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Research Article

Synchronization for a Class of Fractional-Order Hyperchaotic System and Its Application

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A new controller design method is proposed to synchronize the fractional-order hyperchaotic system through the stability theory of fractional calculus; the synchronization between two identical fractional-order Chen hyperchaotic systems is realized by designing only two suitable controllers in the response system. Furthermore, this control scheme can be used in secure communication via the technology of chaotic masking using the complex nonperiodic information as trial message, and the useful information can be recovered at the receiver. Numerical simulations coincide with the theoretical analysis.

1. Introduction

It is well known that the fractional calculus has a long mathematical history, but its applications to physics and engineering are just recent subject of interest [1]. In recent years, more and more researchers focused on the control of fractional-order chaotic systems and its dynamic behavior [2–4]. In [5], chaos and hyperchaos in the fractional-order Rŏssler equations were studied, in which chaos can exist in the fractional-order Rŏssler equation with order as low as 2.4, and hyperchaos exists in the fractional-order Rŏssler hyperchaos equation with order as low as 3.8. In [6], the chaotic behavior with the lowest order 3.72 in the fractional-order hyperchaotic Chen system is presented.

For decades, the complex dynamics and synchronization of chaotic systems have attracted much attention [7–21] since the seminal paper by Pecora and Carroll in 1990 [22]. The chaotic control and synchronization of fractional-order systems are concerned extremely [23–27]. The fractional-order system, compared to the integer-order system, has more universality, large key space, and more complex dynamic behaviors than the low-dimensional chaotic system [28], turning into a more challenging work to solve the problem of its control, synchronization, and antisynchronization. In [29], the synchronization of fractional-order hyperchaotic Lorenz system with unknown parameters is realized via designing adaptive tracking controller. In [30], the antisynchronization of different hyperchaotic systems is studied based on the method of active control. In nature, the usual way to synchronize nonlinear fractional-order hyperchaotic systems is to design controller generally needing three or more based on the stability theory of fractional calculus [9, 29–32]. Therefore, such designed controller must result in unnecessarily complicated structure, to a certain extent, which limits to the application in practice.

More recently, studies have been intensively focused on fractional-order hyperchaotic system due to its potential applications in secure communication and control processing. However, to our best knowledge, in literature [33–35], the authors are all concerned with its applications to secure communication with regular or periodic signals. Yet, the normal signals are too simple, and they are easy to be decoded even mingling the chaotic signals with transmission channel. Usually, the actual signal transmission is irregular; therefore, research on the synchronization of fractional-order hyperchaotic systems holds great significance for its application to secure communication with nonperiodic information signals.

In this paper, we propose a new control design method using only two controllers in response system to synchronize a class of fractional-order hyperchaotic system. The example, the two identical fractional-order hyperchaotic Chen systems, is utilized to illustrate how to design the controllers to simplify the existing control scheme, and the proposed procedure can be applied to secure communication via the technology of chaotic masking.

2. Fractional Derivative and Its Approximation

There are several definitions of fractional derivatives [1]. In our work, we use the best known Caputo derivative [30] defined by

$$D_*^{\alpha}x(t) = J^{n-\alpha}x^{(n)}(t), \qquad \alpha > 0, \tag{2.1}$$

where n is the first integer which is not less than α, and J^{β} is the β-order Riemann-Liouville integer operator which is described as follows:

$$J^{\beta}y(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{y(\tau)}{(t-\tau)^{1-\beta}}d\tau, \tag{2.2}$$

where $\Gamma(\cdot)$ is the gamma function and $0 < \beta \leqslant 1$.

Given a fractional-order chaotic system, the drive (master) system is

$$\frac{d^qX_1}{dt^q} = F(X_1). \tag{2.3}$$

Here $X_1 \in \Re^{n\times 1}$, $F(X_1) = (f(X_1), f(X_2), \ldots, f(X_n))^T$, and $q \in (0,1]$. The fractional-order $q \in \Re^{n\times 1}$ may be unequal. The equilibrium points of system (2.3) can be derived by solving following equation:

$$F(X_1) = 0. \tag{2.4}$$

The stability of fractional-order system has been thoroughly investigated, and necessary and sufficient conditions have been presented in [26]:

$$\frac{d^q X_2}{dt^q} = F(X_2) + u(t), \tag{2.5}$$

where $X_2 \in \Re^{n\times 1}$, and $u(t)$ is the control functions. Suppose that the error between the system (2.3) and the system (2.5) is $e(t) = X_1(t) - X_2(t)$; then the fractional error system can be obtained as

$$\frac{d^q e}{dt^q} = F(X_1) - F(X_2) - u(t). \tag{2.6}$$

The master-slave synchronization of two chaotic systems is tightly associated with stability of the error dynamics; in this paper, the control function method is presented. Before discussing the method, we first give some useful preliminaries which are of great help to the proof of the forthcoming theorem.

Lemma 2.1 (see [36]). *System* (2.3) *is locally asymptotically stable if all the eigenvalues* $(\lambda_1, \ldots, \lambda_n)$ *of the Jacobian matrix of all equilibrium point satisfy*

$$\left|\arg(\lambda)\right| > q > \frac{2}{\pi}. \tag{2.7}$$

In nature, if $q > (2/\pi)|\arctan(\mathrm{Im}(\lambda)/\mathrm{Re}(\lambda))|$, the system (2.3) is locally asymptotically stable and behave chaotic for all the variations where the eigenvalues of the matrix satisfy $|\arg(\lambda)| > q\pi/2$. The stable and unstable regions for $q \in (0,1]$ are depicted in Figure 1. Obviously, the stable region of a fractional-order system is normally larger than its corresponding integer-order system:

$$\dot{X}_1 = F(X_1), \tag{2.8}$$

whose stable region is the left half plane. Based on this, we obtain the following corollary immediately.

Corollary 2.2. *The fractional-order system* (2.3) *with order* $q \in (0,1]$ *is asymptotically stable if the corresponding integer system* (2.8) *is stable.*

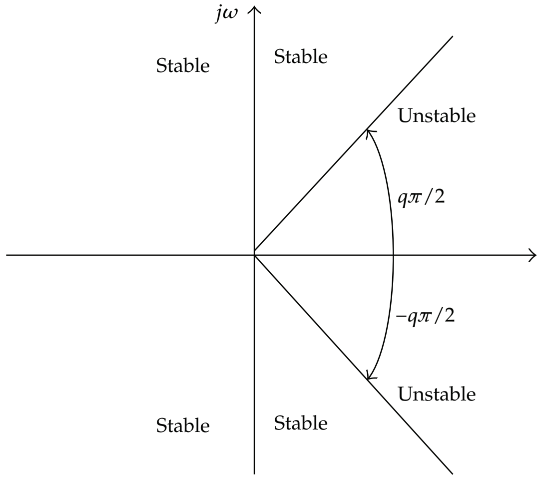

Figure 1: Stability region of fractional-order system.

3. The Fractional-Order Chen System

Now, consider the fractional-order hyperchaotic Chen system as follows:

$$\begin{aligned}\frac{d^\alpha x}{dt^\alpha} &= a(y-x)+w,\\ \frac{d^\alpha y}{dt^\alpha} &= bx - xz + cy,\\ \frac{d^\alpha z}{dt^\alpha} &= xy - dz,\\ \frac{d^\alpha w}{dt^\alpha} &= yz + rw,\end{aligned} \tag{3.1}$$

where $a = 35,\ b = 7,\ c = 12,\ d = 3$, and $r = 0.5$. This system has five equilibria:

$$\begin{aligned} S_0 &= (0,0,0,0),\\ S_1 &= (-82.0531 - 284.8306i, -6.6181 - 1.2245i, -4.2737 + 6.9135i, 12.2500 - 13.5069i),\\ S_2 &= (-82.0531 + 284.8306i, -6.6181 + 1.2245i, -4.2737 - 6.9135i, 12.2500 + 13.5069i),\\ S_3 &= (82.0531 - 284.8306i, 6.6181 - 1.2245i, 4.2737 + 6.9135i, 12.2500 + 13.5069i),\\ S_4 &= (82.0531 + 284.8306i, 6.6181 + 1.2245i, 4.2737 - 6.9135i, 12.2500 - 13.5069i).\end{aligned} \tag{3.2}$$

The corresponding Jacobian matrix is as follows:

$$J = \begin{pmatrix} -35 & 35 & 0 & 1\\ 7 & 12 & -x & 0\\ y & x & -3 & 0\\ z & 0 & x & r\end{pmatrix}. \tag{3.3}$$

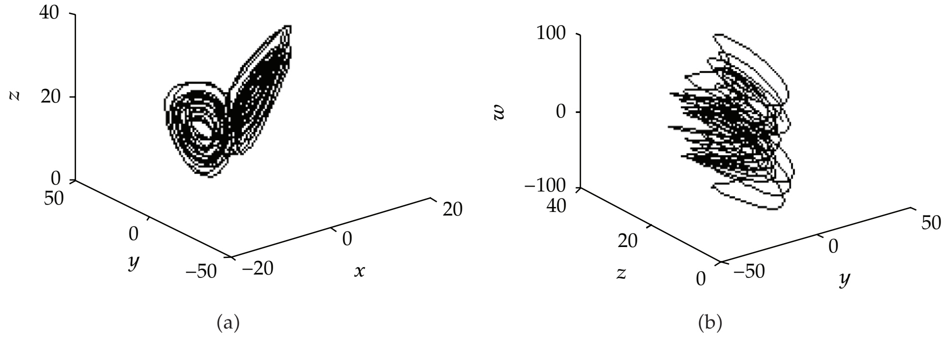

Figure 2: The phase portrait of system (3.1) chaotic attractor ($\alpha = 0.96$) with (a) x, y, z and (b) y, z, w.

Then, the eigenvalues of the Jacobian matrix are obtained:

$$
\begin{aligned}
&S_0\colon \lambda_1 = -39.7356, \quad \lambda_2 = 16.7356, \quad \lambda_3 = -3.0000, \quad \lambda_4 = 0.5000;\\
&S_1\colon \lambda_1 = 289.93 - 82.25i, \quad \lambda_2 = -280.64 + 81.49i, \quad \lambda_3 = -35.29 + 0.77i, \quad \lambda_4 = 0.50 - 0.01i;\\
&S_2\colon \lambda_1 = 289.93 + 82.25i, \quad \lambda_2 = -280.64 - 81.49i, \quad \lambda_3 = -35.29 - 0.77i, \quad \lambda_4 = 0.50 + 0.01i;\\
&S_3\colon \lambda_1 = 289.93 + 82.25i, \quad \lambda_2 = -280.64 - 81.49i, \quad \lambda_3 = -35.29 - 0.77i, \quad \lambda_4 = 0.50;\\
&S_4\colon \lambda_1 = 289.93 - 82.25i, \quad \lambda_2 = -280.64 + 81.49i, \quad \lambda_3 = -35.29 + 0.77i, \quad \lambda_4 = 0.50.
\end{aligned}
\tag{3.4}
$$

It is easy to show that eigenvalues from S_0 to S_4 hold if they satisfy $q > (2/\pi)|\arctan(82.25/289.93)|$ based on Lemma 2.1, and the fractional-order hyperchaotic Chen is chaotic. From the phase portrait of chaotic attractor at $\alpha = 0.96$ as shown in Figure 2 we can find that the system (3.1) exists with chaotic behavior indeed.

4. The Synchronization of the Two Identical Incommensurate Fractional-Order Chen Systems

Now, we will study synchronization between two identical fractional-order hyperchaotic Chen systems. The fractional-order hyperchaotic Chen system as the drive system is expressed by

$$
\begin{aligned}
\frac{d^\alpha x_1}{dt^\alpha} &= a(y_1 - x_1) + w_1,\\
\frac{d^\alpha y_1}{dt^\alpha} &= bx_1 - x_1 z_1 + c y_1,\\
\frac{d^\alpha z_1}{dt^\alpha} &= x_1 y_1 - d z_1,\\
\frac{d^\alpha w_1}{dt^\alpha} &= y_1 z_1 + r w_1.
\end{aligned}
\tag{4.1}
$$

And the corresponding response system is written by

$$\begin{aligned}\frac{d^\alpha x_2}{dt^\alpha} &= a(y_2 - x_2) + w_2,\\ \frac{d^\alpha y_2}{dt^\alpha} &= bx_2 - x_2 z_2 + cy_2 + u_1,\\ \frac{d^\alpha z_2}{dt^\alpha} &= x_2 y_2 - dz_2,\\ \frac{d^\alpha w_2}{dt^\alpha} &= y_2 z_2 + rw_2 + u_2.\end{aligned} \tag{4.2}$$

Here, $u = [u_1, u_2]^T$ is the control function. Our aim is to design the controller $u = [u_1, u_2]^T$ that will make the system (4.2) achieve synchronization with the system (4.1). In order to facilitate the following analysis, we set the errors between the system (4.2) and system (4.1):

$$\begin{aligned}e_1 &= x_2 - x_1,\\ e_2 &= y_2 - y_1,\\ e_3 &= z_2 - z_1,\\ e_4 &= w_2 - w_1.\end{aligned} \tag{4.3}$$

From (4.3), (4.2), and (4.1), we obtain the following error dynamical system:

$$\begin{aligned}\frac{d^\alpha e_1}{dt^\alpha} &= a(e_2 - e_1) + e_4,\\ \frac{d^\alpha e_2}{dt^\alpha} &= (b - z_1)e_1 + ce_2 + x_2(z_1 - z_2) + u_1,\\ \frac{d^\alpha e_3}{dt^\alpha} &= -de_3 + x_1 e_2 + (y_1 + e_2)e_1,\\ \frac{d^\alpha e_4}{dt^\alpha} &= re_4 + z_1 e_2 - y_2(z_1 - z_2) + u_2.\end{aligned} \tag{4.4}$$

Then, consider the following control function:

$$\begin{aligned}u_1 &= (c + k_1)e_2 + (b - z_1)e_1 + x_2(z_1 - z_2),\\ u_2 &= (r + k_2)e_4 - y_2(z_1 - z_2).\end{aligned} \tag{4.5}$$

Substituting controllers (4.5) into (4.4), we have

$$\begin{aligned}
\frac{d^{\alpha}e_1}{dt^{\alpha}} &= a(e_2 - e_1) + e_4,\\
\frac{d^{\alpha}e_2}{dt^{\alpha}} &= -k_1 e_2,\\
\frac{d^{\alpha}e_3}{dt^{\alpha}} &= -de_3 + x_1 e_2 + (y_1 + e_2)e_1,\\
\frac{d^{\alpha}e_4}{dt^{\alpha}} &= -k_2 e_4 + z_1 e_2.
\end{aligned} \tag{4.6}$$

And its corresponding integer system is

$$\begin{aligned}
\frac{de_1}{dt} &= a(e_2 - e_1) + e_4,\\
\frac{de_2}{dt} &= -k_1 e_2,\\
\frac{de_3}{dt} &= -de_3 + x_1 e_2 + (y_1 + e_2)e_1,\\
\frac{de_4}{dt} &= -k_2 e_4 + z_1 e_2.
\end{aligned} \tag{4.7}$$

We can obtain the Jacobian matrix of the error system with linear system (4.7) as follows:

$$A = \begin{pmatrix} -a & a & 0 & 1 \\ 0 & -k_1 & 0 & 0 \\ y_1 + e_2 & x_1 + e_1 & -d & 0 \\ 0 & z_1 & 0 & -k_2 \end{pmatrix}. \tag{4.8}$$

Obviously, the system (4.7), in which all eigenvalues $-a, -k_1, -d$, and $-k_2$ are less than zero with k_1 and k_2 to positive constant, must be stable without a doubt. And the corresponding fractional-order system (4.6) is asymptotically stable according to Corollary 2.2 given in the second section. So, the errors $\lim e_1(t)$, $\lim e_2(t)$, $\lim e_3(t)$, and $\lim e_4(t)$ will converge to zero when $t \to \infty$. Therefore, synchronization of the two identical fractional-order hyperchaotic systems is achieved.

4.1. Simulation Results

In the numerical simulations, we set the parameters of the system (4.1) and (4.2) as $a = 35$, $b = 3$, $c = 28$, and $r = 0.5$ for drive system and response system with $\alpha = 0.96$ and the coefficient of control function $k_1 = 15$ and $k_2 = 20$. The initial conditions of the drive and response systems are taken arbitrarily as $x_1(0) = 10$, $y_1(0) = 10$, $z_1(0) = 10$, and $w_1(0) = 10$; and $x_2(0) = -10$, $y_2(0) = -10$, $z_2(0) = 3$, and $w_2(0) = -10$. Numerical results show that the synchronization of two identical fractional-order hyperchaotic system is achieved as shown in Figure 3. The experiments coincided with the theory analysis.

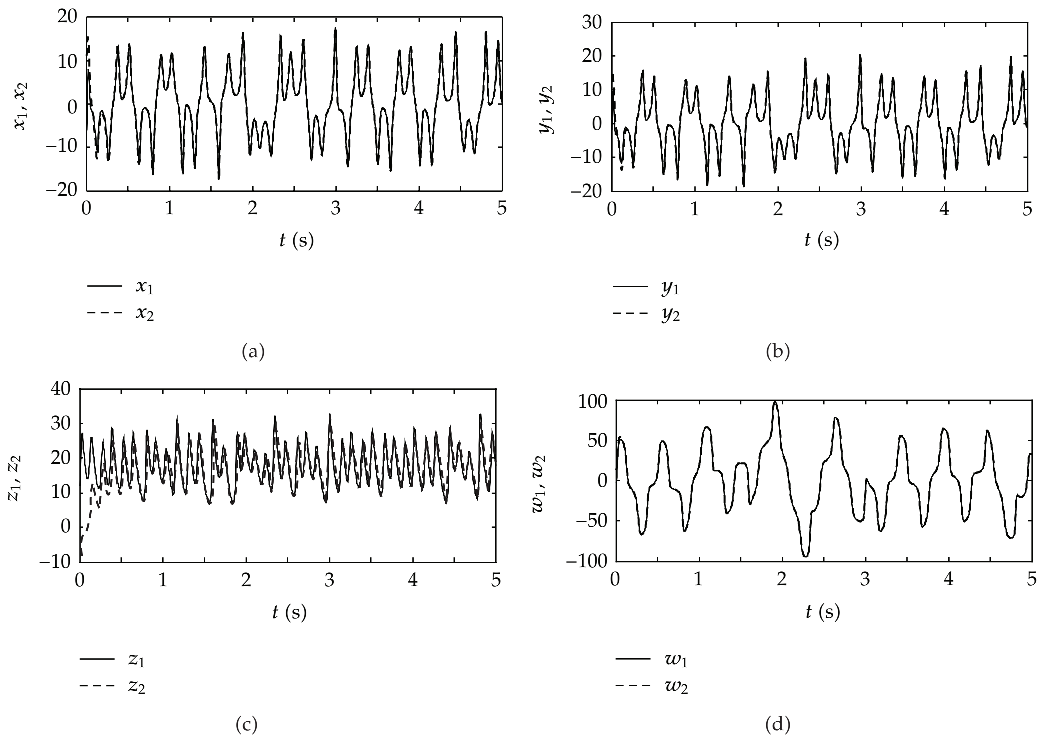

Figure 3: The synchronization of system (4.2) and system (4.1) with $\alpha = 0.96$: (a) the synchronization of $x_1 - x_2$; (b) the synchronization of $y_1 - y_2$; (c) the synchronization of $z_1 - z_2$; (d) the synchronization of $w_1 - w_2$.

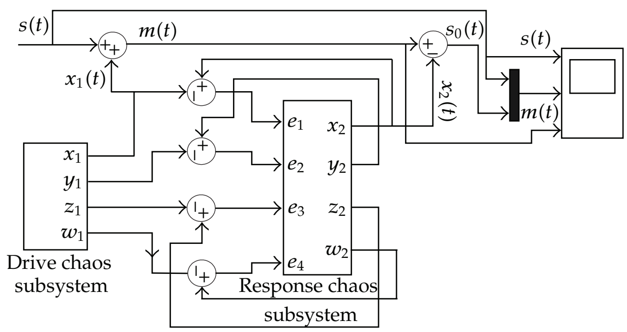

Figure 4: Chaotic masking technology for communication system.

5. Application to Secure Communication

In this section, to verify and demonstrate the effectiveness of the proposed method, we will display the numerical results for fractional-order hyperchaotic Chen systems in secure communication. Based on the theory of the communication, the block schematic of secure communication scheme with the synchronization scheme is depicted in Figure 4, where $x_1(t)$ is the chaotic state variable of drive system for the transmitter, $x_2(t)$ is the chaotic state variable of response system, $s(t)$ is the transmitted message signal with complex nonperiodic

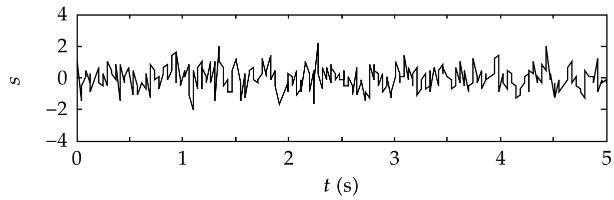

Figure 5: Complex nonperiodic code.

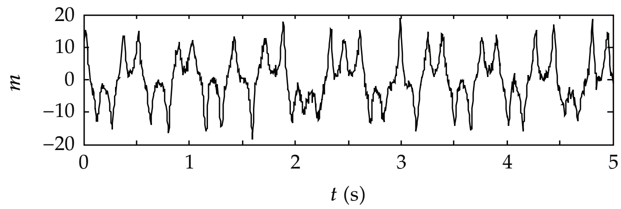

Figure 6: The mixed signals.

mode, which is added to the variable $x_1(t)$, mixed signal $m(t) = s(t) + x_1(t)$, and $s_0(t)$ is the recovered signal after synchronization between the chaotic state variable $x_2(t)$ and $x_1(t)$ at the receiver terminal end.

The complex nonperiodic information, which has the typical representative to accurately simulate the real transmission signal with generally complex and disorder, is chosen as the transmitted useful message during the numerical experiment in order to reinforce the feasibility of the scheme. The mixed signals (Figure 6) have a good masking effectiveness, are completely different from the original signals (Figure 5), which reached the purpose of safety, and are not to be cracked on the processing of signal transmit. Simulation results show that the system effectively restored the useful signals after about 1.2 s as depicted in Figure 7. Therefore, we can find that, even adopting other irregular signals as transmitted information by the synchronization and cover-up technology, the useful signals can be recovered with no distortion at the receiver end; namely, the decoded information $s_0(t)$ coincides with the transmitted signal $s(t)$.

6. Conclusions

In this paper, a new method of designing controller to synchronize a class of fractional-order hyperchaotic system is presented, and the synchronization between two identical fractional-order systems has been realized via designing only two controllers. The simulation results show that the control method is reliable. Moreover, the complex nonperiodic information signals can be recovered with no distortion when the scheme is applied to secure communication. Numerical experiments for the secure communication system indicate that the synchronization works quite well, which may has potential applications in many interdisciplinary fields. Future work on this topic should include transmission of high-frequency digital signal as well as in-depth studies on application to secure communication.

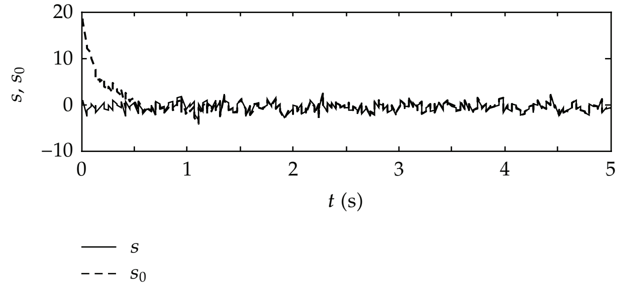

Figure 7: The original message and the recovered message.

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