{Fix}(V) = \text{VI}(C, B_1)$. Accordingly, $F = \bigcap_{i=0}^{\infty} \text{Fix}(S_i) \cap \Omega \cap (\bigcap_{i=0}^{\infty} \text{VI}(C, A_i)) = \bigcap_{i=0}^{\infty} \text{Fix}(S_i) \cap \text{Fix}(V) \cap (\bigcap_{i=0}^{\infty} \text{VI}(C, A_i))$, and

$$Gx_n = ((1-l)I + lV)x_n, \quad \forall n \geq 0. \tag{177}$$

So, scheme (130) reduces to (174). Therefore, the desired result follows from Theorem 31. □

Remark 30. Our Theorems 24 and 28 improve, extend, supplement and develop Ceng and Yao's [10, Theorem 3.2], Cai and Bu's [11, Theorem 3.1], Kangtunyakarn's [38, Theorem 3.1], and Ceng and Yao's [8, Theorem 3.1], in the following aspects.

(i) The problem of finding a point $q \in (\bigcap_{i=0}^{\infty} \text{Fix}(S_i)) \cap \Omega \cap (\bigcap_{i=0}^{\infty} \text{VI}(C, A_i))$ in our Theorems 24 and 28 is more general and more subtle than every one of the problem of finding a point $q \in \bigcap_{i=0}^{\infty} \text{Fix}(T_i)$ in [10, Theorem 3.2], the problem of finding a point $q \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \cap \Omega$ in [11, Theorem 3.1], the problem of finding a point $q \in \text{Fix}(S) \cap \text{Fix}(V) \cap (\bigcap_{i=1}^{N} \text{VI}(C, A_i))$ in [38, Theorem 3.1], and the problem of finding a point $q \in \text{Fix}(T)$ in [8, Theorem 3.1].

(ii) The iterative scheme in [8, Theorem 3.1] is extended to develop the iterative schemes (42) and (130) in our Theorems 24 and 28 by virtue of the iterative schemes of [11, Theorem 3.1] and [10, Theorems 3.2]. The iterative schemes (42) and (130) in our Theorems 24 and 28 are more advantageous and more flexible than the iterative scheme of [8, Theorem 3.1] because they can be applied to solving three problems (i.e., GSVI (13), fixed point problem and infinitely many VIPs), and involve several parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, (and $\{\sigma_n\}$).

(iii) Our Theorems 24 and 28 extend and generalize Ceng and Yao [8, Theorem 3.1] from a nonexpansive mapping to a countable family of nonexpansive mappings, and Ceng and Yao's [10, Theorems 3.2], to the setting of the GSVI (13) and infinitely many VIPs, Kangtunyakarn [38, Theorem 3.1], from finitely many VIPs to infinitely many VIPs, from a nonexpansive mapping to a countable family of nonexpansive mappings and from a strict pseudocontraction to the GSVI (13). In the meantime, our Theorems 24 and 28 extend and generalize Cai and Bu's [11, Theorem 3.1], to the setting of infinitely many VIPs.

(iv) The iterative schemes (42) and (130) in our Theorems 24 and 28 are very different from every one in [10, Theorem 3.2], [11, Theorem 3.1], [38, Theorem 3.1], and [8, Theorem 3.1] because the mappings G and T_n in [11, Theorem 3.1] and the mapping T in [8, Theorem 3.1] are replaced with the same composite mapping $S_n G$ in the iterative schemes (42) and (130) and the mapping W_n in [10, Theorem 3.2] is replaced with B_n.

(v) Cai and Bu's proof in [11, Theorem 3.1] depends on the argument techniques in [14], the inequality in 2-uniformly smooth Banach spaces (see Lemma 4), and the inequality in smooth and uniform convex Banach spaces (see Proposition 6). Because the composite mapping $S_n G$ appears in the iterative scheme (42) of our Theorem 24, the proof of our Theorem 24 depends on the argument techniques in [14], the inequality in 2-uniformly smooth Banach spaces (see Lemma 4), the inequality in smooth and uniform convex Banach spaces (see Proposition 6), the inequality in uniform convex Banach spaces (see Lemma 15 in Section 2 of this paper), and the properties of the W-mapping and the Banach limit (see Lemmas 16–18 in Section 2 of this paper). However, the proof of our Theorem 28 does not depend on the argument techniques in [14], the inequality in 2-uniformly smooth Banach spaces (see Lemma 4), and the inequality in smooth and uniform convex Banach spaces (see Proposition 6). It depends on only the inequality in uniform convex Banach spaces (see Lemma 15 in Section 2 of this paper) and the properties of the W-mapping and the Banach limit (see Lemmas 16–18 in Section 2 of this paper).

(vi) The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [11, Theorem 3.1] is weakened to the one of the uniformly convex Banach space X having a uniformly Gateaux differentiable norm in our Theorem 28. Moreover, the assumption of the uniformly smooth Banach space X in [8, Theorem 3.1] is replaced with the one of the uniformly convex Banach space X having a uniformly Gateaux differentiable norm in our Theorem 28. It is worth emphasizing that there is no assumption on the convergence of parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ (and $\{\sigma_n\}$) to zero in our Theorems 24 and 28.

4. Relaxed Mann Iterations and Their Convergence Criteria

In this section, we introduce our relaxed Mann iteration algorithms in real smooth and uniformly convex Banach spaces and present their convergence criteria.

Theorem 31. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0,1)$ and $A_i : C \to X$ an $\widehat{\alpha}_i$-inverse strongly accretive mapping for each $i = 0, 1, \ldots$. Define a mapping $G_i : C \to C$ by $\Pi_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 0, 1, \ldots$,* *where $\lambda_i \in (0, \widehat{\alpha}_i/\kappa^2]$ and κ is the 2-uniformly smooth constant of X. Let $B_n : C \to C$ be the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$ and $\rho_n, \rho_{n-1}, \ldots, \rho_0$. Let the mapping $B_i : C \to X$ be $\widehat{\beta}_i$-inverse strongly accretive for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)) \cap \Omega \cap (\bigcap_{i=0}^{\infty} \mathrm{VI}(C, A_i)) \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i < \widehat{\beta}_i/\kappa^2$ for $i = 1, 2$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n G x_n, \quad \forall n \geq 0, \tag{178}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$;

(iii) $\lim_{n\to\infty}(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) = 0$;

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1} x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. Then, there hold the following:

(I) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(II) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $q \in F$ which is the unique solution of the variational inequality problem (VIP)*

$$\langle (I - f) q, J(q - p) \rangle \leq 0, \quad \forall p \in F, \tag{179}$$

provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0, 1)$.

Proof. First of all, since $0 < \lambda_i < [\widehat{\alpha}_i/\kappa^2]$ for $i = 0, 1, \ldots$, it is easy to see that G_i is a nonexpansive mapping for each $i = 0, 1, \ldots$. Since $B_n : C \to C$ is the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$, and $\rho_n, \rho_{n-1}, \ldots, \rho_0$, by Lemma 16 we know that, for each $x \in C$ and $k \geq 0$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists. Moreover, one can define a mapping $B : C \to C$ as follows:

$$Bx = \lim_{n\to\infty} B_n x = \lim_{n\to\infty} U_{n,0} x \tag{180}$$

for every $x \in C$. That is, such a B is the W-mapping generated by the sequences $\{G_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$. According to Lemma 17, we know that $\mathrm{Fix}(B) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(G_i)$. From Lemma 21 and the definition of G_i, we have $\mathrm{Fix}(G_i) = \mathrm{VI}(C, A_i)$ for each $i = 0, 1, \ldots$. Hence, we have

$$\mathrm{Fix}(B) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(G_i) = \bigcap_{i=0}^{\infty} \mathrm{VI}(C, A_i). \tag{181}$$

Next, let us show that the sequence $\{x_n\}$ is bounded. Indeed, take a fixed $p \in F$ arbitrarily. Then, we get $p = Gp$, $p = B_n p$, and $p = S_n p$ for all $n \geq 0$. By Lemma 23, we know that G is nonexpansive. Then, from (178), we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad \leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| \\
&\qquad + \gamma_n \|B_n x_n - p\| + \delta_n \|S_n G x_n - p\| \\
&\quad \leq \alpha_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) \\
&\qquad + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|G x_n - p\| \\
&\quad \leq \alpha_n (\rho \|x_n - p\| + \|f(p) - p\|) \\
&\qquad + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|x_n - p\| \\
&\quad = (1 - \alpha_n (1 - \rho)) \|x_n - p\| + \alpha_n (1 - \rho) \frac{\|f(p) - p\|}{1 - \rho} \\
&\quad \leq \max \left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1 - \rho} \right\}.
\end{aligned} \tag{182}$$

By induction, we obtain

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \rho} \right\}, \quad \forall n \geq 0. \tag{183}$$

Hence, $\{x_n\}$ is bounded, and so are the sequences $\{G x_n\}$ and $\{f(x_n)\}$.

Let us show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{184}$$

As a matter of fact, observe that x_{n+1} can be rewritten as follows:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n, \tag{185}$$

where $z_n = (\alpha_n f(x_n) + \gamma_n B_n x_n + \delta_n S_n G x_n)/(1 - \beta_n)$. Observe that

$$\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\quad = \left\| \frac{\alpha_n f(x_n) + \gamma_n B_n x_n + \delta_n S_n G x_n}{1 - \beta_n} \right. \\
&\qquad \left. - \frac{\alpha_{n-1} f(x_{n-1}) + \gamma_{n-1} B_{n-1} x_{n-1} + \delta_{n-1} S_{n-1} G x_{n-1}}{1 - \beta_{n-1}} \right\| \\
&\quad = \left\| \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} - \frac{x_n - \beta_{n-1} x_{n-1}}{1 - \beta_{n-1}} \right\| \\
&\quad = \left\| \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} - \frac{x_n - \beta_{n-1} x_{n-1}}{1 - \beta_n} \right. \\
&\qquad \left. + \frac{x_n - \beta_{n-1} x_{n-1}}{1 - \beta_n} - \frac{x_n - \beta_{n-1} x_{n-1}}{1 - \beta_{n-1}} \right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} - \frac{x_n - \beta_{n-1} x_{n-1}}{1 - \beta_n} \right\| \\
&\quad + \left\| \frac{x_n - \beta_{n-1} x_{n-1}}{1 - \beta_n} - \frac{x_n - \beta_{n-1} x_{n-1}}{1 - \beta_{n-1}} \right\| \\
&= \frac{1}{1 - \beta_n} \left\| x_{n+1} - \beta_n x_n - \left(x_n - \beta_{n-1} x_{n-1} \right) \right\| \\
&\quad + \left| \frac{1}{1 - \beta_n} - \frac{1}{1 - \beta_{n-1}} \right| \left\| x_n - \beta_{n-1} x_{n-1} \right\| \\
&= \frac{1}{1 - \beta_n} \left\| x_{n+1} - \beta_n x_n - \left(x_n - \beta_{n-1} x_{n-1} \right) \right\| \\
&\quad + \frac{\left| \beta_n - \beta_{n-1} \right|}{\left(1 - \beta_{n-1} \right) \left(1 - \beta_n \right)} \left\| x_n - \beta_{n-1} x_{n-1} \right\| \\
&= \frac{1}{1 - \beta_n} \left\| \alpha_n f\left(x_n \right) + \gamma_n B_n x_n + \delta_n S_n G x_n - \alpha_{n-1} f\left(x_{n-1} \right) \right. \\
&\qquad \left. - \gamma_{n-1} B_{n-1} x_{n-1} - \delta_{n-1} S_{n-1} G x_{n-1} \right\| \\
&\quad + \frac{\left| \beta_n - \beta_{n-1} \right|}{\left(1 - \beta_{n-1} \right) \left(1 - \beta_n \right)} \left\| x_n - \beta_{n-1} x_{n-1} \right\| \\
&\leq \frac{1}{1 - \beta_n} \left[\alpha_n \left\| f\left(x_n \right) - f\left(x_{n-1} \right) \right\| \right. \\
&\qquad + \gamma_n \left\| B_n x_n - B_{n-1} x_{n-1} \right\| \\
&\qquad + \delta_n \left\| S_n G x_n - S_{n-1} G x_{n-1} \right\| \\
&\qquad + \left| \alpha_n - \alpha_{n-1} \right| \left\| f\left(x_{n-1} \right) \right\| \\
&\qquad + \left| \gamma_n - \gamma_{n-1} \right| \left\| B_{n-1} x_{n-1} \right\| \\
&\qquad \left. + \left| \delta_n - \delta_{n-1} \right| \left\| S_{n-1} G x_{n-1} \right\| \right] \\
&\quad + \frac{\left| \beta_n - \beta_{n-1} \right|}{\left(1 - \beta_{n-1} \right) \left(1 - \beta_n \right)} \left\| x_n - \beta_{n-1} x_{n-1} \right\| .
\end{aligned} \tag{186}
$$

On the other hand, we note that, for all $n \geq 1$,

$$
\begin{aligned}
&\left\| S_n G x_n - S_{n-1} G x_{n-1} \right\| \\
&\quad \leq \left\| S_n G x_n - S_n G x_{n-1} \right\| + \left\| S_n G x_{n-1} - S_{n-1} G x_{n-1} \right\| \\
&\quad \leq \left\| G x_n - G x_{n-1} \right\| + \left\| S_n G x_{n-1} - S_{n-1} G x_{n-1} \right\| \\
&\quad \leq \left\| x_n - x_{n-1} \right\| + \left\| S_n G x_{n-1} - S_{n-1} G x_{n-1} \right\| .
\end{aligned} \tag{187}
$$

Furthermore, by (CY), since G_i and $U_{n,i}$ are nonexpansive, we deduce that for each $n \geq 1$

$$
\begin{aligned}
&\left\| B_n x_n - B_{n-1} x_{n-1} \right\| \\
&\quad \leq \left\| B_n x_n - B_n x_{n-1} \right\| + \left\| B_n x_{n-1} - B_{n-1} x_{n-1} \right\| \\
&\quad \leq \left\| x_n - x_{n-1} \right\| + \left\| B_n x_{n-1} - B_{n-1} x_{n-1} \right\| \\
&\quad = \left\| x_n - x_{n-1} \right\| + \left\| \lambda_0 G_0 U_{n,1} x_{n-1} - \lambda_0 G_0 U_{n-1,1} x_{n-1} \right\| \\
&\quad \leq \left\| x_n - x_{n-1} \right\| + \lambda_0 \left\| U_{n,1} x_{n-1} - U_{n-1,1} x_{n-1} \right\| \\
&\quad = \left\| x_n - x_{n-1} \right\| + \lambda_0 \left\| \lambda_1 G_1 U_{n,2} x_{n-1} - \lambda_1 G_1 U_{n-1,2} x_{n-1} \right\| \\
&\quad \leq \left\| x_n - x_{n-1} \right\| + \lambda_0 \lambda_1 \left\| U_{n,2} x_{n-1} - U_{n-1,2} x_{n-1} \right\| \\
&\quad \vdots \\
&\quad \leq \left\| x_n - x_{n-1} \right\| + \left(\prod_{i=0}^{n-1} \lambda_i \right) \left\| U_{n,n} x_{n-1} - U_{n-1,n} x_{n-1} \right\| \\
&\quad \leq \left\| x_n - x_{n-1} \right\| + M \prod_{i=0}^{n-1} \lambda_i ,
\end{aligned} \tag{188}
$$

for some constant $M > 0$. Taking into account $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [\widehat{c}, \widehat{d}]$. Utilizing (186)–(188), we have

$$
\begin{aligned}
&\left\| z_n - z_{n-1} \right\| \\
&\leq \frac{1}{1 - \beta_n} \left[\alpha_n \left\| f\left(x_n \right) - f\left(x_{n-1} \right) \right\| \right. \\
&\qquad + \gamma_n \left\| B_n x_n - B_{n-1} x_{n-1} \right\| \\
&\qquad + \delta_n \left\| S_n G x_n - S_{n-1} G x_{n-1} \right\| \\
&\qquad + \left| \alpha_n - \alpha_{n-1} \right| \left\| f\left(x_{n-1} \right) \right\| \\
&\qquad + \left| \gamma_n - \gamma_{n-1} \right| \left\| B_{n-1} x_{n-1} \right\| \\
&\qquad \left. + \left| \delta_n - \delta_{n-1} \right| \left\| S_{n-1} G x_{n-1} \right\| \right] \\
&\quad + \frac{\left| \beta_n - \beta_{n-1} \right|}{\left(1 - \beta_{n-1} \right) \left(1 - \beta_n \right)} \left\| x_n - \beta_{n-1} x_{n-1} \right\| \\
&\leq \frac{1}{1 - \beta_n} \left\{ \alpha_n \rho \left\| x_n - x_{n-1} \right\| + \gamma_n \left[\left\| x_n - x_{n-1} \right\| + M \prod_{i=0}^{n-1} \lambda_i \right] \right. \\
&\qquad + \delta_n \left[\left\| x_n - x_{n-1} \right\| + \left\| S_n G x_{n-1} - S_{n-1} G x_{n-1} \right\| \right] \\
&\qquad + \left| \alpha_n - \alpha_{n-1} \right| \left\| f\left(x_{n-1} \right) \right\| + \left| \gamma_n - \gamma_{n-1} \right| \left\| B_{n-1} x_{n-1} \right\| \\
&\qquad \left. + \left| \delta_n - \delta_{n-1} \right| \left\| S_{n-1} G x_{n-1} \right\| \right\} \\
&\quad + \frac{\left| \beta_n - \beta_{n-1} \right|}{\left(1 - \beta_{n-1} \right) \left(1 - \beta_n \right)} \left\| x_n - \beta_{n-1} x_{n-1} \right\| \\
&= \frac{1}{1 - \beta_n} \left\{ \left(1 - \beta_n - \alpha_n \left(1 - \rho \right) \right) \left\| x_n - x_{n-1} \right\| \right. \\
&\qquad + \gamma_n M \prod_{i=0}^{n-1} \lambda_i + \delta_n \left\| S_n G x_{n-1} - S_{n-1} G x_{n-1} \right\| \\
&\qquad + \left| \alpha_n - \alpha_{n-1} \right| \left\| f\left(x_{n-1} \right) \right\|
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \left|\gamma_n - \gamma_{n-1}\right| \left\|B_{n-1}x_{n-1}\right\| \\
&\qquad\qquad \left. + \left|\delta_n - \delta_{n-1}\right| \left\|S_{n-1}Gx_{n-1}\right\| \right\} \\
&\quad + \frac{\left|\beta_n - \beta_{n-1}\right|}{\left(1-\beta_{n-1}\right)\left(1-\beta_n\right)} \left\|x_n - \beta_{n-1}x_{n-1}\right\| \\
&= \left(1 - \frac{\alpha_n\left(1-\rho\right)}{1-\beta_n}\right) \left\|x_n - x_{n-1}\right\| + \frac{\gamma_n M}{1-\beta_n}\prod_{i=0}^{n-1}\lambda_i \\
&\quad + \frac{\delta_n}{1-\beta_n} \left\|S_nGx_{n-1} - S_{n-1}Gx_{n-1}\right\| + \frac{1}{1-\beta_n} \\
&\quad \times \left[\left|\alpha_n - \alpha_{n-1}\right| \left\|f\left(x_{n-1}\right)\right\| + \left|\gamma_n - \gamma_{n-1}\right| \left\|B_{n-1}x_{n-1}\right\| \right. \\
&\qquad \left. + \left|\delta_n - \delta_{n-1}\right| \left\|S_{n-1}Gx_{n-1}\right\|\right] \\
&\quad + \frac{\left|\beta_n - \beta_{n-1}\right|}{\left(1-\beta_{n-1}\right)\left(1-\beta_n\right)} \left\|x_n - \beta_{n-1}x_{n-1}\right\| \\
&\leq \left\|x_n - x_{n-1}\right\| + M\prod_{i=0}^{n-1}\lambda_i + \left\|S_nGx_{n-1} - S_{n-1}Gx_{n-1}\right\| \\
&\quad + \frac{1}{1-\beta_n} \left[\left|\alpha_n - \alpha_{n-1}\right| \left\|f\left(x_{n-1}\right)\right\| + \left|\gamma_n - \gamma_{n-1}\right| \right. \\
&\qquad\qquad \left. \times \left\|B_{n-1}x_{n-1}\right\| + \left|\delta_n - \delta_{n-1}\right| \left\|S_{n-1}Gx_{n-1}\right\|\right] \\
&\quad + \frac{\left|\beta_n - \beta_{n-1}\right|}{\left(1-\beta_{n-1}\right)\left(1-\beta_n\right)} \\
&\quad \times \left\|\alpha_{n-1}f\left(x_{n-1}\right) + \gamma_{n-1}B_{n-1}x_{n-1} + \delta_{n-1}S_{n-1}Gx_{n-1}\right\| \\
&\leq \left\|x_n - x_{n-1}\right\| \\
&\quad + M_1 \left[\prod_{i=0}^{n-1}\lambda_i + \left|\alpha_n - \alpha_{n-1}\right| \right. \\
&\qquad\qquad \left. + \left|\gamma_n - \gamma_{n-1}\right| + \left|\delta_n - \delta_{n-1}\right| + \left|\beta_n - \beta_{n-1}\right| \right] \\
&\quad + \left\|S_nGx_{n-1} - S_{n-1}Gx_{n-1}\right\|,
\end{aligned}
\tag{189}
$$

where $\sup_{n\geq 0}\{(1/(1-\widehat{d})^2)(\|f(x_n)\|+\|B_nx_n\|+\|S_nGx_n\|+M)\} \leq M_1$ for some $M_1 > 0$. Thus, from (189), conditions (i), (iii) and the assumption on $\{S_n\}$, it follows that (noting that $0 < \lambda_i \leq b < 1$, for all $i \geq 0$)

$$\lim_{n\to\infty}\left(\left\|z_n - z_{n-1}\right\| - \left\|x_n - x_{n-1}\right\|\right) \leq 0. \tag{190}$$

Since $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, by Lemma 20 we get

$$\lim_{n\to\infty}\left\|x_n - z_n\right\| = 0. \tag{191}$$

Consequently,

$$\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = \lim_{n\to\infty}\left(1-\beta_n\right)\left\|z_n - x_n\right\| = 0. \tag{192}$$

Next we show that $\|x_n - Gx_n\| \to 0$ as $n \to \infty$.

Indeed, for simplicity, put $q = \Pi_C(p - \mu_2 B_2 p)$, $u_n = \Pi_C(x_n - \mu_2 B_2 x_n)$ and $v_n = \Pi_C(u_n - \mu_1 B_1 u_n)$. Then, $v_n = Gx_n$ for all $n \geq 0$. From Lemma 26 we have

$$
\begin{aligned}
&\left\|u_n - q\right\|^2 \\
&\quad = \left\|\Pi_C\left(x_n - \mu_2 B_2 x_n\right) - \Pi_C\left(p - \mu_2 B_2 p\right)\right\|^2 \\
&\quad \leq \left\|x_n - p - \mu_2\left(B_2x_n - B_2p\right)\right\|^2 \\
&\quad \leq \left\|x_n - p\right\|^2 - 2\mu_2\left(\widehat{\beta}_2 - \kappa^2\mu_2\right)\left\|B_2x_n - B_2p\right\|^2,
\end{aligned}
\tag{193}
$$

$$
\begin{aligned}
&\left\|v_n - p\right\|^2 \\
&\quad = \left\|\Pi_C\left(u_n - \mu_1 B_1 u_n\right) - \Pi_C\left(q - \mu_1 B_1 q\right)\right\|^2 \\
&\quad \leq \left\|u_n - q - \mu_1\left(B_1u_n - B_1q\right)\right\|^2 \\
&\quad \leq \left\|u_n - q\right\|^2 - 2\mu_1\left(\widehat{\beta}_1 - \kappa^2\mu_1\right)\left\|B_1u_n - B_1q\right\|^2.
\end{aligned}
\tag{194}
$$

Substituting (193) for (194), we obtain

$$
\begin{aligned}
\left\|v_n - p\right\|^2 \leq{}& \left\|x_n - p\right\|^2 - 2\mu_2\left(\widehat{\beta}_2 - \kappa^2\mu_2\right)\left\|B_2x_n - B_2p\right\|^2 \\
& - 2\mu_1\left(\widehat{\beta}_1 - \kappa^2\mu_1\right)\left\|B_1u_n - B_1q\right\|^2.
\end{aligned}
\tag{195}
$$

By Lemma 8, we have from (178) and (195)

$$
\begin{aligned}
&\left\|x_{n+1} - p\right\|^2 \\
&\quad = \left\|\alpha_n\left(f\left(x_n\right) - f\left(p\right)\right) + \beta_n\left(x_n - p\right)\right. \\
&\qquad + \gamma_n\left(B_nx_n - p\right) + \delta_n\left(S_nGx_n - p\right) \\
&\qquad \left. + \alpha_n\left(f\left(p\right) - p\right)\right\|^2 \\
&\quad \leq \left\|\alpha_n\left(f\left(x_n\right) - f\left(p\right)\right) + \beta_n\left(x_n - p\right)\right. \\
&\qquad \left. + \gamma_n\left(B_nx_n - p\right) + \delta_n\left(S_nGx_n - p\right)\right\|^2 \\
&\qquad + 2\alpha_n\left\langle f\left(p\right) - p, J\left(x_{n+1} - p\right)\right\rangle \\
&\quad \leq \alpha_n\left\|f\left(x_n\right) - f\left(p\right)\right\|^2 + \beta_n\left\|x_n - p\right\|^2 \\
&\qquad + \gamma_n\left\|B_nx_n - p\right\|^2 + \delta_n\left\|S_nGx_n - p\right\|^2 \\
&\qquad + 2\alpha_n\left\langle f\left(p\right) - p, J\left(x_{n+1} - p\right)\right\rangle \\
&\quad \leq \alpha_n\rho^2\left\|x_n - p\right\|^2 + \beta_n\left\|x_n - p\right\|^2 \\
&\qquad + \gamma_n\left\|x_n - p\right\|^2 + \delta_n\left\|v_n - p\right\|^2 \\
&\qquad + 2\alpha_n\left\langle f\left(p\right) - p, J\left(x_{n+1} - p\right)\right\rangle
\end{aligned}
$$

$$\begin{aligned}
&\leq \alpha_n\rho\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2\\
&\quad+\delta_n\Big[\|x_n-p\|^2-2\mu_2\left(\widehat{\beta}_2-\kappa^2\mu_2\right)\|B_2x_n-B_2p\|^2\\
&\qquad\quad -2\mu_1\left(\widehat{\beta}_1-\kappa^2\mu_1\right)\|B_1u_n-B_1q\|^2\Big]\\
&\quad+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle\\
&=(1-\alpha_n(1-\rho))\|x_n-p\|^2\\
&\quad-2\delta_n\Big[\mu_2\left(\widehat{\beta}_2-\kappa^2\mu_2\right)\|B_2x_n-B_2p\|^2\\
&\qquad\quad +\mu_1\left(\widehat{\beta}_1-\kappa^2\mu_1\right)\|B_1u_n-B_1q\|^2\Big]\\
&\quad+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle\\
&\leq\|x_n-p\|^2-2\delta_n\Big[\mu_2\left(\widehat{\beta}_2-\kappa^2\mu_2\right)\|B_2x_n-B_2p\|^2\\
&\qquad\quad +\mu_1\left(\widehat{\beta}_1-\kappa^2\mu_1\right)\|B_1u_n-B_1q\|^2\Big]\\
&\quad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|,
\end{aligned}\tag{196}$$

which hence implies that

$$\begin{aligned}
&2\delta_n\Big[\mu_2\left(\widehat{\beta}_2-\kappa^2\mu_2\right)\|B_2x_n-B_2p\|^2\\
&\quad+\mu_1\left(\widehat{\beta}_1-\kappa^2\mu_1\right)\|B_1u_n-B_1q\|^2\Big]\\
&\quad\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\qquad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\
&\quad\leq(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
&\qquad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.
\end{aligned}\tag{197}$$

Since $\|x_n-x_{n+1}\|\to 0$, $0<\mu_i<\widehat{\beta}_i/\kappa^2$ for $i=1,2$, and $\{x_n\}$ is bounded, we obtain from conditions (i), (ii) that

$$\lim_{n\to\infty}\|B_2x_n-B_2p\|=0,\qquad \lim_{n\to\infty}\|B_1u_n-B_1q\|=0.\tag{198}$$

Utilizing Proposition 6 and Lemma 9, we have

$$\begin{aligned}
&\|u_n-q\|^2\\
&\quad=\|\Pi_C(x_n-\mu_2B_2x_n)-\Pi_C(p-\mu_2B_2p)\|^2\\
&\quad\leq\langle x_n-\mu_2B_2x_n-(p-\mu_2B_2p),J(u_n-q)\rangle\\
&\quad=\langle x_n-p,J(u_n-q)\rangle+\mu_2\langle B_2p-B_2x_n,J(u_n-q)\rangle\\
&\quad\leq\frac{1}{2}\Big[\|x_n-p\|^2+\|u_n-q\|^2\\
&\qquad\quad -g_1(\|x_n-u_n-(p-q)\|)\Big]\\
&\qquad+\mu_2\|B_2p-B_2x_n\|\|u_n-q\|,
\end{aligned}\tag{199}$$

which implies that

$$\begin{aligned}
\|u_n-q\|^2&\leq\|x_n-p\|^2-g_1(\|x_n-u_n-(p-q)\|)\\
&\quad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|.
\end{aligned}\tag{200}$$

In the same way, we derive

$$\begin{aligned}
&\|v_n-p\|^2\\
&\quad=\|\Pi_C(u_n-\mu_1B_1u_n)-\Pi_C(q-\mu_1B_1q)\|^2\\
&\quad\leq\langle u_n-\mu_1B_1u_n-(q-\mu_1B_1q),J(v_n-p)\rangle\\
&\quad=\langle u_n-q,J(v_n-p)\rangle+\mu_1\langle B_1q-B_1u_n,J(v_n-p)\rangle\\
&\quad\leq\frac{1}{2}\Big[\|u_n-q\|^2+\|v_n-p\|^2\\
&\qquad\quad -g_2(\|u_n-v_n+(p-q)\|)\Big]\\
&\qquad+\mu_1\|B_1q-B_1u_n\|\|v_n-p\|,
\end{aligned}\tag{201}$$

which implies that

$$\begin{aligned}
\|v_n-p\|^2&\leq\|u_n-q\|^2-g_2(\|u_n-v_n+(p-q)\|)\\
&\quad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.
\end{aligned}\tag{202}$$

Substituting (200) for (202), we get

$$\begin{aligned}
&\|v_n-p\|^2\\
&\quad\leq\|x_n-p\|^2-g_1(\|x_n-u_n-(p-q)\|)\\
&\qquad-g_2(\|u_n-v_n+(p-q)\|)\\
&\qquad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|\\
&\qquad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.
\end{aligned}\tag{203}$$

By Lemma 8, we have from (196) and (203)

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad\leq\alpha_n\rho^2\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2\\
&\qquad+\delta_n\|v_n-p\|^2+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\rho\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2\\
&\quad+\delta_n\big[\|x_n-p\|^2-g_1(\|x_n-u_n-(p-q)\|)\\
&\qquad-g_2(\|u_n-v_n+(p-q)\|)+2\mu_2\|B_2p-B_2x_n\|\\
&\qquad\times\|u_n-q\|+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|\big]\\
&\quad+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle\\
&\le(1-\alpha_n(1-\rho))\|x_n-p\|^2\\
&\quad-\delta_n[g_1(\|x_n-u_n-(p-q)\|)\\
&\qquad+g_2(\|u_n-v_n+(p-q)\|)]\\
&\quad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|+2\mu_1\|B_1q-B_1u_n\|\\
&\quad\times\|v_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\
&\le\|x_n-p\|^2-\delta_n[g_1(\|x_n-u_n-(p-q)\|)\\
&\qquad+g_2(\|u_n-v_n+(p-q)\|)]\\
&\quad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|+2\mu_1\|B_1q-B_1u_n\|\\
&\quad\times\|v_n-p\|+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|,
\end{aligned}\tag{204}$$

which hence leads to

$$\begin{aligned}
&\delta_n[g_1(\|x_n-u_n-(p-q)\|)+g_2(\|u_n-v_n+(p-q)\|)]\\
&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\qquad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|\\
&\qquad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|\\
&\qquad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\\
&\quad\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
&\qquad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|\\
&\qquad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|\\
&\qquad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.
\end{aligned}\tag{205}$$

From (198), (205), conditions (i), (ii) and the boundedness of $\{x_n\}$, $\{u_n\}$, and $\{v_n\}$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}g_1(\|x_n-u_n-(p-q)\|)=0,\\
&\lim_{n\to\infty}g_2(\|u_n-v_n+(p-q)\|)=0.
\end{aligned}\tag{206}$$

Utilizing the properties of g_1 and g_2, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\|x_n-u_n-(p-q)\|=0,\\
&\lim_{n\to\infty}\|u_n-v_n+(p-q)\|=0.
\end{aligned}\tag{207}$$

From (207), we get

$$\begin{aligned}
\|x_n-v_n\|&\le\|x_n-u_n-(p-q)\|\\
&\quad+\|u_n-v_n+(p-q)\|\longrightarrow0\quad\text{as } n\to\infty.
\end{aligned}\tag{208}$$

That is,

$$\lim_{n\to\infty}\|x_n-Gx_n\|=0.\tag{209}$$

Next, let us show that

$$\lim_{n\to\infty}\|S_nGx_n-x_n\|=0,\qquad\lim_{n\to\infty}\|B_nx_n-x_n\|=0.\tag{210}$$

Indeed, observe that x_{n+1} can be rewritten as follows:

$$\begin{aligned}
x_{n+1}&=\alpha_nf(x_n)+\beta_nx_n+\gamma_nB_nx_n+\delta_nS_nGx_n\\
&=\alpha_nf(x_n)+\beta_nx_n+(\gamma_n+\delta_n)\\
&\quad\times\frac{\gamma_nB_nx_n+\delta_nS_nGx_n}{\gamma_n+\delta_n}\\
&=\alpha_nf(x_n)+\beta_nx_n+e_n\widehat{z}_n,
\end{aligned}\tag{211}$$

where $e_n=\gamma_n+\delta_n$ and $\widehat{z}_n=(\gamma_nB_nx_n+\delta_nS_nGx_n)/(\gamma_n+\delta_n)$. Utilizing Lemma 11 and (211), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad=\|\alpha_n(f(x_n)-p)+\beta_n(x_n-p)+e_n(\widehat{z}_n-p)\|^2\\
&\quad\le\alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2\\
&\qquad+e_n\|\widehat{z}_n-p\|^2-\beta_ne_ng_3(\|\widehat{z}_n-x_n\|)\\
&\quad=\alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2\\
&\qquad-\beta_ne_ng_3(\|\widehat{z}_n-x_n\|)+e_n\left\|\frac{\gamma_nB_nx_n+\delta_nS_nGx_n}{\gamma_n+\delta_n}-p\right\|^2\\
&\quad=\alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2-\beta_ne_ng_3(\|\widehat{z}_n-x_n\|)\\
&\qquad+e_n\left\|\frac{\gamma_n}{\gamma_n+\delta_n}(B_nx_n-p)+\frac{\delta_n}{\gamma_n+\delta_n}(S_nGx_n-p)\right\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2-\beta_n e_n g_3(\|\widehat{z}_n-x_n\|)\\
&\quad+e_n\left[\frac{\gamma_n}{\gamma_n+\delta_n}\|B_nx_n-p\|^2+\frac{\delta_n}{\gamma_n+\delta_n}\|S_nGx_n-p\|^2\right]\\
&\leq \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2-\beta_n e_n g_3(\|\widehat{z}_n-x_n\|)\\
&\quad+e_n\left[\frac{\gamma_n}{\gamma_n+\delta_n}\|x_n-p\|+\frac{\delta_n}{\gamma_n+\delta_n}\|Gx_n-p\|^2\right]\\
&\leq \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2-\beta_n e_n g_3(\|\widehat{z}_n-x_n\|)\\
&\quad+e_n\left[\frac{\gamma_n}{\gamma_n+\delta_n}\|x_n-p\|+\frac{\delta_n}{\gamma_n+\delta_n}\|x_n-p\|^2\right]\\
&=\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&\quad-\beta_n e_n g_3(\|\widehat{z}_n-x_n\|)\\
&\leq \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-\beta_n e_n g_3(\|\widehat{z}_n-x_n\|),
\end{aligned}\tag{212}$$

which hence implies that

$$\begin{aligned}
&\beta_n e_n g_3(\|\widehat{z}_n-x_n\|)\\
&\quad\leq \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad\leq \alpha_n\|f(x_n)-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\qquad\times\|x_n-x_{n+1}\|.
\end{aligned}\tag{213}$$

Utilizing (184), conditions (i), (ii), (iv), and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we get

$$\lim_{n\to\infty} g_3(\|\widehat{z}_n-x_n\|)=0. \tag{214}$$

From the properties of g_3, we have

$$\lim_{n\to\infty}\|\widehat{z}_n-x_n\|=0. \tag{215}$$

Utilizing Lemma 15 and the definition of $\widehat{z}_n$, we have

$$\begin{aligned}
&\|\widehat{z}_n-p\|^2\\
&\quad=\left\|\frac{\gamma_nB_nx_n+\delta_nS_nGx_n}{\gamma_n+\delta_n}-p\right\|^2\\
&\quad=\left\|\frac{\gamma_n}{\gamma_n+\delta_n}(B_nx_n-p)+\frac{\delta_n}{\gamma_n+\delta_n}(S_nGx_n-p)\right\|^2\\
&\quad\leq\frac{\gamma_n}{\gamma_n+\delta_n}\|B_nx_n-p\|^2+\frac{\delta_n}{\gamma_n+\delta_n}\|S_nGx_n-p\|^2\\
&\qquad-\frac{\gamma_n\delta_n}{(\gamma_n+\delta_n)^2}g_4(\|S_nGx_n-B_nx_n\|)\\
&\quad\leq\frac{\gamma_n}{\gamma_n+\delta_n}\|x_n-p\|^2+\frac{\delta_n}{\gamma_n+\delta_n}\|Gx_n-p\|^2\\
&\qquad-\frac{\gamma_n\delta_n}{(\gamma_n+\delta_n)^2}g_4(\|S_nGx_n-B_nx_n\|)\\
&\quad\leq\frac{\gamma_n}{\gamma_n+\delta_n}\|x_n-p\|^2+\frac{\delta_n}{\gamma_n+\delta_n}\|x_n-p\|^2\\
&\qquad-\frac{\gamma_n\delta_n}{(\gamma_n+\delta_n)^2}g_4(\|S_nGx_n-B_nx_n\|)\\
&\quad=\|x_n-p\|^2-\frac{\gamma_n\delta_n}{(\gamma_n+\delta_n)^2}g_4(\|S_nGx_n-B_nx_n\|),
\end{aligned}\tag{216}$$

which hence yields

$$\begin{aligned}
&\frac{\gamma_n\delta_n}{(\gamma_n+\delta_n)^2}g_4(\|S_nGx_n-B_nx_n\|)\\
&\qquad\leq\|x_n-p\|^2-\|\widehat{z}_n-p\|^2\\
&\qquad\leq(\|x_n-p\|+\|\widehat{z}_n-p\|)\|x_n-\widehat{z}_n\|.
\end{aligned}\tag{217}$$

Since $\{x_n\}$ and $\{\widehat{z}_n\}$ are bounded and $\|\widehat{z}_n-x_n\|\to 0$ as $n\to\infty$, we deduce from condition (ii) that

$$\lim_{n\to\infty} g_4(\|S_nGx_n-B_nx_n\|)=0. \tag{218}$$

From the properties of g_4, we have

$$\lim_{n\to\infty}\|S_nGx_n-B_nx_n\|=0. \tag{219}$$

On the other hand, x_{n+1} can also be rewritten as follows:

$$\begin{aligned}
x_{n+1}&=\alpha_nf(x_n)+\beta_nx_n+\gamma_nB_nx_n+\delta_nS_nGx_n\\
&=\beta_nx_n+\gamma_nB_nx_n+(\alpha_n+\delta_n)\frac{\alpha_nf(x_n)+\delta_nS_nGx_n}{\alpha_n+\delta_n}\\
&=\beta_nx_n+\gamma_nB_nx_n+d_n\widetilde{z}_n,
\end{aligned}\tag{220}$$

where $d_n=\alpha_n+\delta_n$ and $\widetilde{z}_n=(\alpha_nf(x_n)+\delta_nS_nGx_n)/(\alpha_n+\delta_n)$. Utilizing Lemma 11 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\quad=\|\beta_n(x_n-p)+\gamma_n(B_nx_n-p)+d_n(\widetilde{z}_n-p)\|^2\\
&\quad\leq\beta_n\|x_n-p\|^2+\gamma_n\|B_nx_n-p\|^2\\
&\qquad+d_n\|\widetilde{z}_n-p\|^2-\beta_n\gamma_ng_5(\|x_n-B_nx_n\|)
\end{aligned}$$

$$
\begin{aligned}
&= \beta_n\|x_n - p\|^2 + \gamma_n\|B_nx_n - p\|^2 \\
&\quad + d_n\left\|\frac{\alpha_n f(x_n) + \delta_n S_nGx_n}{\alpha_n + \delta_n} - p\right\|^2 \\
&\quad - \beta_n\gamma_n g_5(\|x_n - B_nx_n\|) \\
&= \beta_n\|x_n - p\|^2 + \gamma_n\|B_nx_n - p\|^2 \\
&\quad + d_n\left\|\frac{\alpha_n}{\alpha_n + \delta_n}(f(x_n) - p) + \frac{\delta_n}{\alpha_n + \delta_n}(S_nGx_n - p)\right\|^2 \\
&\quad - \beta_n\gamma_n g_5(\|x_n - B_nx_n\|) \\
&\leq \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\
&\quad + d_n\left[\frac{\alpha_n}{\alpha_n + \delta_n}\|f(x_n) - p\|^2 + \frac{\delta_n}{\alpha_n + \delta_n}\|S_nGx_n - p\|^2\right] \\
&\quad - \beta_n\gamma_n g_5(\|x_n - B_nx_n\|) \\
&\leq \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\
&\quad + d_n\left[\frac{\alpha_n}{\alpha_n + \delta_n}\|f(x_n) - p\|^2 + \frac{\delta_n}{\alpha_n + \delta_n}\|Gx_n - p\|^2\right] \\
&\quad - \beta_n\gamma_n g_5(\|x_n - B_nx_n\|) \\
&\leq \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\
&\quad + d_n\left[\frac{\alpha_n}{\alpha_n + \delta_n}\|f(x_n) - p\|^2 + \frac{\delta_n}{\alpha_n + \delta_n}\|x_n - p\|^2\right] \\
&\quad - \beta_n\gamma_n g_5(\|x_n - B_nx_n\|) \\
&= \alpha_n\|f(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\
&\quad - \beta_n\gamma_n g_5(\|x_n - B_nx_n\|) \\
&\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \beta_n\gamma_n g_5(\|x_n - B_nx_n\|),
\end{aligned}
\tag{221}
$$

which hence implies that

$$
\begin{aligned}
&\beta_n\gamma_n g_5(\|x_n - B_nx_n\|) \\
&\quad \leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad \leq \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\qquad \times \|x_n - x_{n+1}\|.
\end{aligned}
\tag{222}
$$

From (184), conditions (i), (ii), (iv), and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we have

$$\lim_{n\to\infty} g_5(\|x_n - B_nx_n\|) = 0. \tag{223}$$

Utilizing the properties of g_5, we have

$$\lim_{n\to\infty}\|x_n - B_nx_n\| = 0, \tag{224}$$

which, together with (219), implies that

$$
\begin{aligned}
\|S_nGx_n - x_n\| \leq \|S_nGx_n - B_nx_n\| + \|B_nx_n - x_n\| \longrightarrow 0& \\
\text{as } n \to \infty.&
\end{aligned}
\tag{225}
$$

That is,

$$\lim_{n\to\infty}\|S_nGx_n - x_n\| = 0. \tag{226}$$

We note that

$$
\begin{aligned}
&\|x_n - Sx_n\| \\
&\quad \leq \|x_n - S_nGx_n\| + \|S_nGx_n - S_nx_n\| + \|S_nx_n - Sx_n\| \\
&\quad \leq \|x_n - S_nGx_n\| + \|Gx_n - x_n\| + \|S_nx_n - Sx_n\|.
\end{aligned}
\tag{227}
$$

So, in terms of (209), (226), and Lemma 12, we have

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{228}$$

Suppose that $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$ such that $\alpha_n + \beta + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Define a mapping $Vx = (1 - \theta_1 - \theta_2)Sx + \theta_1 Bx + \theta_2 Gx$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then by Lemmas 14 and 17, we have that $\text{Fix}(V) = \text{Fix}(S) \cap \text{Fix}(B) \cap \text{Fix}(G) = F$. For each $k \geq 1$, let $\{p_k\}$ be a unique element of C such that

$$p_k = \frac{1}{k}f(p_k) + \left(1 - \frac{1}{k}\right)Vp_k. \tag{229}$$

From Lemma 13, we conclude that $p_k \to q \in \text{Fix}(V) = F$ as $k \to \infty$. Observe that for every n, k

$$
\begin{aligned}
&\|x_{n+1} - Bp_k\| \\
&\quad = \|\alpha_n(f(x_n) - Bp_k) + \beta(x_n - Bp_k) \\
&\qquad + \gamma_n(B_nx_n - Bp_k) + \delta_n(S_nGx_n - Bp_k)\| \\
&\quad \leq \alpha_n\|f(x_n) - Bp_k\| + \beta\|x_n - Bp_k\| + \gamma_n\|B_nx_n - Bp_k\| \\
&\qquad + \delta_n(\|S_nGx_n - B_nx_n\| + \|B_nx_n - Bp_k\|) \\
&\quad = \alpha_n\|f(x_n) - Bp_k\| + \beta\|x_n - Bp_k\| \\
&\qquad + (\gamma_n + \delta_n)\|B_nx_n - Bp_k\| + \delta_n\|S_nGx_n - B_nx_n\| \\
&\quad = \alpha_n\|f(x_n) - Bp_k\| + \beta\|x_n - Bp_k\| \\
&\qquad + (1 - \alpha_n - \beta)\|B_nx_n - Bp_k\| + \delta_n\|S_nGx_n - B_nx_n\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \alpha_n \|f(x_n) - Bp_k\| + \beta \|x_n - Bp_k\| + (1 - \alpha_n - \beta) \\
&\quad \times [\|B_n x_n - B_n p_k\| + \|B_n p_k - Bp_k\|] \\
&\quad + \delta_n \|S_n G x_n - B_n x_n\| \\
&\leq \alpha_n \|f(x_n) - Bp_k\| + \beta \|x_n - Bp_k\| \\
&\quad + (1 - \alpha_n - \beta) [\|x_n - p_k\| + \|B_n p_k - Bp_k\|] \\
&\quad + \delta_n \|S_n G x_n - B_n x_n\| \\
&\leq \alpha_n \|f(x_n) - Bp_k\| + \beta \|x_n - Bp_k\| \\
&\quad + (1 - \beta) [\|x_n - p_k\| + \|B_n p_k - Bp_k\|] \\
&\quad + \delta_n \|S_n G x_n - B_n x_n\| \\
&= \theta_n + \beta \|x_n - Bp_k\| + (1 - \beta) \|x_n - p_k\|,
\end{aligned} \tag{230}$$

where $\theta_n = \alpha_n\|f(x_n) - Bp_k\| + (1 - \beta)\|B_n p_k - Bp_k\| + \delta_n\|S_n G x_n - B_n x_n\|$. Since $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\|B_n p_k - Bp_k\| = \lim_{n\to\infty}\|S_n G x_n - B_n x_n\| = 0$, we know that $\theta_n \to 0$ as $n \to \infty$.

From (230), we obtain

$$\begin{aligned}
&\|x_{n+1} - Bp_k\|^2 \\
&\quad \leq (\beta \|x_n - Bp_k\| + (1 - \beta) \|x_n - p_k\|)^2 \\
&\qquad + \theta_n [2(\beta \|x_n - Bp_k\| + (1 - \beta) \|x_n - p_k\|) + \theta_n] \\
&\quad = \beta^2\|x_n - Bp_k\|^2 + (1 - \beta)^2\|x_n - p_k\|^2 \\
&\qquad + 2\beta(1 - \beta) \|x_n - Bp_k\| \|x_n - p_k\| + \tau_n \\
&\quad \leq \beta^2\|x_n - Bp_k\|^2 + (1 - \beta)^2\|x_n - p_k\|^2 \\
&\qquad + \beta(1 - \beta) (\|x_n - Bp_k\|^2 + \|x_n - p_k\|^2) + \tau_n \\
&\quad = \beta\|x_n - Bp_k\|^2 + (1 - \beta) \|x_n - p_k\|^2 + \tau_n,
\end{aligned} \tag{231}$$

where $\tau_n = \theta_n[2(\beta\|x_n - Bp_k\| + (1 - \beta)\|x_n - p_k\|) + \theta_n] \to 0$ as $n \to \infty$.

For any Banach limit μ, from (231) we derive

$$\mu_n\|x_n - Bp_k\|^2 = \mu_n\|x_{n+1} - Bp_k\|^2 \leq \mu_n\|x_n - p_k\|^2. \tag{232}$$

In addition, note that

$$\begin{aligned}
&\|x_n - Gp_k\|^2 \\
&\quad \leq \|x_n - Gx_n + Gx_n - Gp_k\|^2 \\
&\quad \leq (\|x_n - Gx_n\| + \|x_n - p_k\|)^2 \\
&\quad = \|x_n - p_k\|^2 + \|x_n - Gx_n\| \\
&\qquad \times (2 \|x_n - p_k\| + \|x_n - Gx_n\|),
\end{aligned}$$

$$\begin{aligned}
&\|x_n - Sp_k\|^2 \\
&\quad \leq \|x_n - Sx_n + Sx_n - Sp_k\|^2 \\
&\quad \leq (\|x_n - Sx_n\| + \|x_n - p_k\|)^2 \\
&\quad = \|x_n - p_k\|^2 + \|x_n - Sx_n\| \\
&\qquad \times (2 \|x_n - p_k\| + \|x_n - Sx_n\|).
\end{aligned} \tag{233}$$

It is easy to see from (209) and (228) that

$$\begin{aligned}
\mu_n\|x_n - Gp_k\|^2 &\leq \mu_n\|x_n - p_k\|^2, \\
\mu_n\|x_n - Sp_k\|^2 &\leq \mu_n\|x_n - p_k\|^2.
\end{aligned} \tag{234}$$

Utilizing (232) and (234), we deduce that

$$\begin{aligned}
&\mu_n\|x_n - Vp_k\|^2 \\
&\quad = \mu_n \|(1 - \theta_1 - \theta_2) (x_n - Sp_k) \\
&\qquad + \theta_1 (x_n - Bp_k) + \theta_2 (x_n - Gp_k)\|^2 \\
&\quad \leq (1 - \theta_1 - \theta_2) \mu_n\|x_n - Sp_k\|^2 \\
&\qquad + \theta_1\mu_n\|x_n - Bp_k\|^2 + \theta_2\mu_n\|x_n - Gp_k\|^2 \\
&\quad \leq \mu_n\|x_n - p_k\|^2.
\end{aligned} \tag{235}$$

Also, observe that

$$x_n - p_k = \frac{1}{k} (x_n - f(p_k)) + \left(1 - \frac{1}{k}\right) (x_n - Vp_k); \tag{236}$$

that is,

$$\left(1 - \frac{1}{k}\right) (x_n - Vp_k) = x_n - p_k - \frac{1}{k} (x_n - f(p_k)). \tag{237}$$

It follows from Lemma 8(ii) and (237) that

$$\begin{aligned}
&\left(1 - \frac{1}{k}\right)^2 \|x_n - Vp_k\|^2 \\
&\quad \geq \|x_n - p_k\|^2 - \frac{2}{k} \langle x_n - p_k + p_k - f(p_k), J(x_n - p_k)\rangle \\
&\quad = \left(1 - \frac{2}{k}\right) \|x_n - p_k\|^2 + \frac{2}{k} \langle f(p_k) - p_k, J(x_n - p_k)\rangle.
\end{aligned} \tag{238}$$

So by (235) and (238), we have

$$\begin{aligned}
&\left(1 - \frac{1}{k}\right)^2 \mu_n\|x_n - p_k\|^2 \\
&\quad \geq \left(1 - \frac{2}{k}\right) \mu_n\|x_n - p_k\|^2 \\
&\qquad + \frac{2}{k}\mu_n \langle f(p_k) - p_k, J(x_n - p_k)\rangle,
\end{aligned} \tag{239}$$

and hence

$$\frac{1}{k^2}\mu_n\|x_n - p_k\|^2 \geq \frac{2}{k}\mu_n \langle f(p_k) - p_k, J(x_n - p_k)\rangle. \quad (240)$$

This implies that

$$\frac{1}{2k}\mu_n\|x_n - p_k\|^2 \geq \mu_n \langle f(p_k) - p_k, J(x_n - p_k)\rangle. \quad (241)$$

Since $p_k \to q \in \text{Fix}(V) = F$ as $k \to \infty$, by the uniform Frechet differentiability of the norm of X, we have

$$\mu_n \langle f(q) - q, J(x_n - q)\rangle \leq 0. \quad (242)$$

On the other hand, from (184) and the norm-to-norm uniform continuity of J on bounded subsets of X, it follows that

$$\lim_{n\to\infty} |\langle f(q) - q, J(x_{n+1} - q)\rangle - \langle f(q) - q, J(x_n - q)\rangle| = 0. \quad (243)$$

So, utilizing Lemma 18 we deduce from (242) and (243) that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \leq 0, \quad (244)$$

which, together with (184) and the norm-to-norm uniform continuity of J on bounded subsets of X, implies that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_{n+1} - q)\rangle \leq 0. \quad (245)$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. Utilizing Lemma 8 (i), from (178) and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \|\alpha_n(f(x_n) - f(q)) + \beta_n(x_n - q) + \gamma_n(B_nx_n - q) \\
&\qquad + \delta_n(S_nGx_n - q) + \alpha_n(f(q) - q)\|^2 \\
&\quad \leq \|\alpha_n(f(x_n) - f(q)) + \beta_n(x_n - q) \\
&\qquad + \gamma_n(B_nx_n - q) + \delta_n(S_nGx_n - q)\|^2 \\
&\qquad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad \leq \alpha_n\|f(x_n) - f(q)\|^2 + \beta_n\|x_n - q\|^2 \\
&\qquad + \gamma_n\|B_nx_n - q\|^2 + \delta_n\|S_nGx_n - q\|^2 \\
&\qquad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad \leq \alpha_n\|f(x_n) - f(q)\|^2 + \beta_n\|x_n - q\|^2 \\
&\qquad + \gamma_n\|x_n - q\|^2 + \delta_n\|Gx_n - q\|^2 \\
&\qquad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad \leq \alpha_n\rho\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 + \gamma_n\|x_n - q\|^2 \\
&\qquad + \delta_n\|x_n - q\|^2 + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad = (1 - \alpha_n(1-\rho))\|x_n - q\|^2 \\
&\qquad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad = (1 - \alpha_n(1-\rho))\|x_n - q\|^2 \\
&\qquad + \alpha_n(1-\rho)\frac{2\langle f(q) - q, J(x_{n+1} - q)\rangle}{1-\rho}.
\end{aligned} \quad (246)$$

Applying Lemma 7 to (246), we obtain that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 32. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0,1)$ and $A_i : C \to X$ an $\widehat{\alpha}_i$-inverse strongly accretive mapping for each $i = 0, 1, \ldots$. Define a mapping $G_i : C \to C$ by $\Pi_C(I - \lambda_i A_i)x = G_ix$ for all $x \in C$ and $i = 0, 1, \ldots$, where $\lambda_i \in (0, \widehat{\alpha}_i/\kappa^2]$ and κ is the 2-uniformly smooth constant of X. Let $B_n : C \to C$ be the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$, and $\rho_n, \rho_{n-1}, \ldots, \rho_0$. Let $V : C \to C$ be an α-strictly pseudocontractive mapping. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = (\bigcap_{i=0}^{\infty} \text{Fix}(S_i)) \cap \text{Fix}(V) \cap (\bigcap_{i=0}^{\infty} \text{VI}(C, A_i)) \neq \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n \\
&\quad + \delta_n S_n((1-l)I + lV)x_n, \quad \forall n \geq 0,
\end{aligned} \quad (247)$$

where $0 < l < \alpha/\kappa^2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c, d]$ *for some* $c, d \in (0,1)$;

(iii) $\lim_{n\to\infty}(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) = 0$;

(iv) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Assume that $\sum_{n=1}^{\infty}\sup_{x\in D}\|S_nx - S_{n-1}x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined *by* $Sx = \lim_{n\to\infty} S_n x$ *for all* $x \in C$ *and suppose that* $\operatorname{Fix}(S) = \bigcap_{i=0}^{\infty} \operatorname{Fix}(S_i)$. *Then, there hold the following:*

(I) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;

(II) *the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to some* $q \in F$ *which is the unique solution of the variational inequality problem (VIP)*

$$\langle (I - f) q, J(q - p)\rangle \le 0, \quad \forall p \in F, \tag{248}$$

provided $\beta_n \equiv \beta$ *for some fixed* $\beta \in (0,1)$.

Proof. In Theorem 31, we put $B_1 = I - V$, $B_2 = 0$ and $\mu_1 = l$ where $0 < l < \alpha/\kappa^2$. Then GSVI (13) is equivalent to the VIP of finding $x^* \in C$ such that

$$\langle B_1 x^*, J(x - x^*)\rangle \ge 0, \quad \forall x \in C. \tag{249}$$

In this case, $B_1 : C \to X$ is α-inverse strongly accretive. Repeating the same arguments as those in the proof of Corollary 25, we can infer that $\operatorname{Fix}(V) = \operatorname{VI}(C, B_1)$. Accordingly, we know that $F = \bigcap_{i=0}^{\infty} \operatorname{Fix}(T_i) \cap \Omega \cap A^{-1}0 = \bigcap_{i=0}^{\infty} \operatorname{Fix}(T_i) \cap \operatorname{Fix}(V) \cap A^{-1}0$, and

$$\begin{aligned} &\Pi_C (I - \mu_1 B_1)\Pi_C (I - \mu_2 B_2) x_n \\ &\quad = \Pi_C (I - \mu_1 B_1) x_n \\ &\quad = \Pi_C ((1 - l) x_n + l V x_n) \\ &\quad = ((1 - l) I + l V) x_n. \end{aligned} \tag{250}$$

So, scheme (178) reduces to (247). Therefore, the desired result follows from Theorem 31. □

Theorem 33. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* X *which has a uniformly Gateaux differentiable norm. Let* Π_C *be a sunny nonexpansive retraction from* X *onto* C. *Let* $\{\rho_n\}_{n=0}^{\infty}$ *be a sequence of positive numbers in* $(0, b]$ *for some* $b \in (0,1)$ *and* $A_i : C \to X\xi_i$*-strictly pseudocontractive and* $\widehat{\alpha}_i$*-strongly accretive with* $\xi_i + \widehat{\alpha}_i \ge 1$ *for each* $i = 0, 1, \ldots$. *Define a mapping* $G_i : C \to C$ *by* $\Pi_C(I - \lambda_i A_i)x = G_i x$ *for all* $x \in C$ *and* $i = 0, 1, \ldots$, *where* $1 - (\xi_i/(1+\xi_i))(1 - \sqrt{(1 - \widehat{\alpha}_i)/\xi_i}) \le \lambda_i \le 1$ *for all* $i = 0, 1, \ldots$. *Let* $B_n : C \to C$ *be the* W*-mapping generated by* $G_n, G_{n-1}, \ldots, G_0$ *and* $\rho_n, \rho_{n-1}, \ldots, \rho_0$. *Let the mapping* $B_i : C \to X$ *be* ζ_i*-strictly pseudocontractive and* $\widehat{\beta}_i$*-strongly accretive with* $\zeta_i + \widehat{\beta}_i \ge 1$ *for* $i = 1, 2$. *Let* $f : C \to C$ *be a contraction with coefficient* $\rho \in (0,1)$. *Let* $\{S_i\}_{i=0}^{\infty}$ *be a countable family of nonexpansive mappings of* C *into itself such that* $F = (\bigcap_{i=0}^{\infty} \operatorname{Fix}(S_i)) \cap \Omega \cap (\bigcap_{i=0}^{\infty} \operatorname{VI}(C, A_i)) \ne \emptyset$, *where* Ω *is the fixed point set of the mapping* $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ *with* $1 - (\zeta_i/(1 + \zeta_i))(1 - \sqrt{(1 - \widehat{\beta}_i)/\zeta_i}) \le \mu_i \le 1$ *for* $i = 1, 2$. *For arbitrarily given* $x_0 \in C$, *let* $\{x_n\}$ *be the sequence generated by*

$$\begin{aligned} x_{n+1} &= \sigma_n G x_n + (1 - \sigma_n) \\ &\quad \times [\alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n G x_n], \quad \forall n \ge 0, \end{aligned} \tag{251}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, *and* $\{\sigma_n\}$ *are the sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *for all* $n \ge 0$. *Suppose that the following conditions hold:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c, d]$ *for some* $c, d \in (0,1)$;

(iii) $\sum_{n=1}^{\infty}(|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$;

(iv) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ *and* $0 < \liminf_{n\to\infty}\sigma_n \le \limsup_{n\to\infty}\sigma_n < 1$.

Assume that $\sum_{n=1}^{\infty}\sup_{x\in D}\|S_n x - S_{n-1}x\| < \infty$ *for any bounded subset* D *of* C *and let* S *be a mapping of* C *into itself defined by* $Sx = \lim_{n\to\infty} S_n x$ *for all* $x \in C$ *and suppose that* $\operatorname{Fix}(S) = \bigcap_{i=0}^{\infty} \operatorname{Fix}(S_i)$. *Then, there hold the following:*

(I) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;

(II) *the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to some* $q \in F$ *which is the unique solution of the variational inequality problem (VIP)*

$$\langle (I - f) q, J(q - p)\rangle \le 0, \quad \forall p \in F, \tag{252}$$

provided $\beta_n \equiv \beta$ *for some fixed* $\beta \in (0,1)$.

Proof. First of all, it is easy to see that (251) can be rewritten as follows:

$$\begin{aligned} y_n &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n G x_n, \\ x_{x+1} &= \sigma_n G x_n + (1 - \sigma_n) y_n, \quad \forall n \ge 0. \end{aligned} \tag{253}$$

Take a fixed $p \in F$ arbitrarily. Then, we obtain $p = Gp$, $p = B_n p$ and $S_n p = p$ for all $n \ge 0$. Thus, we get from (253)

$$\begin{aligned} &\|y_n - p\| \\ &\quad \le \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| \\ &\qquad + \gamma_n \|B_n x_n - p\| + \delta_n \|S_n G x_n - p\| \\ &\quad \le \alpha_n (\rho \|x_n - p\| + \|f(p) - p\|) \\ &\qquad + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|x_n - p\| \\ &\quad = (1 - \alpha_n (1 - \rho)) \|x_n - p\| + \alpha_n \|f(p) - p\|, \end{aligned} \tag{254}$$

and hence

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\quad\le \sigma_n\|Gx_n-p\|+(1-\sigma_n)\|y_n-p\| \\
&\quad\le \sigma_n\|x_n-p\|+(1-\sigma_n) \\
&\qquad\times[(1-\alpha_n(1-\rho))\|x_n-p\|+\alpha_n\|f(p)-p\|] \\
&\quad=(1-(1-\sigma_n)\alpha_n(1-\rho))\|x_n-p\| \\
&\qquad+(1-\sigma_n)\alpha_n\|f(p)-p\| \\
&\quad=(1-(1-\sigma_n)\alpha_n(1-\rho))\|x_n-p\| \\
&\qquad+(1-\sigma_n)\alpha_n(1-\rho)\frac{\|f(p)-p\|}{1-\rho} \\
&\quad\le \max\left\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\}.
\end{aligned} \tag{255}$$

By induction, we have

$$\|x_n-p\|\le \max\left\{\|x_0-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\},\quad \forall n\ge 0. \tag{256}$$

which implies that $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{Gx_n\}$ and $\{f(x_n)\}$.

Let us show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{257}$$

As a matter of fact, observe that y_n can be rewritten as follows:

$$y_n=\beta_nx_n+(1-\beta_n)z_n, \tag{258}$$

where $z_n=(\alpha_nf(x_n)+\gamma_nB_nx_n+\delta_nS_nGx_n)/(1-\beta_n)$. Observe that

$$\begin{aligned}
&\|z_n-z_{n-1}\| \\
&\quad=\Bigg\|\frac{\alpha_nf(x_n)+\gamma_nB_nx_n+\delta_nS_nGx_n}{1-\beta_n} \\
&\qquad-\frac{\alpha_{n-1}f(x_{n-1})+\gamma_{n-1}B_{n-1}x_{n-1}+\delta_{n-1}S_{n-1}Gx_{n-1}}{1-\beta_{n-1}}\Bigg\| \\
&\quad=\left\|\frac{y_n-\beta_nx_n}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad=\Bigg\|\frac{y_n-\beta_nx_n}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_n} \\
&\qquad+\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\Bigg\| \\
&\quad\le\left\|\frac{y_n-\beta_nx_n}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_n}\right\| \\
&\qquad+\left\|\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_n}-\frac{y_{n-1}-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad=\frac{1}{1-\beta_n}\|y_n-\beta_nx_n-(y_{n-1}-\beta_{n-1}x_{n-1})\| \\
&\qquad+\left|\frac{1}{1-\beta_n}-\frac{1}{1-\beta_{n-1}}\right|\|y_{n-1}-\beta_{n-1}x_{n-1}\| \\
&\quad=\frac{1}{1-\beta_n}\|y_n-\beta_nx_n-(y_{n-1}-\beta_{n-1}x_{n-1})\| \\
&\qquad+\frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|y_{n-1}-\beta_{n-1}x_{n-1}\| \\
&\quad=\frac{1}{1-\beta_n}\|\alpha_nf(x_n)+\gamma_nB_nx_n+\delta_nS_nGx_n \\
&\qquad\quad-\alpha_{n-1}f(x_{n-1})-\gamma_{n-1}B_{n-1}x_{n-1} \\
&\qquad\quad-\delta_{n-1}S_{n-1}Gx_{n-1}\| \\
&\qquad+\frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|y_{n-1}-\beta_{n-1}x_{n-1}\| \\
&\quad\le\frac{1}{1-\beta_n}[\alpha_n\|f(x_n)-f(x_{n-1})\| \\
&\qquad\quad+\gamma_n\|B_nx_n-B_{n-1}x_{n-1}\| \\
&\qquad\quad+\delta_n\|S_nGx_n-S_{n-1}Gx_{n-1}\| \\
&\qquad\quad+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\| \\
&\qquad\quad+|\gamma_n-\gamma_{n-1}|\|B_{n-1}x_{n-1}\| \\
&\qquad\quad+|\delta_n-\delta_{n-1}|\|S_{n-1}Gx_{n-1}\|] \\
&\qquad+\frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|y_{n-1}-\beta_{n-1}x_{n-1}\|.
\end{aligned} \tag{259}$$

On the other hand, repeating the same arguments as those of (52) and (54) in the proof of Theorem 24, we can deduce that for all $n\ge 1$

$$\begin{aligned}
&\|S_nGx_n-S_{n-1}Gx_{n-1}\| \\
&\quad\le\|x_n-x_{n-1}\|+\|S_nGx_{n-1}-S_{n-1}Gx_{n-1}\|, \\
&\|B_nx_n-B_{n-1}x_{n-1}\|\le\|x_n-x_{n-1}\|+M\prod_{i=0}^{n-1}\lambda_i,
\end{aligned} \tag{260}$$

for some constant $M>0$. Taking into account $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [\widehat{c}, \widehat{d}]$. Utilizing (259)-(260) we have

$$\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\quad \le \frac{1}{1-\beta_n} \\
&\qquad \times \Bigg\{\alpha_n \rho \|x_n - x_{n-1}\| + \gamma_n \left[\|x_n - x_{n-1}\| + M \prod_{i=0}^{n-1} \lambda_i\right] \\
&\qquad\quad + \delta_n [\|x_n - x_{n-1}\| + \|S_n G x_{n-1} - S_{n-1} G x_{n-1}\|] \\
&\qquad\quad + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + |\gamma_n - \gamma_{n-1}| \|B_{n-1} x_{n-1}\| \\
&\qquad\quad + |\delta_n - \delta_{n-1}| \|S_{n-1} G x_{n-1}\| \Bigg\} \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1} x_{n-1}\| \\
&\quad = \frac{1}{1-\beta_n} \Bigg\{(1 - \beta_n - \alpha_n (1-\rho)) \|x_n - x_{n-1}\| \\
&\qquad\quad + \gamma_n M \prod_{i=0}^{n-1} \lambda_i + \delta_n \|S_n G x_{n-1} - S_{n-1} G x_{n-1}\| \\
&\qquad\quad + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\qquad\quad + |\gamma_n - \gamma_{n-1}| \|B_{n-1} x_{n-1}\| \\
&\qquad\quad + |\delta_n - \delta_{n-1}| \|S_{n-1} G x_{n-1}\| \Bigg\} \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1} x_{n-1}\| \\
&\quad = \left(1 - \frac{\alpha_n (1-\rho)}{1-\beta_n}\right) \|x_n - x_{n-1}\| + \frac{\gamma_n M}{1-\beta_n} \prod_{i=0}^{n-1} \lambda_i \\
&\qquad + \frac{\delta_n}{1-\beta_n} \|S_n G x_{n-1} - S_{n-1} G x_{n-1}\| \\
&\qquad + \frac{1}{1-\beta_n} [|\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + |\gamma_n - \gamma_{n-1}| \\
&\qquad\qquad \times \|B_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|S_{n-1} G x_{n-1}\|] \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1} x_{n-1}\| \\
&\quad \le \left(1 - \frac{\alpha_n (1-\rho)}{1-\beta_n}\right) \|x_n - x_{n-1}\| + M \prod_{i=0}^{n-1} \lambda_i \\
&\qquad + \|S_n G x_{n-1} - S_{n-1} G x_{n-1}\| + \frac{1}{1-\beta_n} \\
&\qquad \times [|\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
&\qquad\quad + |\gamma_n - \gamma_{n-1}| \|B_{n-1} x_{n-1}\| \\
&\qquad\quad + |\delta_n - \delta_{n-1}| \|S_{n-1} G x_{n-1}\|] + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \\
&\qquad \times \|\alpha_{n-1} f(x_{n-1}) + \gamma_{n-1} B_{n-1} x_{n-1} + \delta_{n-1} S_{n-1} G x_{n-1}\| \\
&\quad \le \left(1 - \frac{\alpha_n (1-\rho)}{1-\beta_n}\right) \|x_n - x_{n-1}\| \\
&\qquad + M_1 \Bigg[\prod_{i=0}^{n-1} \lambda_i + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\
&\qquad\quad + |\delta_n - \delta_{n-1}| \Bigg] + \|S_n G x_{n-1} - S_{n-1} G x_{n-1}\|,
\end{aligned} \tag{261}$$

where $\sup_{n\ge 0}\{(1/(1-\widehat{d})^2)(\|f(x_n)\| + \|B_n x_n\| + \|S_n G x_n\| + M)\} \le M_1$ for some $M_1 > 0$. In the meantime, observe that

$$\begin{aligned}
x_{n+1} - x_n &= \sigma_n (G x_n - G x_{n-1}) + (\sigma_n - \sigma_{n-1})(G x_{n-1} - z_{n-1}) \\
&\quad + (1-\sigma_n)(z_n - z_{n-1}).
\end{aligned} \tag{262}$$

This together with (261), implies that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \sigma_n \|G x_n - G x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|G x_{n-1} - z_{n-1}\| \\
&\qquad + (1-\sigma_n) \|z_n - z_{n-1}\| \\
&\quad \le \sigma_n \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|G x_{n-1} - z_{n-1}\| \\
&\qquad + (1-\sigma_n) \Bigg\{\left(1 - \frac{\alpha_n (1-\rho)}{1-\beta_n}\right) \|x_n - x_{n-1}\| \\
&\qquad\quad + M_1 \Bigg[\prod_{i=0}^{n-1} \lambda_i + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\
&\qquad\qquad + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}| \Bigg] \\
&\qquad\quad + \|S_n G x_{n-1} - S_{n-1} G x_{n-1}\| \Bigg\} \\
&\quad \le \left(1 - \frac{(1-\sigma_n)\alpha_n (1-\rho)}{1-\beta_n}\right) \|x_n - x_{n-1}\| \\
&\qquad + |\sigma_n - \sigma_{n-1}| \|G x_{n-1} - z_{n-1}\| \\
&\qquad + M_1 \Bigg[\prod_{i=0}^{n-1} \lambda_i + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\
&\qquad\quad + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}| \Bigg]
\end{aligned}$$

$$\begin{aligned}
&+\|S_nGx_{n-1}-S_{n-1}Gx_{n-1}\| \\
&\leq\left(1-\frac{(1-\sigma_n)\alpha_n(1-\rho)}{1-\beta_n}\right)\|x_n-x_{n-1}\| \\
&+M_2\left[\prod_{i=0}^{n-1}\lambda_i+|\sigma_n-\sigma_{n-1}|+|\alpha_n-\alpha_{n-1}|\right. \\
&\qquad\left.+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\delta_n-\delta_{n-1}|\right] \\
&+\|S_nGx_{n-1}-S_{n-1}Gx_{n-1}\|,
\end{aligned}\tag{263}$$

where $\sup_{n\geq 0}\{M_1+\|Gx_n-z_n\|\}\leq M_2$ for some $M_2>0$. Since $\sum_{n=0}^{\infty}\alpha_n=\infty$ and $(1-\sigma_n)\alpha_n(1-\rho)/(1-\beta_n)\geq(1-\sigma_n)\alpha_n(1-\rho)$, we obtain from conditions (i) and (iv) that $\sum_{n=0}^{\infty}((1-\sigma_n)\alpha_n(1-\rho)/(1-\beta_n))=\infty$. Thus, applying Lemma 7 to (263), we deduce from condition (iii) and the assumption on $\{S_n\}$ that (noting that $0<\lambda_i\leq b<1$, for all $i\geq 0$)

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{264}$$

Next, we show that $\|x_n-Gx_n\|\to 0$ as $n\to\infty$.

Indeed, according to Lemma 8 we have from (253)

$$\begin{aligned}
&\|y_n-p\|^2 \\
&=\|\alpha_n(f(x_n)-f(p))+\beta_n(x_n-p)+\gamma_n(B_nx_n-p) \\
&\quad+\delta_n(S_nGx_n-p)+\alpha_n(f(p)-p)\|^2 \\
&\leq\|\alpha_n(f(x_n)-f(p))+\beta_n(x_n-p) \\
&\quad+\gamma_n(B_nx_n-p)+\delta_n(S_nGx_n-p)\|^2 \\
&\quad+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle \\
&\leq\alpha_n\|f(x_n)-f(p)\|^2+\beta_n\|x_n-p\|^2 \\
&\quad+\gamma_n\|B_nx_n-p\|^2+\delta_n\|S_nGx_n-p\|^2 \\
&\quad+2\alpha_n\langle f(p)-p,J(x_{n+1}-p)\rangle \\
&\leq\alpha_n\rho\|x_n-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2 \\
&\quad+\delta_n\|x_n-p\|^2+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\| \\
&=(1-\alpha_n(1-\rho))\|x_n-p\|^2 \\
&\quad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\| \\
&\leq\|x_n-p\|^2+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.
\end{aligned}\tag{265}$$

Utilizing Lemma 15 we get from (253) and (265)

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&=\|\sigma_n(Gx_n-p)+(1-\sigma_n)(y_n-p)\|^2 \\
&\leq\sigma_n\|Gx_n-p\|^2+(1-\sigma_n)\|y_n-p\|^2 \\
&\quad-\sigma_n(1-\sigma_n)g(\|Gx_n-y_n\|) \\
&\leq\sigma_n\|x_n-p\|^2+(1-\sigma_n) \\
&\quad\times\left[\|x_n-p\|^2+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|\right] \\
&\quad-\sigma_n(1-\sigma_n)g(\|Gx_n-y_n\|) \\
&\leq\|x_n-p\|^2+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\| \\
&\quad-\sigma_n(1-\sigma_n)g(\|Gx_n-y_n\|),
\end{aligned}\tag{266}$$

which hence yields

$$\begin{aligned}
&\sigma_n(1-\sigma_n)g(\|Gx_n-y_n\|) \\
&\quad\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\qquad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\| \\
&\quad\leq(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\| \\
&\qquad+2\alpha_n\|f(p)-p\|\|x_{n+1}-p\|.
\end{aligned}\tag{267}$$

Since $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$, from condition (iv) and the boundedness of $\{x_n\}$, it follows that

$$\lim_{n\to\infty}g(\|Gx_n-y_n\|)=0.\tag{268}$$

Utilizing the properties of g, we have

$$\lim_{n\to\infty}\|Gx_n-y_n\|=0,\tag{269}$$

which, together with (253) and (257), implies that

$$\begin{aligned}
&\|x_n-y_n\| \\
&\quad\leq\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\| \\
&\quad=\|x_n-x_{n+1}\|+\sigma_n\|Gx_n-y_n\|\longrightarrow 0 \\
&\qquad\text{as } n\longrightarrow\infty.
\end{aligned}\tag{270}$$

That is,

$$\lim_{n\to\infty}\|x_n-y_n\|=0.\tag{271}$$

Since

$$\|x_n-Gx_n\|\leq\|x_n-y_n\|+\|y_n-Gx_n\|,\tag{272}$$

it immediately follows from (269) and (271) that

$$\lim_{n\to\infty}\|x_n-Gx_n\|=0.\tag{273}$$

On the other hand, observe that y_n can be rewritten as follows:

$$\begin{aligned} y_n &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n G x_n \\ &= \alpha_n f(x_n) + \beta_n x_n + (\gamma_n + \delta_n) \frac{\gamma_n B_n x_n + \delta_n S_n G x_n}{\gamma_n + \delta_n} \\ &= \alpha_n f(x_n) + \beta_n x_n + e_n \hat{z}_n, \end{aligned} \tag{274}$$

where $e_n = \gamma_n + \delta_n$ and $\hat{z}_n = (\gamma_n B_n x_n + \delta_n S_n G x_n)/(\gamma_n + \delta_n)$. Utilizing Lemma 11, we have

$$\begin{aligned} &\|y_n - p\|^2 \\ &\quad = \|\alpha_n (f(x_n) - p) + \beta_n (x_n - p) + e_n (\hat{z}_n - p)\|^2 \\ &\quad \le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\ &\qquad + e_n \|\hat{z}_n - p\|^2 - \beta_n e_n g_1 (\|\hat{z}_n - x_n\|) \\ &\quad = \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\ &\qquad - \beta_n e_n g_1 (\|\hat{z}_n - x_n\|) \\ &\qquad + e_n \left\| \frac{\gamma_n B_n x_n + \delta_n S_n G x_n}{\gamma_n + \delta_n} - p \right\|^2 \\ &\quad = \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\ &\qquad - \beta_n e_n g_1 (\|\hat{z}_n - x_n\|) \\ &\qquad + e_n \left\| \frac{\gamma_n}{\gamma_n + \delta_n} (B_n x_n - p) + \frac{\delta_n}{\gamma_n + \delta_n} (S_n G x_n - p) \right\|^2 \\ &\quad \le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\ &\qquad - \beta_n e_n g_1 (\|\hat{z}_n - x_n\|) \\ &\qquad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n} \|B_n x_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n} \|S_n G x_n - p\|^2 \right] \\ &\quad \le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\ &\qquad - \beta_n e_n g_1 (\|\hat{z}_n - x_n\|) \\ &\qquad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n} \|x_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n} \|x_n - p\|^2 \right] \\ &\quad = \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 \\ &\qquad - \beta_n e_n g_1 (\|\hat{z}_n - x_n\|) \\ &\quad \le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 \\ &\qquad - \beta_n e_n g_1 (\|\hat{z}_n - x_n\|), \end{aligned} \tag{275}$$

which hence implies that

$$\begin{aligned} &\beta_n e_n g_1 (\|\hat{z}_n - x_n\|) \\ &\quad \le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad \le \alpha_n \|f(x_n) - p\|^2 + (\|x_n - p\| + \|y_n - p\|) \|x_n - y_n\|. \end{aligned} \tag{276}$$

Utilizing (271), conditions (i), (ii), (iv), and the boundedness of $\{x_n\}$, $\{y_n\}$ and $\{f(x_n)\}$, we get

$$\lim_{n\to\infty} g_1 (\|\hat{z}_n - x_n\|) = 0. \tag{277}$$

From the properties of g_1, we have

$$\lim_{n\to\infty} \|\hat{z}_n - x_n\| = 0. \tag{278}$$

Utilizing Lemma 15 and the definition of $\hat{z}_n$, we have

$$\begin{aligned} &\|\hat{z}_n - p\|^2 \\ &\quad = \left\| \frac{\gamma_n B_n x_n + \delta_n S_n G x_n}{\gamma_n + \delta_n} - p \right\|^2 \\ &\quad = \left\| \frac{\gamma_n}{\gamma_n + \delta_n} (B_n x_n - p) + \frac{\delta_n}{\gamma_n + \delta_n} (S_n G x_n - p) \right\|^2 \\ &\quad \le \frac{\gamma_n}{\gamma_n + \delta_n} \|B_n x_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n} \|S_n G x_n - p\|^2 \\ &\qquad - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2 (\|S_n G x_n - B_n x_n\|) \\ &\quad \le \|x_n - p\|^2 - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2 (\|S_n G x_n - B_n x_n\|), \end{aligned} \tag{279}$$

which leads to

$$\begin{aligned} &\frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2 (\|S_n G x_n - B_n x_n\|) \\ &\quad \le \|x_n - p\|^2 - \|\hat{z}_n - p\|^2 \\ &\quad \le (\|x_n - p\| + \|\hat{z}_n - p\|) \|x_n - \hat{z}_n\|. \end{aligned} \tag{280}$$

Since $\{x_n\}$ and $\{\hat{z}_n\}$ are bounded, we deduce from (278) and condition (ii) that

$$\lim_{n\to\infty} g_2 (\|B_n x_n - S_n G x_n\|) = 0. \tag{281}$$

From the properties of g_2, we have

$$\lim_{n\to\infty} \|B_n x_n - S_n G x_n\| = 0. \tag{282}$$

Furthermore, y_n can also be rewritten as follows:

$$\begin{aligned} y_n &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n G x_n \\ &= \beta_n x_n + \gamma_n B_n x_n + (\alpha_n + \delta_n) \frac{\alpha_n f(x_n) + \delta_n S_n G x_n}{\alpha_n + \delta_n} \\ &= \beta_n x_n + \gamma_n B_n x_n + d_n \tilde{z}_n, \end{aligned} \tag{283}$$

where $d_n = \alpha_n + \delta_n$ and $\tilde{z}_n = (\alpha_n f(x_n) + \delta_n S_n G x_n)/(\alpha_n + \delta_n)$. Utilizing Lemma 11 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \|\beta_n (x_n - p) + \gamma_n (B_n x_n - p) + d_n (\tilde{z}_n - p)\|^2 \\
&\quad \le \beta_n \|x_n - p\|^2 + \gamma_n \|B_n x_n - p\|^2 \\
&\qquad + d_n \|\tilde{z}_n - p\|^2 - \beta_n \gamma_n g_3 (\|x_n - B_n x_n\|) \\
&\quad = \beta_n \|x_n - p\|^2 + \gamma_n \|B_n x_n - p\|^2 \\
&\qquad + d_n \left\| \frac{\alpha_n f(x_n) + \delta_n S_n G x_n}{\alpha_n + \delta_n} - p \right\|^2 \\
&\qquad - \beta_n \gamma_n g_3 (\|x_n - B_n x_n\|) \\
&\quad = \beta_n \|x_n - p\|^2 + \gamma_n \|B_n x_n - p\|^2 \\
&\qquad + d_n \left\| \frac{\alpha_n}{\alpha_n + \delta_n} (f(x_n) - p) + \frac{\delta_n}{\alpha_n + \delta_n} (S_n G x_n - p) \right\|^2 \\
&\qquad - \beta_n \gamma_n g_3 (\|x_n - B_n x_n\|) \\
&\quad \le \beta_n \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 \\
&\qquad + d_n \left[\frac{\alpha_n}{\alpha_n + \delta_n} \|f(x_n) - p\|^2 + \frac{\delta_n}{\alpha_n + \delta_n} \|S_n G x_n - p\|^2 \right] \\
&\qquad - \beta_n \gamma_n g_3 (\|x_n - B_n x_n\|) \\
&\quad \le \alpha_n \|f(x_n) - p\|^2 + (\beta_n + \gamma_n) \|x_n - p\|^2 \\
&\qquad + \delta_n \|x_n - p\|^2 - \beta_n \gamma_n g_3 (\|x_n - B_n x_n\|) \\
&\quad = \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 \\
&\qquad - \beta_n \gamma_n g_3 (\|x_n - B_n x_n\|) \\
&\quad \le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 \\
&\qquad - \beta_n \gamma_n g_3 (\|x_n - B_n x_n\|),
\end{aligned} \tag{284}$$

which hence implies that

$$\begin{aligned}
&\beta_n \gamma_n g_3 (\|x_n - B_n x_n\|) \\
&\quad \le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\quad \le \alpha_n \|f(x_n) - p\|^2 + (\|x_n - p\| + \|y_n - p\|) \\
&\qquad \times \|x_n - y_n\|.
\end{aligned} \tag{285}$$

Utilizing (271), conditions (i), (ii), (iv), and the boundedness of $\{x_n\}$, $\{y_n\}$ and $\{f(x_n)\}$, we get

$$\lim_{n\to\infty} g_3 (\|x_n - B_n x_n\|) = 0. \tag{286}$$

From the properties of g_3, we have

$$\lim_{n\to\infty} \|x_n - B_n x_n\| = 0. \tag{287}$$

Thus, from (282) and (287), we get

$$\begin{aligned}
\|x_n - S_n G x_n\| &\le \|x_n - B_n x_n\| \\
&\quad + \|B_n x_n - S_n G x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{288}$$

That is,

$$\lim_{n\to\infty} \|x_n - S_n G x_n\| = 0. \tag{289}$$

Therefore, from Lemma 12, (273), and (289), it follows that

$$\begin{aligned}
&\|x_n - S x_n\| \\
&\quad \le \|x_n - S_n G x_n\| + \|S_n G x_n - S_n x_n\| + \|S_n x_n - S x_n\| \\
&\quad \le \|x_n - S_n G x_n\| + \|G x_n - x_n\| \\
&\qquad + \|S_n x_n - S x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{290}$$

That is,

$$\lim_{n\to\infty} \|x_n - S x_n\| = 0. \tag{291}$$

Suppose that $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$ such that $\alpha_n + \beta + \gamma_n + \delta_n = 1$ for all $n \ge 0$. Define a mapping $Vx = (1 - \theta_1 - \theta_2) S x + \theta_1 B x + \theta_2 G x$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then, by Lemmas 14 and 17, we have that $\mathrm{Fix}(V) = \mathrm{Fix}(S) \cap \mathrm{Fix}(B) \cap \mathrm{Fix}(G) = F$. For each $k \ge 1$, let $\{p_k\}$ be a unique element of C such that

$$p_k = \frac{1}{k} f(p_k) + \left(1 - \frac{1}{k}\right) V p_k. \tag{292}$$

From Lemma 13, we conclude that $p_k \to q \in \mathrm{Fix}(V) = F$ as $k \to \infty$. Repeating the same arguments as those of (81) in the proof of Theorem 24, we can conclude that for every n, k

$$\begin{aligned}
&\|y_n - B p_k\| \\
&\quad \le \alpha_n \|f(x_n) - B p_k\| + \beta \|x_n - B p_k\| \\
&\qquad + \gamma_n \|B_n x_n - B p_k\| + \delta_n \|S_n G x_n - B p_k\| \\
&\quad \le \theta_n + \beta \|x_n - B p_k\| + (1 - \beta) \|x_n - p_k\|,
\end{aligned} \tag{293}$$

where $\theta_n = \alpha_n \|f(x_n) - B p_k\| + (1 - \beta) \|B_n p_k - B p_k\| + \delta_n \|S_n G x_n - B_n x_n\|$. Since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \|B_n p_k - B p_k\| = \lim_{n\to\infty} \|S_n G x_n - B_n x_n\| = 0$, we know that $\theta_n \to 0$ as

$n \to \infty$. So, it immediately follows that

$$\begin{aligned}
&\|x_{n+1} - Bp_k\|^2 \\
&\quad = \|y_n - Bp_k\|^2 + \|x_{n+1} - y_n\| \\
&\qquad \times [2\|y_n - Bp_k\| + \|x_{n+1} - y_n\|] \\
&\quad \leq (\beta\|x_n - Bp_k\| + (1-\beta)\|x_n - p_k\|)^2 \\
&\qquad + \theta_n [2(\beta\|x_n - Bp_k\| + (1-\beta)\|x_n - p_k\|) + \theta_n] \\
&\qquad + \|x_{n+1} - y_n\| [2\|y_n - Bp_k\| + \|x_{n+1} - y_n\|] \\
&\quad = \beta^2\|x_n - Bp_k\|^2 + (1-\beta)^2\|x_n - p_k\|^2 \\
&\qquad + 2\beta(1-\beta)\|x_n - Bp_k\|\|x_n - p_k\| + \tau_n \\
&\quad \leq \beta^2\|x_n - Bp_k\|^2 + (1-\beta)^2\|x_n - p_k\|^2 \\
&\qquad + \beta(1-\beta)(\|x_n - Bp_k\|^2 + \|x_n - p_k\|^2) + \tau_n \\
&\quad = \beta\|x_n - Bp_k\|^2 + (1-\beta)\|x_n - p_k\|^2 + \tau_n,
\end{aligned} \tag{294}$$

where $\tau_n = \theta_n[2(\beta\|x_n - Bp_k\| + (1-\beta)\|x_n - p_k\|) + \theta_n] + \|x_{n+1} - y_n\|[2\|y_n - Bp_k\| + \|x_{n+1} - y_n\|] \to 0$ as $n \to \infty$.

For any Banach limit μ, from (294), we derive

$$\mu_n\|x_n - Bp_k\|^2 = \mu_n\|x_{n+1} - Bp_k\|^2 \leq \mu_n\|x_n - p_k\|^2. \tag{295}$$

In addition, note that

$$\begin{aligned}
&\|x_n - Gp_k\|^2 \\
&\quad \leq \|x_n - Gx_n + Gx_n - Gp_k\|^2 \\
&\quad \leq (\|x_n - Gx_n\| + \|x_n - p_k\|)^2 \\
&\quad = \|x_n - p_k\|^2 + \|x_n - Gx_n\| \\
&\qquad \times (2\|x_n - p_k\| + \|x_n - Gx_n\|), \\
&\|x_n - Sp_k\|^2 \\
&\quad \leq \|x_n - Sx_n + Sx_n - Sp_k\|^2 \\
&\quad \leq (\|x_n - Sx_n\| + \|x_n - p_k\|)^2 \\
&\quad = \|x_n - p_k\|^2 + \|x_n - Sx_n\| \\
&\qquad \times (2\|x_n - p_k\| + \|x_n - Sx_n\|).
\end{aligned} \tag{296}$$

It is easy to see from (273) and (291) that

$$\begin{aligned}
\mu_n\|x_n - Gp_k\|^2 &\leq \mu_n\|x_n - p_k\|^2, \\
\mu_n\|x_n - Sp_k\|^2 &\leq \mu_n\|x_n - p_k\|^2.
\end{aligned} \tag{297}$$

Utilizing (295) and (297), we deduce that

$$\begin{aligned}
&\mu_n\|x_n - Vp_k\|^2 \\
&\quad = \mu_n\|(1-\theta_1-\theta_2)(x_n - Sp_k) \\
&\qquad\quad + \theta_1(x_n - Bp_k) + \theta_2(x_n - Gp_k)\|^2 \\
&\quad \leq (1-\theta_1-\theta_2)\mu_n\|x_n - Sp_k\|^2 + \theta_1\mu_n\|x_n - Bp_k\|^2 \\
&\qquad + \theta_2\mu_n\|x_n - Gp_k\|^2 \\
&\quad \leq \mu_n\|x_n - p_k\|^2.
\end{aligned} \tag{298}$$

Repeating the same arguments as those of (99) in the proof of Theorem 24, we can obtain that

$$\frac{1}{2k}\mu_n\|x_n - p_k\|^2 \geq \mu_n\langle f(p_k) - p_k, J(x_n - p_k)\rangle. \tag{299}$$

Since $p_k \to q \in \text{Fix}(V) = F$ as $k \to \infty$, by the uniform Gateaux differentiability of the norm of X we have

$$\mu_n\langle f(q) - q, J(x_n - q)\rangle \leq 0. \tag{300}$$

On the other hand, from (257) and the norm-to-weak* uniform continuity of J on bounded subsets of X, it follows that

$$\lim_{n\to\infty}|\langle f(q) - q, J(x_{n+1} - q)\rangle - \langle f(q) - q, J(x_n - q)\rangle| = 0. \tag{301}$$

So, utilizing Lemma 18, we deduce from (300) and (301) that

$$\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \leq 0, \tag{302}$$

which together with (271) and the norm-to-weak* uniform continuity of J on bounded subsets of X, implies that

$$\limsup_{n\to\infty}\langle f(q) - q, J(y_n - q)\rangle \leq 0. \tag{303}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. Utilizing Lemma 8 (i), from (253) and the convexity of $\|\cdot\|$, we get

$$\begin{aligned}
&\|y_n - q\|^2 \\
&\quad \leq \|\alpha_n(f(x_n) - f(q)) + \beta_n(x_n - q) \\
&\qquad + \gamma_n(B_n x_n - q) + \delta_n(S_n G x_n - q)\|^2 \\
&\qquad + 2\alpha_n\langle f(q) - q, J(y_n - q)\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \|f(x_n) - f(q)\|^2 + \beta_n \|x_n - q\|^2 \\
&\quad + \gamma_n \|B_n x_n - q\|^2 + \delta_n \|S_n G x_n - q\|^2 \\
&\quad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle \\
&\leq \alpha_n \rho \|x_n - q\|^2 + \beta_n \|x_n - q\|^2 \\
&\quad + \gamma_n \|x_n - q\|^2 + \delta_n \|x_n - q\|^2 \\
&\quad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle \\
&= (1 - \alpha_n(1-\rho)) \|x_n - q\|^2 \\
&\quad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle,
\end{aligned} \tag{304}$$

and hence

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \leq \sigma_n \|G x_n - q\|^2 + (1-\sigma_n)\|y_n - q\|^2 \\
&\quad \leq \sigma_n \|x_n - q\|^2 + (1-\sigma_n) \\
&\qquad \times \left[(1 - \alpha_n(1-\rho))\|x_n - q\|^2 \right. \\
&\qquad\quad \left. + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle \right] \\
&\quad = [1 - (1-\sigma_n)\alpha_n(1-\rho)]\|x_n - q\|^2 \\
&\qquad + 2(1-\sigma_n)\alpha_n \langle f(q) - q, J(y_n - q)\rangle \\
&\quad = [1 - (1-\sigma_n)\alpha_n(1-\rho)]\|x_n - q\|^2 \\
&\qquad + (1-\sigma_n)\alpha_n(1-\rho)\frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}.
\end{aligned} \tag{305}$$

From conditions (i) and (iv), it is easy to see that $\sum_{n=0}^{\infty}(1-\sigma_n)\alpha_n(1-\rho) = \infty$. Applying Lemma 7 to (305), we infer that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 34. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gateaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0,b]$ for some $b \in (0,1)$ and $A_i : C \to X \xi_i$-strictly pseudocontractive and $\widehat{\alpha}_i$-strongly accretive with $\xi_i + \widehat{\alpha}_i \geq 1$ for each $i = 0,1,\ldots$. Define a mapping $G_i : C \to C$ by $\Pi_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 0,1,\ldots$, where $1-(\xi_i/(1+\xi_i))(1-\sqrt{(1-\widehat{\alpha}_i)/\xi_i}) \leq \lambda_i \leq 1$ for all $i = 0,1,\ldots$. Let $B_n : C \to C$ be the W-mapping generated by $G_n, G_{n-1}, \ldots, G_0$ and $\rho_n, \rho_{n-1}, \ldots, \rho_0$. Let $V : C \to C$ be a self-mapping such that $I - V : C \to X$ is ζ-strictly pseudocontractive and θ-strongly accretive with $\theta + \zeta \geq 1$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings of C into itself such that $F = (\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)) \cap \mathrm{Fix}(V) \cap (\bigcap_{i=0}^{\infty}\mathrm{VI}(C,A_i)) \neq \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
&x_{n+1} \\
&\quad = \sigma_n((1-l)I + lV)x_n + (1-\sigma_n) \\
&\qquad \times [\alpha_n f(x_n) + \beta_n x_n + \gamma_n B_n x_n + \delta_n S_n((1-l)I + lV)x_n], \\
&\qquad\qquad \forall n \geq 0,
\end{aligned} \tag{306}$$

where $1-(\zeta/(1+\zeta))(1-\sqrt{(1-\theta)/\zeta}) \leq l \leq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, and $\{\sigma_n\}$ are the sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c,d]$ *for some* $c, d \in (0,1)$;

(iii) $\sum_{n=1}^{\infty}(|\sigma_n-\sigma_{n-1}|+|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\delta_n-\delta_{n-1}|) < \infty$;

(iv) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ *and* $0 < \liminf_{n\to\infty}\sigma_n \leq \limsup_{n\to\infty}\sigma_n < 1$.

Assume that $\sum_{n=1}^{\infty}\sup_{x\in D}\|S_n x - S_{n-1}x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty}S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)$. Then, there hold the following:

(I) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;

(II) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $q \in F$ which is the unique solution of the variational inequality problem (VIP)*

$$\langle (I - f)q, J(q-p)\rangle \leq 0, \quad \forall p \in F, \tag{307}$$

provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$.

Proof. In Theorem 33, we put $B_1 = I - V$, $B_2 = 0$ and $\mu_1 = l$ where $1-(\zeta/(1+\zeta))(1-\sqrt{(1-\theta)/\zeta}) \leq l \leq 1$. Then, GSVI (13) is equivalent to the VIP of finding $x^* \in C$ such that

$$\langle B_1 x^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{308}$$

In this case, $B_1 : C \to X$ is ζ-strictly pseudocontractive and θ-strongly accretive. Repeating the same arguments as those in the proof of Corollary 25, we can infer that $\mathrm{Fix}(V) = \mathrm{VI}(C,B_1)$. Accordingly, $F = (\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)) \cap \Omega \cap (\bigcap_{i=0}^{\infty}\mathrm{VI}(C,A_i)) = (\bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i)) \cap \mathrm{Fix}(V) \cap (\bigcap_{i=0}^{\infty}\mathrm{VI}(C,A_i))$,

$$Gx_n = ((1-l)I + lV)x_n, \tag{309}$$

So, the scheme (251) reduces to (306). Therefore, the desired result follows from Theorem 33. □

Remark 35. Our Theorems 31 and 33 improve, extend, supplement and develop Ceng and Yao's [10, Theorem 3.2], Cai and Bu's [11, Theorem 3.1], Kangtunyakarn's [38, Theorem 3.1], and Ceng and Yao's [8, Theorem 3.1], in the following aspects.

(i) The problem of finding a point $q \in (\bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)) \cap \Omega \cap (\bigcap_{i=0}^{\infty} \mathrm{VI}(C, A_i))$ in our Theorems 31 and 33 is more general and more subtle than every one of the problem of finding a point $q \in \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i)$ in [10, Theorem 3.2], the problem of finding a point $q \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \Omega$ in [11, Theorem 3.1], the problem of finding a point $q \in \mathrm{Fix}(S) \cap \mathrm{Fix}(V) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(C, A_i))$ in [38, Theorem 3.1], and the problem of finding a point $q \in \mathrm{Fix}(T)$ in [8, Theorem 3.1].

(ii) The iterative scheme in [38, Theorem 3.1] is extended to develop the iterative scheme (178) of our Theorem 31, and the iterative scheme in [11, Theorem 3.1] is extended to develop the iterative scheme (251) of our Theorem 33. Iterative schemes (178) and (181) in our Theorems 31 and 33 are more advantageous and more flexible than the iterative scheme of [11, Theorem 3.1] because they both are one-step iteration schemes and involve several parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, (and $\{\sigma_n\}$).

(iii) Our Theorems 31 and 33 extend and generalize Ceng and Yao's [8, Theorem 3.1] from a nonexpansive mapping to a countable family of nonexpansive mappings, and Ceng and Yao's [10, Theorems 3.2] to the setting of the GSVI (13) and infinitely many VIPs, Kangtunyakarn's [38, Theorem 3.1] from finitely many VIPs to infinitely many VIPs, from a nonexpansive mapping to a countable family of nonexpansive mappings and from a strict pseudocontraction to the GSVI (13). In the meantime, our Theorems 31 and 33 extend and generalize Cai and Bu's [11, Theorem 3.1] to the setting of infinitely many VIPs.

(iv) The iterative schemes (178) and (251) in our Theorems 31 and 33 are very different from every one in [10, Theorem 3.2], [11, Theorem 3.1], [38, Theorem 3.1], and [8, Theorem 3.1] because the mappings G and T_n in [11, Theorem 3.1] and the mapping T in [8, Theorem 3.1] are replaced with the same composite mapping S_nG in the iterative schemes (42) and (130) and the mapping W_n in [10, Theorem 3.2] is replaced by B_n.

(v) Cai and Bu's proof in [11, Theorem 3.1] depends on the argument techniques in [14], the inequality in 2-uniformly smooth Banach spaces (see Lemma 4), and the inequality in smooth and uniform convex Banach spaces (see Proposition 6). Because the composite mapping S_nG appears in the iterative scheme (178) of our Theorem 31, the proof of our Theorem 31 depends on the argument techniques in [14], the inequality in 2-uniformly smooth Banach spaces (see Lemma 4), the inequality in smooth and uniform convex Banach spaces (see Proposition 6), the inequalities in uniform convex Banach spaces (see Lemmas 11 and 15 in Section 2 of this paper), and the properties of the W-mapping and the Banach limit (see Lemmas 16, 17, and 18 in Section 2 of this paper). However, the proof of our Theorem 33 does not depend on the argument techniques in [14], the inequality in 2-uniformly smooth Banach spaces (see Lemma 4), and the inequality in smooth and uniform convex Banach spaces (see Proposition 6). It depends on only the inequalities in uniform convex Banach spaces (see Lemmas 11 and 15 in Section 2 of this paper) and the properties of the W-mapping and the Banach limit (see Lemmas 16–18 in Section 2 of this paper).

(vi) The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [11, Theorem 3.1] is weakened to the one of the uniformly convex Banach space X having a uniformly Gateaux differentiable norm in our Theorem 33. Moreover, the assumption of the uniformly smooth Banach space X in [8, Theorem 3.1] is replaced with the one of the uniformly convex Banach space X having a uniformly Gateaux differentiable norm in our Theorem 33.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University under Grant no. HiCi/15-130-1433. The authors, therefore, acknowledge technical and financial support of KAU. The authors would like to thank Professor J. C. Yao for motivation and many fruitful discussions regarding this work.

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