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Research Article

Inverse Source Identification by the Modified Regularization Method on Poisson Equation

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This paper deals with an inverse problem for identifying an unknown source which depends only on one variable in two-dimensional Poisson equation, with the aid of an extra measurement at an internal point. Since this problem is illposed, we obtain the regularization solution by the modified regularization method. Furthermore, we obtain the Hölder-type error estimate between the regularization solution and the exact solution. The numerical results show that the proposed method is stable and the unknown source is recovered very well.

1. Introduction

Inverse source problem is an ill posed problem that has received considerable attention from many researches in a variety of fields, such as heat conduction, crack identification, electromagnetic theory, geophysical prospecting, and pollutant detection. For the heat source identification, there have been a large number of research results for different forms of heat source [1–8]. To the authors' knowledge, there were also a lot of researches on identification of the unknown source in the Poisson equation adopted numerical algorithms, such as the logarithmic potential method [9], the projective method [10], the Green's function method [11], the dual reciprocity boundary element method [12], the dual reciprocity method [13, 14], and the method of fundamental solution (MFS) [15]. But, by the regularization method, there are a few papers with strict theoretical analysis on identifying the unknown source.

In this paper, we consider the following inverse problem: to find a pair of functions $(u(x,y), f(x))$ which satisfy the Poisson equation on half unbounded domain as follows:

$$
\begin{gathered}
-u_{xx} - u_{yy} = f(x), \quad -\infty < x < +\infty,\ 0 < y < +\infty, \\
u(x,0) = 0, \qquad u(x,y)|_{y\to\infty} \text{ bounded}, \quad -\infty < x < +\infty, \\
u(x,1) = g(x), \quad -\infty < x < +\infty,
\end{gathered} \tag{1.1}
$$

where $f(x)$ is the unknown source depending only on one spatial variable and $u(x,1) = g(x)$ is the supplementary condition. In applications, input data $g(x)$ can only be measured, and there will be measured data function $g_\delta(x)$ which is merely in $L^2(\mathbb{R})$ and satisfies

$$
\|g - g_\delta\|_{L^2(\mathbb{R})} \leq \delta, \tag{1.2}
$$

where the constant $\delta > 0$ represents a noise level of input data.

The problem (1.1) is mildly ill posed, and the degree of the ill posedness is equivalent to the second-order numerical differentiation. It is impossible to solve the problem (1.1) using classical methods. The major object of this paper is to use the modified regularization method to obtain the regularization solution. Meanwhile, the Hölder-type stability estimate between the regularization solution and the exact solution is obtained. In [16], the authors ever identified the unknown source on the Poisson equation on half band domain using separation of variables. But in this paper, we identified the unknown source on the Poisson equation on half unbounded domain using the Fourier Transform.

This paper is organized as follows. Section 2 analyzes the ill posedness of the identification of the unknown source and gives some auxiliary results. Section 3 gives a regularization solution and error estimate. Section 4 gives several numerical examples including both nonsmooth and discontinuous cases for the problem (1.1). Section 5 ends this paper with a brief conclusion.

2. Some Auxiliary Results

The ill posedness can be seen by solving the problem (1.1) in the Fourier domain. Let $\widehat{f}(\xi)$ denote the Fourier transform of $f(x) \in L^2(\mathbb{R})$ which is defined by

$$
\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx. \tag{2.1}
$$

The problem (2.2) can now be formulated in frequency space as follows:

$$
\begin{gathered}
\xi^2 \widehat{u}(\xi,y) - \widehat{u}_{yy}(\xi,y) = \widehat{f}(\xi), \quad y > 0,\ \xi \in \mathbb{R}, \\
\widehat{u}(\xi,0) = 0, \quad \xi \in \mathbb{R}, \\
\widehat{u}(\xi,y)|_{y\to\infty} \text{ bounded}, \quad \xi \in \mathbb{R}, \\
\widehat{u}(\xi,1) = \widehat{g}(\xi), \quad \xi \in \mathbb{R}.
\end{gathered} \tag{2.2}
$$

The solution of the problem (2.2) is given by

$$
\widehat{f}(\xi) = \frac{\xi^2}{1 - e^{-\xi}} \widehat{g}(\xi). \tag{2.3}
$$

So,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\xi^2}{1 - e^{-\xi}} \widehat{g}(\xi) d\xi. \tag{2.4}$$

The unbounded function $\xi^2/(1 - e^{-\xi})$ in (2.3) or (2.4) can be seen as an amplification factor of $\widehat{g}(\xi)$ when $\xi \to \infty$. Therefore, when we consider our problem in $L^2(\mathbb{R})$, the exact data function $\widehat{g}(\xi)$ must decay. But, in the applications, the input data $g(x)$ can only be measured and can never be exact. Thus, if we try to obtain the unknown source $f(x)$, high-frequency components in the error are magnified and can destroy the solution. In general, for an ill posed problem, the convergence rates of the regularization solution can only be given under prior assumptions on the exact solution; we impose an a priori bound on the exact solution $f(x)$ as follows:

$$\|f(\cdot)\|_{H^p(\mathbb{R})} \le E, \quad p > 0, \tag{2.5}$$

where $E > 0$ is a constant and $\|\cdot\|_{H^p(\mathbb{R})}$ denotes the norm in the Sobolev space $H^p(\mathbb{R})$ defined by

$$\|f(\cdot)\|_{H^p(\mathbb{R})} := \left(\int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^2 \left(1 + \xi^2\right)^p d\xi \right)^{1/2}. \tag{2.6}$$

Now we give some important lemmas as follows.

Lemma 2.1. *If $x > 1$, the following inequality:*

$$\frac{1}{1 - e^{-x}} < 2 \tag{2.7}$$

holds.

Lemma 2.2. *As $0 < \mu < 1$, one obtains the following inequalities:*

$$\begin{gathered} \sup_{\xi \in \mathbb{R}} \left| \left(1 - \frac{1}{1 + \xi^2 \mu^2}\right) \left(1 + \xi^2\right)^{-p/2} \right| \le \max\left\{\mu^p, \mu^2\right\}, \\ \sup_{\xi \in \mathbb{R}} \left| \frac{\xi^2}{(1 - e^{-\xi})(1 + \mu^2 \xi^2)} \right| \le \frac{2}{\mu^2}. \end{gathered} \tag{2.8}$$

Proof. Let

$$G(\xi) := \left(1 - \frac{1}{1 + \xi^2 \mu^2}\right) \left(1 + \xi^2\right)^{-p/2}. \tag{2.9}$$

The proof of the first inequality of (2.8) can be divided into three cases.

Case 1 ($|\xi| \ge \xi_0 := 1/\mu$). We obtain

$$G(\xi) \le \left(1 + \xi^2\right)^{-p/2} \le |\xi|^{-p} \le \xi_0^{-p} = \mu^p. \tag{2.10}$$

Case 2 ($1 < |\xi| < \xi_0$). We get

$$G(\xi) = \frac{\xi^2\mu^2}{1+\xi^2\mu^2}\left(1+\xi^2\right)^{-p/2} \leq \frac{\xi^{2-p}\mu^2}{1+\xi^2\mu^2} \leq \xi^{2-p}\mu^2. \tag{2.11}$$

If $0 < p \leq 2$, the above inequality becomes

$$G(\xi) \leq \xi_0^{2-p}\mu^2 = \mu^p. \tag{2.12}$$

If $p > 2$, we get

$$G(\xi) \leq \xi^{2-p}\mu^2 = \mu^2. \tag{2.13}$$

Case 3 ($|\xi| \leq 1$). We obtain

$$G(\xi) \leq \xi^2\mu^2\left(1+\xi^2\right)^{-p/2} \leq \mu^2. \tag{2.14}$$

Combining (2.10) with (2.12), (2.13), and (2.14), we obtain the first inequality equation.

Let

$$B(\xi) := \frac{\xi^2}{(1-e^{-\xi})(1+\xi^2\mu^2)}, \qquad D(\xi) := \frac{\xi^2}{1-e^{-\xi}}. \tag{2.15}$$

The proof of the second inequality of (2.8) can also be divided into two cases.

Case 1 ($|\xi| \leq \xi_0 := 1/\mu$). We obtain

$$D(\xi) \leq D\left(\frac{1}{\mu}\right) \leq \frac{2}{\mu^2}, \quad \text{if } 0 < \mu < 1. \tag{2.16}$$

So,

$$B(\xi) \leq \frac{2}{\mu^2}. \tag{2.17}$$

Case 2 ($|\xi| > \xi_0$). We obtain

$$\begin{gathered} D(\xi) \leq 2\xi^2, \\ B(\xi) \leq \frac{2\xi^2}{1+\xi^2\mu^2} \leq \frac{2}{\mu^2}. \end{gathered} \tag{2.18}$$

Combining (2.17) with (2.18), (2.8) holds. □

3. A Modified Regularization Method and Error Estimate

We modify (1.1), where a two-order derivation of $f(x)$, is added, that is,

$$-u_{xx} - u_{yy} + \mu^2 f_{xx}(x) = f(x). \tag{3.1}$$

This is based on the modified regularization method which we learned from Eldén [17] who considered a standard inverse heat conduction problem and the idea initially came from Weber [18]. This method has been studied for solving various types of inverse problems [19–24]. We obtain a stable approximate solution of problem (1.1), that is,

$$\begin{gathered}
-u_{xx} - u_{yy} + \mu^2 f_{xx}(x) = f(x), \quad -\infty < x < +\infty,\ 0 < y < +\infty, \\
u(x,0) = 0, \qquad u(x,y)|_{y\to\infty} \text{ bounded}, \quad -\infty < x < +\infty, \\
u(x,1) = g_\delta(x), \quad -\infty < x < +\infty,
\end{gathered} \tag{3.2}$$

where the parameter μ is regarded as a regularization parameter. The problem (3.2) can be formulated in frequency space as follows:

$$\begin{gathered}
\xi^2 \widehat{u}(\xi,y) - \widehat{u}_{yy}(\xi,y) - \mu^2\xi^2 \widehat{f}(\xi) = \widehat{f}(\xi), \quad \xi \in \mathbb{R},\ 0 < y < +\infty, \\
\widehat{u}(\xi,0) = 0, \quad \xi \in \mathbb{R}, \\
\widehat{u}(\xi,y)|_{y\to\infty} \text{ bounded}, \quad \xi \in \mathbb{R}, \\
\widehat{u}(\xi,1) = \widehat{g}_\delta(\xi), \quad \xi \in \mathbb{R}.
\end{gathered} \tag{3.3}$$

The solution to this problem is given by

$$\widehat{f}(\xi) = \frac{\xi^2}{(1-e^{-\xi})(1+\xi^2\mu^2)}\widehat{g}_\delta(\xi) := \widehat{f}_{\delta,\mu}(\xi). \tag{3.4}$$

So

$$f_{\delta,\mu}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\frac{\xi^2}{(1-e^{-\xi})(1+\xi^2\mu^2)}\widehat{g}_\delta(\xi)d\xi. \tag{3.5}$$

Note that, for small μ, $\xi^2/(1+\xi^2\mu^2)$ is close to ξ^2. On the contrary, if $|\xi|$ becomes large, $|\xi^2/(1+\xi^2\mu^2)|$ is bounded. So, $f_{\delta,\mu}(x)$ is considered as an approximation of $f(x)$.

Now we will give an error estimate between the regularization solution and the exact solution by the following theorem.

Theorem 3.1. *Suppose $f(x)$ is an exact solution of* (1.1) *given by* (2.4) *and $f_{\delta,\mu}(x)$ is the regularized approximation to $f(x)$ given by* (3.5). *Let $g_\delta(x)$ be the measured data at $y = 1$ satisfying* (1.2). *Moreover, one assumes the a priori bound* (2.5) *holds. If one selects*

$$\mu = \left(\frac{\delta}{E}\right)^{1/(p+2)}, \tag{3.6}$$

then one obtains the following error estimate:

$$\|f(\cdot)-f_{\delta,\mu}(\cdot)\| \le 2\delta^{p/(p+2)}E^{2/(p+2)}\left(1+\frac{1}{2}\max\left\{1,\left(\frac{\delta}{E}\right)^{(2-p)/(p+2)}\right\}\right). \tag{3.7}$$

Proof. From the Parseval formula and (2.3), (3.4), (2.6), (2.7), (2.8), (1.2), (2.5), and (3.6), we obtain

$$\begin{aligned}
\|f(\cdot)-f_{\delta,\mu}(\cdot)\| &= \left\|\widehat{f}(\cdot)-\widehat{f}_{\delta,\mu}(\cdot)\right\| \\
&= \left\|\frac{\xi^2}{1-e^{-\xi}}\widehat{g}(\xi)-\frac{\xi^2}{(1+\xi^2\mu^2)(1-e^{-\xi})}\widehat{g}_\delta(\xi)\right\| \\
&\le \left\|\frac{\xi^2}{1-e^{-\xi}}\widehat{g}(\xi)-\frac{\xi^2}{(1+\xi^2\mu^2)(1-e^{-\xi})}\widehat{g}(\xi)\right\| \\
&\quad + \left\|\frac{\xi^2}{(1+\xi^2\mu^2)(1-e^{-\xi})}\widehat{g}(\xi)-\frac{\xi^2}{(1+\xi^2\mu^2)(1-e^{-\xi})}\widehat{g}_\delta(\xi)\right\| \\
&= \left\|\frac{\xi^2\widehat{g}(\xi)}{1-e^{-\xi}}\left(1-\frac{1}{1+\xi^2\mu^2}\right)\right\| + \left\|\frac{\xi^2}{(1+\xi^2\mu^2)(1-e^{-\xi})}\left(\widehat{g}(\xi)-\widehat{g}_\delta(\xi)\right)\right\| \\
&\le \left\|\widehat{f}(\xi)\left(1+\xi^2\right)^{p/2}\left(1+\xi^2\right)^{-p/2}\left(1-\frac{1}{1+\xi^2\mu^2}\right)\right\| \\
&\quad + \sup_{\xi\in\mathbb{R}}\left|\frac{\xi^2}{(1+\xi^2\mu^2)(1-e^{-\xi})}\right|\left\|\widehat{g}(\xi)-\widehat{g}_\delta(\xi)\right\| \\
&\le \sup_{\xi\in\mathbb{R}}\left|\left(1-\frac{1}{1+\xi^2\mu^2}\right)\left(1+\xi^2\right)^{-p/2}\right|\left\|\widehat{f}(\xi)\left(1+\xi^2\right)^{p/2}\right\| \\
&\quad + \sup_{\xi\in\mathbb{R}}\left|\frac{\xi^2}{(1+\xi^2\mu^2)(1-e^{-\xi})}\right|\left\|\widehat{g}(\xi)-\widehat{g}_\delta(\xi)\right\| \\
&\le \max\left\{\mu^p,\mu^2\right\}E+\frac{2}{\mu^2}\delta = \max\left\{\left(\frac{\delta}{E}\right)^{p/(p+2)},\left(\frac{\delta}{E}\right)^{2/(p+2)}\right\}E+2\left(\frac{\delta}{E}\right)^{-2/(p+2)}\delta \\
&= 2\delta^{p/(p+2)}E^{2/(p+2)}\left(1+\frac{1}{2}\max\left\{1,\left(\frac{\delta}{E}\right)^{(2-p)/(p+2)}\right\}\right).
\end{aligned} \tag{3.8}$$

□

Remark 3.2. *If* $0 < p \leq 2$,

$$\|f(\cdot) - f_{\delta,\mu}(\cdot)\| \leq 3\delta^{p/(p+2)} E^{2/(p+2)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0. \tag{3.9}$$

If $p > 2$,

$$\|f(\cdot) - f_{\delta,\mu}(\cdot)\| \leq 2\delta^{p/(p+2)} E^{2/(p+2)} + \delta^{p/(p+2)} E^{2/(p+2)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0. \tag{3.10}$$

Hence, $f_{\delta,\mu}(x)$ can be regarded as the approximation of the exact solution $f(x)$.

Remark 3.3. In general, the a priori bound E in (2.5) is unknown exactly in practice. But, if we choose $\mu = \delta^{1/(p+2)}$, we can also obtain

$$\|f(\cdot) - f_{\delta,\mu}(\cdot)\| \longrightarrow 0, \quad \text{as } \delta \longrightarrow 0. \tag{3.11}$$

This choice is useful in concrete computation.

4. Several Numerical Examples

In this section, we present three numerical examples intended to illustrate the usefulness of the proposed method. The numerical results are presented, which verify the validity of the theoretical results of this method.

The numerical examples were constructed in the following way. First we selected the exact solution $f(x)$ of problem (1.1) and obtained the exact data function $g(x)$ using (2.3) or (2.4). Then, we added a normally distributed perturbation to each data function giving vectors g_δ. Finally, we obtained the regularization solutions using (3.4) or (3.5).

In the following, we first give an example which has the exact expression of the solutions $(u(x,y), f(x))$.

Example 4.1. It is easy to see that the function

$$u(x,y) = \left(1 - e^{-y}\right) \sin x \tag{4.1}$$

and the function

$$f(x) = \sin x \tag{4.2}$$

are satisfied with the problem (1.1) with exact data

$$g(x) = \left(1 - e^{-1}\right) \sin x. \tag{4.3}$$

Suppose that the sequence $\{g_k\}_{k=0}^{n}$ represents samples from the function $g(x)$ on an equidistant grid and n is even. Then we add a random uniformly perturbation to each data, which forms the vector g_δ, that is,

$$g_\delta = g + \varepsilon \text{ randn}(\text{size}(g)), \tag{4.4}$$

where

$$g = g(g(x_1), \ldots, g(x_n))^T, \qquad x_i = (i-1)\Delta x, \quad \Delta x = \frac{1}{n-1}, \ i = 1, 2, \ldots, n. \tag{4.5}$$

The function "randn(·)" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$. "Randn(size(g))" returns an array of random entries that is of the same size as g. The total noise level δ can be measured in the sense of root mean square error (RMSE) according to

$$\delta = \|g_\delta - g\|_{l^2} = \left(\frac{1}{n}\sum_{i=1}^{n}(g_i - g_{i,\delta})^2\right)^{1/2}. \tag{4.6}$$

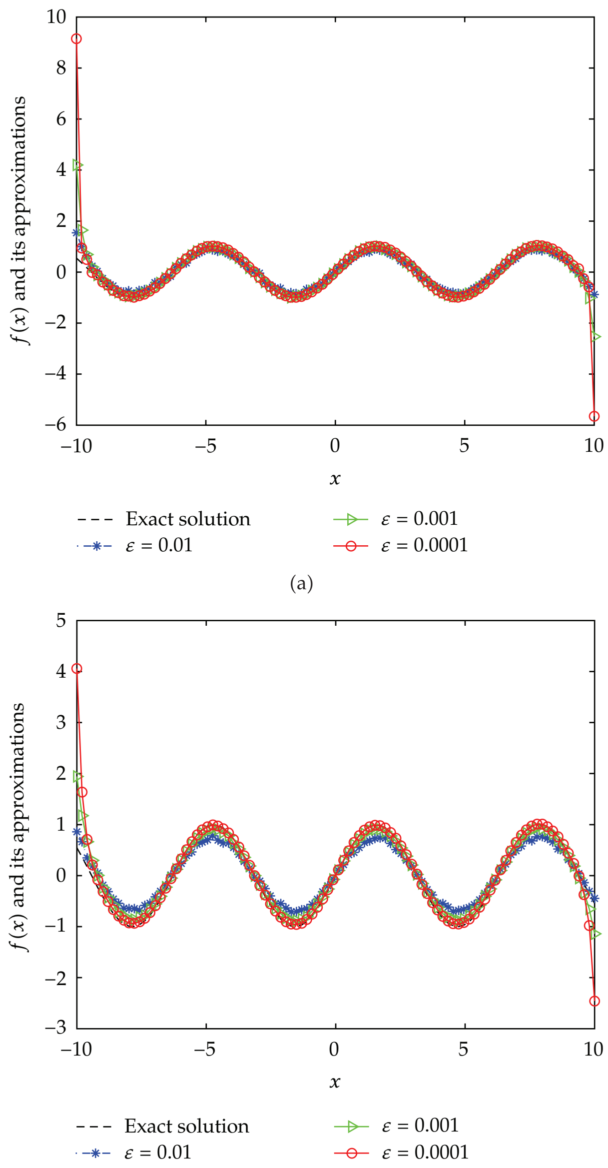
Moreover, we need to make the vector g_δ periodical [25], and then we take the discrete Fourier transform for the vector g_δ. The approximation of the regularization solution is computed by using FFT algorithm [25], and the range of variable x in the numerical experiment is $[-10, 10]$.

Example 4.2. Consider a piecewise smooth source:

$$f(x) = \begin{cases} 0, & -10 \le x \le -5, \\ x + 5, & -5 < x \le 0, \\ -x + 5, & 0 < x \le 5, \\ 0, & 5 < x \le 10. \end{cases} \tag{4.7}$$

Example 4.3. Consider the following discontinuous case:

$$f(x) = \begin{cases} -1, & -10 \le x \le -5, \\ 1, & -5 < x \le 0, \\ -1, & 0 < x \le 5, \\ 1, & 5 < x \le 10. \end{cases} \tag{4.8}$$



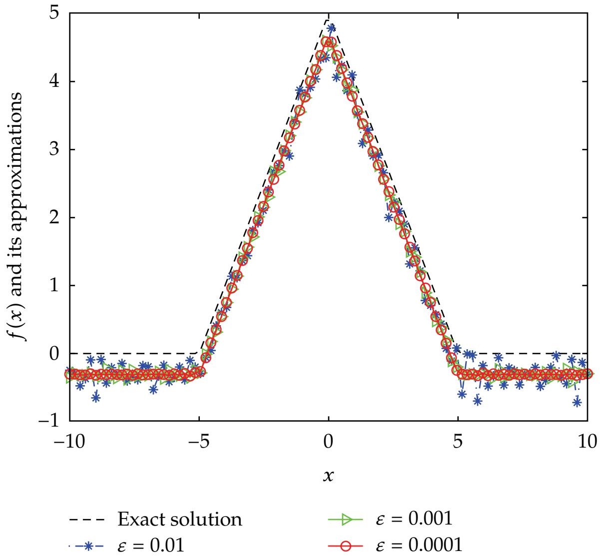
(a)

(b)

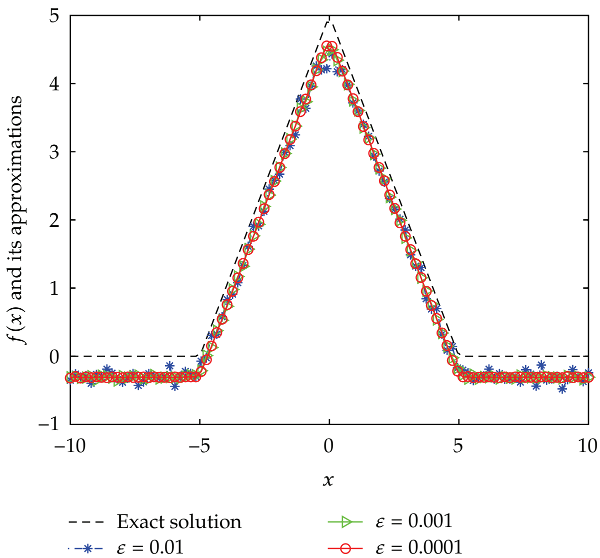
Figure 1: Comparison between the exact solution and its computed approximations with various levels of noise for Example 4.1: (a) $p = 1$, (b) $p = 2$.

From Figures 1–3, we can see that the smaller the ε, the better the computed approximation $f_{\delta,\mu}(x)$.

In Examples 4.2 and 4.3, since the direct problem with the source $f(x)$ does not have an analytical solution, the data $g(x)$ is obtained by solving the direct problem. From Figures 2 and 3, we can see that the numerical solutions are less ideal than that of Example 4.1. It is not difficult to see that the well-known Gibbs phenomenon and the recovered data near the nonsmooth and discontinuities points are not accurate. Taking into consideration the ill posedness of the problem, the results presented in Figures 2 and 3 are reasonable.



(a)

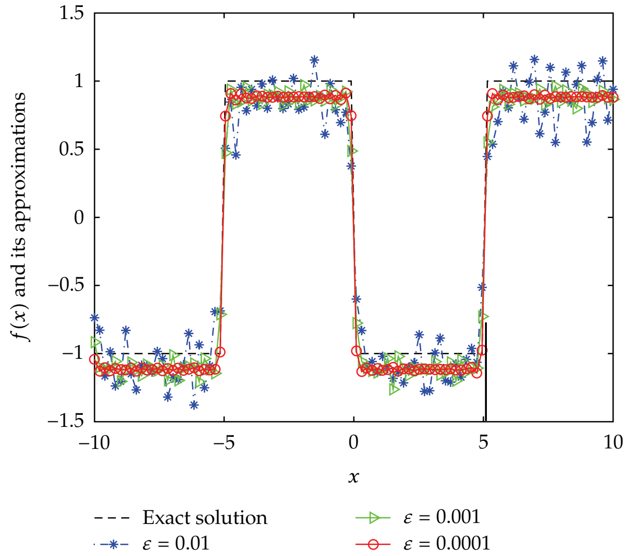


(b)

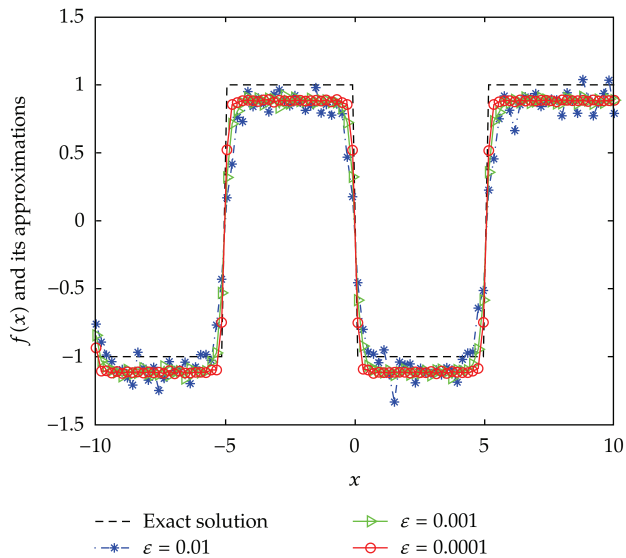
Figure 2: Comparison between the exact solution and its computed approximations with various levels of noise for Example 4.2: (a) $p = 1$, (b) $p = 2$.

5. Conclusions

In this paper, we consider the identification of an unknown source term depending only on one variable in two-dimensional Poisson equation. This problem is ill posed, that is, the solution (if it exists) does not depend on the input data. We obtain the regularization solution and a Hölder-type error estimate. Through the comparison between [16] and this paper, as the degree inverse problem of the ill posedness of identifying the unknown source dependent only on one variable in two-dimensional Poisson equation is equivalent to the



(a)



(b)

Figure 3: . Comparison between the exact solution and its computed approximations with various levels of noise for Example 4.3: (a) $p = 1$, (b) $p = 2$.

second-order numerical differentiation, we obtain the same error estimate $2\delta^{p/(p+2)}E^{2/(p+2)}(1+1/2\max\{1,(\delta/E)^{(2-p)/(p+2)}\})$. According to [26], this Hölder-type error estimate is order optimal.

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References

[1] J. R. Cannon and P. DuChateau, "Structural identification of an unknown source term in a heat equation," *Inverse Problems*, vol. 14, no. 3, pp. 535–551, 1998.

[2] T. Johansson and D. Lesnic, "Determination of a spacewise dependent heat source," *Journal of Computational and Applied Mathematics*, vol. 209, no. 1, pp. 66–80, 2007.

[3] G. A. Kriegsmann and W. E. Olmstead, "Source identification for the heat equation," *Applied Mathematics Letters*, vol. 1, no. 3, pp. 241–245, 1988.

[4] L. Yang, M. Dehghan, J.-N. Yu, and G.-W. Luo, "Inverse problem of time-dependent heat sources numerical reconstruction," *Mathematics and Computers in Simulation*, vol. 81, no. 8, pp. 1656–1672, 2011.

[5] N. Magnoli and G. A. Viano, "The source identification problem in electromagnetic theory," *Journal of Mathematical Physics*, vol. 38, no. 5, pp. 2366–2388, 1997.

[6] Z. Yi and D. A. Murio, "Identification of source terms in 2-D IHCP," *Computers & Mathematics with Applications*, vol. 47, no. 10-11, pp. 1517–1533, 2004.

[7] M. N. Ahmadabadi, M. Arab, and F. M. Maalek Ghaini, "The method of fundamental solutions for the inverse space-dependent heat source problem," *Engineering Analysis with Boundary Elements*, vol. 33, no. 10, pp. 1231–1235, 2009.

[8] F. Yang and C.-L. Fu, "The method of simplified Tikhonov regularization for dealing with the inverse time-dependent heat source problem," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1228–1236, 2010.

[9] T. Ohe and K. Ohnaka, "A precise estimation method for locations in an inverse logarithmic potential problem for point mass models," *Applied Mathematical Modelling*, vol. 18, no. 8, pp. 446–452, 1994.

[10] T. Nara and S. Ando, "A projective method for an inverse source problem of the Poisson equation," *Inverse Problems*, vol. 19, no. 2, pp. 355–369, 2003.

[11] Y. C. Hon, M. Li, and Y. A. Melnikov, "Inverse source identification by Green's function," *Engineering Analysis with Boundary Elements*, vol. 34, no. 4, pp. 352–358, 2010.

[12] A. Farcas, L. Elliott, D. B. Ingham, D. Lesnic, and N. S. Mera, "A dual reciprocity boundary element method for the regularized numerical solution of the inverse source problem associated to the poisson equation," *Inverse Problems in Engineering*, vol. 11, no. 2, pp. 123–139, 2003.

[13] Y. Kagawa, Y. Sun, and O. Matsumoto, "Inverse solution of poisson equation using DRM boundary element model-Identification of space charge distribution," *Inverse Problems in Engineering*, vol. 1, no. 2, pp. 247–265, 1995.

[14] Y. Sun and Y. Kagawa, "Identification of electric charge distribution using dual reciprocity boundary element models," *IEEE Transactions on Magnetics*, vol. 33, no. 2, pp. 1970–1973, 1997.

[15] B. Jin and L. Marin, "The method of fundamental solutions for inverse source problems associated with the steady-state heat conduction," *International Journal for Numerical Methods in Engineering*, vol. 69, no. 8, pp. 1570–1589, 2007.

[16] F. Yang and C. L. Fu, "The modified regularization method for identifying the unknown source on Poisson equation," *Applied Mathematical Modelling*, vol. 36, no. 2, pp. 756–763, 2012.

[17] L. Eldén, "Approximations for a Cauchy problem for the heat equation," *Inverse Problems*, vol. 3, no. 2, pp. 263–273, 1987.

[18] C. F. Weber, "Analysis and solution of the ill-posed inverse heat conduction problem," *International Journal of Heat and Mass Transfer*, vol. 24, no. 11, pp. 1783–1792, 1981.

[19] Z. Qian, C.-L. Fu, and R. Shi, "A modified method for a backward heat conduction problem," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 564–573, 2007.

[20] Z. Qian, C.-L. Fu, and X.-L. Feng, "A modified method for high order numerical derivatives," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1191–1200, 2006.

[21] Z. Qian, C.-L. Fu, and X.-T. Xiong, "A modified method for determining the surface heat flux of IHCP," *Inverse Problems in Science and Engineering*, vol. 15, no. 3, pp. 249–265, 2007.

[22] Z. Qian, C.-L. Fu, and X.-T. Xiong, "A modified method for a non-standard inverse heat conduction problem," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 453–468, 2006.

[23] Z. Qian, C.-L. Fu, and X.-T. Xiong, "Fourth-order modified method for the Cauchy problem for the Laplace equation," *Journal of Computational and Applied Mathematics*, vol. 192, no. 2, pp. 205–218, 2006.

[24] X.-T. Xiong, C.-L. Fu, and Z. Qian, "Two numerical methods for solving a backward heat conduction problem," *Applied Mathematics and Computation*, vol. 179, no. 1, pp. 370–377, 2006.

[25] L. Eldén, F. Berntsson, and T. Regińska, "Wavelet and Fourier methods for solving the sideways heat equation," *SIAM Journal on Scientific Computing*, vol. 21, no. 6, pp. 2187–2205, 2000.

[26] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic, Boston, Mass, USA, 1996.