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Research Article

A Granular Reduction Algorithm Based on Covering Rough Sets

Tian Yang,[1] Zhaowen Li,[2] and Xiaoqing Yang[3]

[1] *College of Science, Central South University of Forestry and Technology, Changsha 410004, China*
[2] *College of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning 530006, China*
[3] *School of Economics and Management, Changsha University of Science and Technology, Changsha 410004, China*

Correspondence should be addressed to Zhaowen Li, math_yangtian@126.com

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The granular reduction is to delete dispensable elements from a covering. It is an efficient method to reduce granular structures and get rid of the redundant information from information systems. In this paper, we develop an algorithm based on discernability matrixes to compute all the granular reducts of covering rough sets. Moreover, a discernibility matrix is simplified to the minimal format. In addition, a heuristic algorithm is proposed as well such that a granular reduct is generated rapidly.

1. Introduction

With the development of technology, the gross of information increases in a surprising way. It is a great challenge to extract valuable knowledge from the massive information. Rough set theory was raised by Pawlak [1, 2] to deal with uncertainty and vagueness, and it has been applied to the information processing in various areas [3–8].

One of the most important topics in rough set theory is to design reduction algorithms. The reduction of Pawlak's rough sets is to reduce dispensable elements from a family of equivalence relations which induce the equivalence classes, or a partition.

Covering generalized rough set [9–19] and binary relation generalized rough set [20–26] are two main extensions of Pawlak's rough set. The reduction theory of covering rough sets [10, 11, 15, 23, 27, 28] plays an important role in practice. A partition is no longer a partition if any of its elements is deleted, while a covering may still be a covering with invariant set approximations after dropping some elements. Therefore, there are two types of reduction on covering rough sets. One is to reduce redundant coverings from a family of coverings, referred to as the *attribute reduction.* The other is to reduce redundant elements from a covering, noted as the *granular reduction.* It is to find the minimal subsets of a covering which generate the same set approximations with the original covering. Employed to reduce granular structures and databases as well as interactive with the attribute reduction, we think the granular reduction should be ignored by no means. In this paper, we devote to investigate granular reduction of covering rough sets.

In order to compute all attribute reducts for Pawlak's rough sets, discernibility matrix is initially presented [29]. Tsang et al. [15] develop an algorithm of discernibility matrices to compute attribute reducts for one type of covering rough sets. Zhu and Wang [17] and Zhu [18] build one type of granular reduction for two covering rough set models initially. In addition, Yang et al. systematically examine the granular reduction in [30] and the relationship between reducts and topology in [31]. Unfortunately, no effective algorithm for granular reduction has hitherto been proposed.

In this paper, we bridge the gap by constructing an algorithm based on discernibility matrixes which is applicable to all granular reducts of covering rough sets. This algorithm can reduce granular structures and get rid of the redundant information from information systems. Then a discernibility matrix is simplified to the minimal format. Meanwhile, based on a simplification of discernibility matrix, a heuristic algorithm is proposed as well.

The remainder of this paper proceeds as follows. Section 2 reviews the relevant background knowledge about the granular reduction. Section 3 constructs the algorithm based on discernibility matrix. Section 4 simplifies the discernibility matrix and proposes a heuristic algorithm. Section 5 concludes the study.

2. Background

Our aim in this section is to give a glimpse of rough set theory.

Let U be a finite and nonempty set, and let R be an equivalence relation on U. R generates a partition $U/R = \{[x]_R \mid x \in X\}$ on U, where $[x]_R$ is an equivalence class of x generated by the equivalence relation R. We call it elementary sets of R in rough set theory. For any set X, we describe X by the elementary sets of R, and the two sets

$$R_* = \cup\{[x]_R \mid [x]_R \subseteq X\}, \qquad R^* = \cup\{[x]_R \mid [x]_R \cap X \neq \varnothing\} \tag{2.1}$$

are called the lower and upper approximations of X, respectively. If $R_*(X) = R^*(X), X$ is an R-exact set. Otherwise, it is an R-rough set.

Let $\mathbb{R}$ be a family of equivalence relations, and let $A \in \mathbb{R}$, denoted as $\text{IND}(\mathbb{R}) = \cap\{R : R \in \mathbb{R}\}$. A is dispensable in $\mathbb{R}$ if and only if $\text{IND}(\mathbb{R}) = \text{IND}(\mathbb{R} - A)$. Otherwise, A is indispensable in $\mathbb{R}$. The family $\mathbb{R}$ is independent if every $A \in \mathbb{R}$ is indispensable in $\mathbb{R}$. Otherwise, $\mathbb{R}$ is dependent. $\mathbb{Q} \in \mathbb{P}$ is a reduct of $\mathbb{P}$ if $\mathbb{Q}$ is independent and $\text{IND}(\mathbb{Q}) = \text{IND}(\mathbb{P})$. The sets of all indispensable relations in $\mathbb{P}$ are called the core of $\mathbb{P}$, denoted as $\text{CORE}(\mathbb{P})$. Evidently, $\text{CORE}(\mathbb{P}) = \cap\text{RED}(\mathbb{P})$, where $\text{RED}(\mathbb{P})$ is the family of all reducts of $\mathbb{P}$. The discernibility matrix method is proposed to compute all reducts of information systems and relative reducts of decision systems [29].

$\mathcal{C}$ is called a covering of U, where U is a nonempty domain of discourse, and $\mathcal{C}$ is a family of nonempty subsets of U and $\cup\mathcal{C} = U$.

It is clear that a partition of U is certainly a covering of U, so the concept of a covering is an extension of the concept of a partition.

Definition 2.1 (minimal description [9]). Let $\mathcal{C}$ be a covering of U,

$$Md_{\mathcal{C}}(x) = \{K \in \mathcal{C} \mid x \in K \wedge (\forall S \in \mathcal{C} \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\} \tag{2.2}$$

is called the minimal description of x. When there is no confusion, we omit the $\mathcal{C}$ from the subscript.

Definition 2.2 (neighborhood [9, 19]). Let $\mathcal{C}$ be a covering of U, and $N_{\mathcal{C}}(x) = \cap\{C \in \mathcal{C} \mid x \in C\}$ is called the neighborhood of x. Generally, we omit the subscript $\mathcal{C}$ when there is no confusion.

Minimal description and neighborhood are regarded as related information granules to describe x, which are used as approximation elements in rough sets (as shown in Definition 2.3). It shows that $N(x) = \cap\{C \in \mathcal{C} \mid x \in C\} = \cap Md(x)$. The neighborhood of x can be seen as the minimum description of x, and it is the most precise description (more details are referred to [9]).

Definition 2.3 (covering lower and upper approximation operations [19]). Let $\mathcal{C}$ be a covering of U. The operations $CL_{\mathcal{C}} : P(U) \rightarrow P(U)$ and $CL'_{\mathcal{C}} : P(U) \rightarrow P(U)$ are defined as follows: for all $X \in P(U)$,

$$\begin{gathered} CL_{\mathcal{C}}(X) = \cup\{K \in \mathcal{C} \mid K \subseteq X\} = \cup\{K \mid \exists x, \text{ s.t. } (K \in Md(x)) \wedge (K \subseteq X)\}, \\ CL'_{\mathcal{C}}(X) = \{x \mid N(x) \subseteq X\} = \cup\{N(x) \mid N(x) \subseteq X\}. \end{gathered} \tag{2.3}$$

We call $CL_{\mathcal{C}}$ the first, the second, the third, or the fourth covering lower approximation operations and $CL'_{\mathcal{C}}$ the fifth, the sixth, or the seventh covering lower approximation operations, with respect to the covering $\mathcal{C}$.

The operations FH, SH, TH, RH, IH, XH, and $VH : P(U) \rightarrow P(U)$ are defined as follows: for all $X \in P(U)$,

$$\begin{gathered} FH_{\mathcal{C}}(X) = CL(X) \cup (\cup\{Md(x) \mid x \in X - CL(X)\}), \\ SH_{\mathcal{C}}(X) = \cup\{K \mid K \in \mathcal{C}, K \cap X \neq \varnothing\}, \\ TH_{\mathcal{C}}(X) = \cup\{Md(x) \mid x \in X\}, \\ RH_{\mathcal{C}}(X) = CL(X) \cup (\cup\{K \mid K \cap (X - CL(X)) \neq \varnothing\}), \\ IH_{\mathcal{C}}(X) = CL(X) \cup (\cup\{N(x) \mid x \in X - CL(X)\} = \cup\{N(x) \mid x \in X\}), \\ XH_{\mathcal{C}}(X) = \{x \mid N(x) \cap X \neq \varnothing\}, \\ VH_{\mathcal{C}}(X) = \cup\{N(x) \mid N(x) \cap X \neq \varnothing\}. \end{gathered} \tag{2.4}$$

$FH_{\mathcal{C}}$, $SH_{\mathcal{C}}$, $TH_{\mathcal{C}}$, $RH_{\mathcal{C}}$, $IH_{\mathcal{C}}$, $XH_{\mathcal{C}}$, and $VH_{\mathcal{C}}$ are called the first, the second, the third, the fourth, the fifth, the sixth, and the seventh covering upper approximation operations with respect to $\mathcal{C}$, respectively. We leave out $\mathcal{C}$ at the subscript when there is no confusion.

As shown in [32], every approximation operation in Definition 2.3 may be applied in certain circumstance. We choose the suitable approximation operation according to the specific situation. So it is important to design the granular reduction algorithms for all of these models.

More precise approximation spaces are proposed in [30]. As a further result, a reasonable granular reduction of coverings is also introduced. Let $\mathcal{M}_{\mathcal{C}} = \cup\{Md(x) \mid x \in U\}$, $\mathcal{N}_{\mathcal{C}} = \{N(x) \mid x \in U\}$. $\langle U, \mathcal{M}_{\mathcal{C}}\rangle$ is the approximation space of the first and the third types of covering rough sets, $\langle U, \mathcal{C}\rangle$ is the approximation space of the second and the fourth types of covering rough sets, and $\langle U, \mathcal{N}_{\mathcal{C}}\rangle$ is the approximation space of the fifth, the sixth, and the seventh types of covering rough sets (referred to [30] for the details). In this paper, we design the algorithm of granular reduction for the fifth, the sixth, and the seventh type of covering rough sets.

Let $\mathcal{C}$ be a covering of U, denoting a covering approximation space. $\mathcal{M}_{\mathcal{C}}$ denotes an $\mathcal{M}$-approximation space. $\mathcal{N}_{\mathcal{C}}$ represents an $\mathcal{N}$-approximation space. We omit $\mathcal{C}$ at the subscript when there is no confusion (referred to [30] for the details).

3. Discernibility Matrixes Based on Covering Granular Reduction

In the original Pawlak's rough sets, a family of equivalence classes induced by equivalence relations is a partition. Once any of its elements are deleted, a partition is no longer a partition. The granular reduction refers to the method of reducing granular structures and to get rid of redundant information in databases. Therefore, granular reduction is not applicable to the original Pawlak's rough sets. However, as one of the most extensions of Pawlak's rough sets, a covering is still working even subject to the omission of its elements, as long as the set approximations are invariant. The purpose of covering granular reduction is to find minimal subsets keeping the same set approximations. It is meaningful and necessary to develop the algorithm for covering granular reduction.

The quintuple $(U, \mathcal{C}, CL, CH)$ is called a covering rough set system (CRSS), where $\mathcal{C}$ is a covering of U, CL and CH are the lower and upper approximation operations with respect to the covering $\mathcal{C}$, and $\langle U, \mathcal{A}_{\mathcal{C}}\rangle$ is the approximation space. According to the categories of covering approximation operations in [30], there are two kinds of situations as follows.

(1) If $\mathcal{A}_{\mathcal{C}} = \mathcal{C}$ or $\mathcal{A}_{\mathcal{C}} = \mathcal{M}_{\mathcal{C}}$, then $\mathcal{A}_{\mathcal{C}} \subseteq \mathcal{C}$: thus; $\mathcal{A}_{\mathcal{C}}$ is the unique granular reduct of $\mathcal{C}$. There is no need to develop an algorithm to compute granular reducts for the first, the second, the third, and the fourth type of the covering rough sets.

(2) If $\mathcal{A}_{\mathcal{C}} = \mathcal{N}_{\mathcal{C}}$, generally, $\mathcal{A}_{\mathcal{C}}$ is not a subset of $\mathcal{C}$. Consequently, an algorithm is needed to compute all granular reducts of $\mathcal{C}$ for the fifth, the sixth, and the seventh type of covering rough set models.

Next we examine the algorithm of granular reduction for the fifth, the sixth, and the seventh type of covering rough sets. Let $\mathcal{C}$ be a covering of U, since $\mathcal{N}_{\mathcal{C}} = \{N(x) \mid x \in U\}$, and $\mathcal{N}_{\mathcal{C}}$ is the collection of all approximation elements of the fifth, the sixth, or the seventh type of lower/upper approximation operations. $\mathcal{N}_{\mathcal{C}}$ is called the $\mathcal{N}$-approximation space of $\mathcal{C}$. Given a pair of approximation operations, the set approximations of any $X \subseteq U$ are determined by the $\mathcal{N}$-approximation spaces. Thus, for the fifth, the sixth, and the seventh type of covering rough set models, the purpose of granular reduction is to find the minimal subsets $\mathcal{C}'$ of $\mathcal{C}$ such that $\mathcal{N}_{\mathcal{C}} = \mathcal{N}_{\mathcal{C}'}$. The granular reducts based on the $\mathcal{N}$-approximation spaces are called the $\mathcal{N}$-reducts. $N\text{red}(\mathcal{C})$ is the set of all $\mathcal{N}$-reducts of $\mathcal{C}$, and $NI(\mathcal{C})$ is the set of all $\mathcal{N}$-irreducible elements of $\mathcal{C}$ (referred to [30] for the details).

In Pawlak's rough set theory, for every pair of $x, y \in U$, if y belongs to the equivalence class containing x, we say that x and y are indiscernible. Otherwise, they are discernible. Let $\mathbb{R} = \{R_1, R_2, \ldots, R_n\}$ be a family of equivalence relation on U, $R_i \in \mathbb{R}$. R_i is indispensable in $\mathbb{R}$ if and only if there is a pair of $x, y \in U$ such that the relation between x and y is altered after deleting R_i from $\mathbb{R}$. The attribute reduction of Pawlak's rough sets is to find minimal subsets of $\mathbb{R}$ which keep the relations invariant for any $x,\ y \in U$. Based on this statement, the method of discernibility matrix to compute all reducts of Pawlak's rough sets was proposed in [29]. In covering rough sets, however, the discernibility relation between $x,\ y \in U$ is different from that in Pawlak's rough sets.

Let $\mathcal{C}$ be a covering on U, $(x, y) \in U \times U$. Then we call (x, y) indiscernible if $y \in N(x)$, that is, $N(y) \subseteq N(x)$. Otherwise, (x, y) is discernible. When $\mathcal{C}$ is a partition, the new discernibility relation coincides with that in Pawlak's. It is an extension of Pawlak's discernibility relation. In Pawlak's rough sets, (x, y) is indiscernible if and only if (y, x) is indiscernible. However, for a general covering, if $N(y) \subseteq N(x)$ and $N(y) \neq N(x)$, that is, $y \in N(x)$ and $x \notin N(y)$, (y, x) is discernible while (x, y) is indiscernible. Thereafter, we call these relations the relations of (x, y) with respect to $\mathcal{C}$. The following theorem characterizes these relations.

Proposition 3.1. *Let $\mathcal{C} = \{C_i \mid i = 1, 2, 3, \ldots, n\}$ be a covering on U, and let $\mathcal{C}_x = \{C_i \in \mathcal{C} \mid x \in C_i\}$.*

(1) $y \in N(x)$ *if and only if* $\mathcal{C}_x \subseteq \mathcal{C}_y$.

(2) $y \notin N(x)$ *if and only if there is* $C_i \in \mathcal{C}$ *such that* $x \in C_i$ *and* $y \notin C_i$.

Proof. (1) $y \in N(x) = \cap\mathcal{C}_x \Leftrightarrow$ for any $C_i \in \mathcal{C}_x$, $y \in C_i \Leftrightarrow$ for any $C_i \in \mathcal{C}_x$, $C_i \in \mathcal{C}_y \Leftrightarrow \mathcal{C}_x \subseteq \mathcal{C}_y$.

(2) It is evident from (1). □

Theorem 3.2. *Let $\mathcal{C}$ be a covering on U, $C_i \in \mathcal{C}$. Then $\mathcal{N}_{\mathcal{C}} \neq \mathcal{N}_{\mathcal{C}-\{C_i\}}$ if and only if there is $(x, y) \in U \times U$ whose discernibility relation with respect to $\mathcal{C}$ is changed after deleting C_i from $\mathcal{C}$.*

Proof. Suppose that $\mathcal{N}_{\mathcal{C}} \neq \mathcal{N}_{\mathcal{C}-\{C_i\}}$, then there is at least one element $x \in U$ such that $N_{\mathcal{C}}(x) \neq N_{\mathcal{C}-\{C_i\}}(x)$, that is, $N_{\mathcal{C}}(x) \subset N_{\mathcal{C}-\{C_i\}}(x)$. Since $N_{\mathcal{C}-\{C_i\}}(x) - N_{\mathcal{C}}(x) \neq \varnothing$, suppose that $y \in N_{\mathcal{C}-\{C_i\}}(x) - N_{\mathcal{C}}(x)$, then $y \in N_{\mathcal{C}-\{C_i\}}(x)$ and $y \notin N_{\mathcal{C}}(x)$. Namely, (x, y) is discernible with respect to $\mathcal{C}$, while (x, y) is indiscernible with respect to $\mathcal{C} - \{C_i\}$.

Suppose that there is $(x, y) \in U \times U$ whose discernibility relation with respect to $\mathcal{C}$ is changed after deleting C_i from $\mathcal{C}$. Put differently, (x, y) is discernible with respect to $\mathcal{C}$, while (x, y) is indiscernible with respect to $\mathcal{C} - \{C_i\}$. Then we have $y \in N_{\mathcal{C}-\{C_i\}}(x)$ and $y \notin N_{\mathcal{C}}(x)$, so $y \in N_{\mathcal{C}-\{C_i\}}(x) - N_{\mathcal{C}}(x)$. Thus, $N_{\mathcal{C}}(x) \neq N_{\mathcal{C}-\{C_i\}}(x)$. It implies $\mathcal{N}_{\mathcal{C}} \neq \mathcal{N}_{\mathcal{C}-\{C_i\}}$. □

The purpose of granular reducts of a covering $\mathcal{C}$ is to find the minimal subsets of $\mathcal{C}$ which keep the same classification ability as $\mathcal{C}$ or, put differently, keep $\mathcal{N}_{\mathcal{C}}$ invariant. In Theorem 3.2, $\mathcal{N}_{\mathcal{C}}$ is kept unchanged to make the discernibility relations of any $(x, y) \in U \times U$ invariant. Based on this statement, we are able to compute granular reducts with discernibility matrix.

Definition 3.3. Let $U = \{x_1, x_2, \ldots, x_n\}$, $\mathcal{C}$ be a covering on U. $M(U, \mathcal{C})$ is an $n \times n$ matrix $(c_{ij})_{n\times n}$ called a discernibility matrix of $(U, \mathcal{C})$, where

(1) $c_{ij} = \varnothing$, $x_j \in N(x_i)$,

(2) $c_{ij} = \{C \in \mathcal{C} \mid x_i \in C,\ x_j \notin C\}$, $x_j \notin N(x_i)$.

This definition of discernibility matrix is more concise than the one in [11, 15] due to the reasonable statement of the discernibility relations. Likewise, we restate the characterizations of $\mathcal{N}$-reduction.

Proposition 3.4. *Consider that* $NI(\mathcal{C}) = \{C \mid c_{ij} = \{C\}$ *for some* $c_{ij} \in M(U, \mathcal{C})\}$.

Proof. For any $C \in NI(\mathcal{C})$, $\mathcal{N}_{\mathcal{C}} \neq \mathcal{N}_{\mathcal{C}-\{C\}}$, then there is $(x_i, x_j) \in U \times U$ such that $x_j \in N_{\mathcal{C}-\{C\}}(x_i)$ and $x_j \notin N_{\mathcal{C}}(x_i)$. It implies that $x_i \in C$ and $x_j \notin C$. Moreover, for any $C' \in \mathcal{C} - \{C\}$, since $x_j \in N_{\mathcal{C}-\{C\}}(x_i)$, we have $x_i \in C'$ if $x_i \in C'$. Thus, $c_{ij} = \{C\}$.

If $c_{ij} = \{C\}$ for some $c_{ij} \in M(U, \mathcal{C})$, then $x_i \in C$ and $x_j \notin C$. And for any $C' \in \mathcal{C} - \{C\}$, if $x_i \in C'$, then $x_i \in C'$, that is, $x_j \in N_{\mathcal{C}-\{C\}}(x_i)$ and $x_j \notin N_{\mathcal{C}}(x_i)$, then $N_{\mathcal{C}-\{C\}}(x_i) \neq N_{\mathcal{C}}(x_i)$. Namely, $\mathcal{N}_{\mathcal{C}} \neq \mathcal{N}_{\mathcal{C}-\{C\}}$, which implies $C \in NI(\mathcal{C})$. □

Proposition 3.5. *Suppose that* $\mathcal{C}' \subseteq \mathcal{C}$, *then* $\mathcal{N}_{\mathcal{C}} = \mathcal{N}_{\mathcal{C}'}$ *if and only if* $\mathcal{C}' \cap c_{ij} \neq \varnothing$ *for every* $c_{ij} \neq \varnothing$.

Proof. $\mathcal{N}_{\mathcal{C}} = \mathcal{N}_{\mathcal{C}'}$

$\Leftrightarrow$ for any $(x_i, x_j) \in U \times U$, $x_j \notin N_{\mathcal{C}}(x_i)$ if and only if $x_j \notin N_{\mathcal{C}'}(x_i)$,

$\Leftrightarrow$ for any $(x_i, x_j) \in U \times U$, there is $C \in \mathcal{C}$ such that $x_i \in C$ and $x_j \notin C$ if and only if there is $C' \in \mathcal{C}'$ such that $x_i \in C'$ and $x_j \notin C'$,

$\Leftrightarrow$ for any $c_{ij} \neq \varnothing$, $\mathcal{C}' \neq \varnothing$.

□

Proposition 3.6. *Suppose that* $\mathcal{C}' \subseteq \mathcal{C}$, *then* $\mathcal{C}' \in N\mathrm{red}(\mathcal{C})$ *if and only if* $\mathcal{C}'$ *is a minimal set satisfying* $\mathcal{C}' \cap c_{ij} \neq \varnothing$ *for every* $c_{ij} \neq \varnothing$.

Definition 3.7. Let $U = \{x_1, x_2, \ldots, x_n\}$, let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be a covering of U, and let $M(U, \mathcal{C}) = (c_{ij})_{n\times n}$ be the discernibility matrix of $(U, \mathcal{C})$. A discernibility function $f(U, \mathcal{C})$ is a Boolean function of m Boolean variables, $\overline{C_1}, \overline{C_2}, \ldots, \overline{C_m}$, corresponding to the covering elements $C_1, C_2, \ldots, C_m$, respectively, defined as $f(U, \mathcal{C})(\overline{C_1}, \overline{C_2}, \ldots, \overline{C_m}) = \wedge\{\vee(c_{ij}) \mid c_{ij} \in M(U, \mathcal{C}), c_{ij} \neq \varnothing\}$.

Theorem 3.8. *Let* $\mathcal{C}$ *be a family of covering on* U, *let* $f(U, \mathcal{C})$ *be the discernibility function, and let* $g(U, \mathcal{C})$ *be the reduced disjunctive form of* $f(U, \mathcal{C})$ *by applying the multiplication and absorption laws. If* $g(U, \mathcal{C}) = (\wedge\mathcal{C}_1) \vee \cdots \vee (\wedge\mathcal{C}_l)$, *where* $\mathcal{C}_k \subseteq \mathcal{C}$, $k = 1, 2, \ldots, l$ *and every element in* $\mathcal{C}_k$ *only appears once, then* $N\mathrm{red}(\mathcal{C}) = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l\}$.

Proof. For every $k = 1, 2, \ldots, l$, $\wedge\mathcal{C}_k \leq \vee c_{ij}$ for any $c_{ij} \in M(U, \mathcal{C})$, so $\mathcal{C}_k \cap c_{ij} \neq \varnothing$. Let $\mathcal{C}'_k = \mathcal{C}_k - \{C\}$ for any $C \in \mathcal{C}_k$, then $g(U, \mathcal{C}) \gneqq \vee_{t=1}^{k-1}(\wedge\mathcal{C}_t) \vee (\wedge\mathcal{C}'_k) \vee (\vee_{t=k+1}^{l}(\wedge\mathcal{C}_t))$. If for every $c_{ij} \in M(U, \mathcal{C})$, we have $\mathcal{C}'_k \cap c_{ij} \neq \varnothing$, then $\wedge\mathcal{C}'_k \leq \vee c_{ij}$ for every $c_{ij} \in M(U, \mathcal{C})$, that is, $g(U, \mathcal{C}) \geq \vee_{t=1}^{k-1}(\wedge\mathcal{C}_t) \vee (\wedge\mathcal{C}'_k) \vee (\vee_{t=k+1}^{l}(\wedge\mathcal{C}_t))$, which is a contradiction. It implies that there is $c_{i_0 j_0} \in M(U, \mathcal{C})$ such that $\mathcal{C}'_k \cap c_{i_0 j_0} = \varnothing$. Thus, $\mathcal{C}_k$ is a reduct of $\mathcal{C}$.

Table 1

Objects	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_1	*	*		*	*		*
x_2	*						
x_3	*		*			*	
x_4		*	*	*	*	*	*
x_5			*	*			*
x_6					*		*

For any $\mathcal{C}' \in \text{Red}(\mathcal{C})$, we have $\mathcal{C}' \cap c_{ij} \neq \emptyset$ for every $c_{ij} \in M(U, \mathcal{C})$, so $f(U, \mathcal{C}) \wedge (\wedge\mathcal{C}') = \wedge(\vee c_{ij}) \wedge (\wedge\mathcal{C}') = \wedge\mathcal{C}'$, which implies $\wedge\mathcal{C}' \leq f(U, \mathcal{C}) = g(U, \mathcal{C})$. Suppose that, for every $k = 1, 2, \ldots, l$, we have $\mathcal{C}_k - \mathcal{C}' \neq \emptyset$, then for every k, there is $C_k \in \mathcal{C}_k - \mathcal{C}'$. By rewriting $g(U, \mathcal{C}) = (\vee_{k=1}^{l} C_k) \wedge \Phi$, $\wedge\mathcal{C}' \leq \vee_{k=1}^{l} C_k$. Thus, there is C_{k_0} such that $\wedge\mathcal{C}' \leq C_{k_0}$, that is, $C_{k_0} \in \mathcal{C}'$, which is a contradiction. So $\mathcal{C}_{k_0} \subseteq \mathcal{C}'$ for some k_0, since both $\mathcal{C}'$ and $\mathcal{C}_{k_0}$ are reducts, and it is evident that $\mathcal{C}' = \mathcal{C}_{k_0}$. Consequently, $\text{Red}(\mathcal{C}) = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l\}$. □

Algorithm 3.9. Consider the following:

input: $\langle U, \mathcal{C} \rangle$,

output: $N\text{red}(\mathcal{C})$ and $NI(\mathcal{C})$// The set of all granular reducts and the set of all $\mathcal{N}$-irreducible elements.

Step 1: $M(U, \mathcal{C})=(c_{ij})_{n \times n}$, for each c_{ij}, let $c_{ij} = \emptyset$.

Step 2: for each $x_i \in U$, compute $N(x_i) = \cap\{C \in \mathcal{C} \mid x_i \in C\}$.

If $x_j \notin N(x_i), c_{ij} = \{C \in \mathcal{C} \mid x_i \in C, x_j \notin C\}$.

Step 3: $f(U, \mathcal{C})(\overline{C_1}, \overline{C_2}, \ldots, \overline{C_m}) = \wedge\{\vee(c_{ij}) \mid c_{ij} \in M(U, \mathcal{C}), c_{ij} \neq \emptyset\}$.

Step 4: compute $f(U, \mathcal{C})$ to $g(U, \mathcal{C}) = (\wedge\mathcal{C}_1) \vee \cdots \vee (\wedge\mathcal{C}_l)$// where $\mathcal{C}_k \subseteq \mathcal{C}$,

$k = 1, 2, \ldots, l$, and every element in $\mathcal{C}_k$ only appears once.

Step 4: output $N\text{red}(\mathcal{C}) = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l\}$, $NI(\mathcal{C}) = \cap N\text{red}(\mathcal{C})$.

Step 5: end.

The following example is used to illustrate our idea.

Example 3.10. Suppose that $U = \{x_1, x_2, \ldots, x_6\}$, where $x_i,\ i = 1, 2, \ldots, 6$ denote six objects, and let $C_i,\ i = 1, 2, \ldots, 7$ denote seven properties; the information is presented in Table 1, that is, the ith object possesses the jth attribute is indicated by a $*$ in the ij-position of the table.

$\{x_1, x_2, x_3\}$ is the set of all objects possessing the attribute C_1, and it is denoted by $C_1 = \{x_1, x_2, x_3\}$. Similarly, $C_2 = \{x_1, x_4\}$, $C_3 = \{x_3, x_4, x_5\}$, $C_4 = \{x_1, x_4, x_5\}$, $C_5 = \{x_1, x_4, x_6\}$, $C_6 = \{x_3, x_4\}$, and $C_7 = \{x_1, x_4, x_5, x_6\}$. Evidently, $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7\}$ is a covering on U.

Then, $N(x_1) = \{x_1\}$, $N(x_2) = \{x_1, x_2, x_3\}$, $N(x_3) = \{x_3\}$, $N(x_4) = \{x_4\}$, $N(x_5) = \{x_4, x_5\}$, and $N(x_6) = \{x_4, x_6\}$.

Table 2

Objects	C_1	C_3	C_4	C_5
x_1	*		*	*
x_2	*			
x_3	*	*		
x_4		*	*	*
x_5		*	*	
x_6				*

Table 3

Objects	C_1	C_3	C_5	C_7
x_1	*		*	*
x_2	*			
x_3	*	*		
x_4		*	*	*
x_5		*		*
x_6			*	*

The discernibility matrix of $(U, \mathcal{C})$ is exhibited as follows:

$$\begin{pmatrix} \varnothing & \{C_2, C_4, C_5, C_7\} & \{C_2, C_4, C_5, C_7\} & \{C_1\} & \{C_1, C_2, C_5\} & \{C_1, C_2, C_4\} \\ \varnothing & \varnothing & \varnothing & \{C_1\} & \{C_1\} & \{C_1\} \\ \{C_3, C_6\} & \{C_3, C_6\} & \varnothing & \{C_1\} & \{C_1, C_6\} & \{C_1, C_3, C_6\} \\ \{C_3, C_6\} & \{C_2, C_3, C_4, C_5, C_6, C_7\} & \{C_2, C_4, C_5, C_7\} & \varnothing & \{C_2, C_5, C_6\} & \{C_2, C_3, C_4, C_6\} \\ \{C_3\} & \{C_3, C_4, C_7\} & \{C_4, C_7\} & \varnothing & \varnothing & \{C_3, C_4\} \\ \varnothing & \{C_5, C_7\} & \{C_5, C_7\} & \varnothing & \{C_5\} & \varnothing \end{pmatrix}, \quad (3.1)$$

$$\begin{aligned} & f(U, \Delta)\left(\overline{C_1}, \overline{C_2}, \overline{C_3}, \overline{C_4}, \overline{C_5}, \overline{C_6}, \overline{C_7}\right) \\ &\quad = \wedge\left\{\vee\left(c_{ij}\right) \mid i, j = 1, 2, \ldots, 6, c_{ij} \neq \varnothing\right\} \\ &\quad = (C_2 \vee C_4 \vee C_5 \vee C_7) \wedge (C_2 \vee C_4 \vee C_5 \vee C_7) \wedge C_1 \wedge (C_1 \vee C_2 \vee C_5) \\ &\qquad \wedge (C_1 \vee C_2 \vee C_4) \wedge C_1 \wedge C_1 \wedge C_1 \wedge (C_3 \vee C_6) \wedge (C_3 \vee C_6) \wedge (C_1) \wedge (C_1 \vee C_6) \\ &\qquad \wedge (C_1 \vee C_3 \vee C_6) \wedge (C_3 \vee C_6) \wedge (C_2 \vee C_3 \vee C_4 \vee C_5 \vee C_6 \vee C_7) \wedge (C_2 \vee C_4 \vee C_5 \vee C_7) \\ &\qquad \wedge (C_2 \vee C_5 \vee C_6) \wedge (C_2 \vee C_3 \vee C_4 \vee C_6) \wedge C_3 \wedge (C_3 \vee C_4 \vee C_7) \wedge (C_4 \vee C_7) \\ &\qquad \wedge (C_3 \vee C_4) \wedge (C_5 \vee C_7) \wedge (C_5 \vee C_7) \wedge C_5 \\ &\quad = (C_5 \wedge C_1 \wedge C_3 \wedge C_4) \vee (C_5 \wedge C_1 \wedge C_3 \wedge C_7). \end{aligned} \quad (3.2)$$

So $N\text{red}(\mathcal{C}) = \{\{C_1, C_3, C_4, C_5\}, \{C_1, C_3, C_5, C_7\}\}$, $NI(\mathcal{C}) = \{C_1, C_3, C_5\}$. As a result, Table 1 can be simplified into Table 2 or Table 3, and the ability of classification is invariant. Obviously, the granular reduction algorithm can reduce data sets as shown.

4. The Simplification of Discernibility Matrixes

For the purpose of finding the set of all granular reducts, we have proposed the method by discernibility matrix. Unfortunately, it is at least an NP problem, since the discernibility matrix in this paper is more complex than the one in [33]. Accordingly, we simplify the discernibility matrixes in this section. In addition, a heuristic algorithm is presented to avoid the NP hard problem.

Definition 4.1. Let $M(\mathcal{U},\mathcal{C}) = (c_{ij})_{n\times n}$ be the discernibility matrix of $(\mathcal{U},\mathcal{C})$. For any $c_{ij} \in M(\mathcal{U},\mathcal{C})$, if there is an nonempty element $c_{i0j0} \in M(\mathcal{U},\mathcal{C}) - \{c_{ij}\}$ such that $c_{i0j0} \subseteq c_{ij}$, let $c'_{ij} = \emptyset$; otherwise, $c'_{ij} = c_{ij}$, then we get a new discernibility matrix $\mathrm{SIM}(\mathcal{U},\mathcal{C}) = (c'_{ij})_{n\times n}$, which called the simplification discernibility matrix of $(\mathcal{U},\mathcal{C})$.

Theorem 4.2. *Let $M(\mathcal{U},\mathcal{C})$ be the discernibility matrix of $(\mathcal{U},\mathcal{C})$, and $\mathrm{SIM}(\mathrm{U},\mathcal{C})$ is the simplification discernibility matrix, $\mathcal{C}' \subseteq \mathcal{C}$. Then $\mathcal{C}' \cap c_{ij} \neq \emptyset$ for any nonempty element $c_{ij} \in M(\mathcal{U},\mathcal{C})$ if and only if $\mathcal{C}' \cap c'_{ij} \neq \emptyset$ for any nonempty element $c'_{ij} \in \mathrm{SIM}(\mathrm{U},\mathcal{C})$.*

Proof. If $\mathcal{C}' \cap c_{ij} \neq \emptyset$ for every $c_{ij} \neq \emptyset$ and $c_{ij} \in M(\mathcal{U},\mathcal{C})$, it is evident that $\mathcal{C}' \cap c'_{ij} \neq \emptyset$ for every $c'_{ij} \neq \emptyset$ and $c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})$.

Suppose that $\mathcal{C}' \cap c'_{ij} \neq \emptyset$ for every $c'_{ij} \neq \emptyset$ and $c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})$. For any nonempty $c_{ij} \in M(\mathcal{U},\mathcal{C})$, if there is an nonempty element $c_{i0j0} \in M(\mathcal{U},\mathcal{C}) - \{c_{ij}\}$ such that $c_{i0j0} \subseteq c_{ij}$, and for any nonempty element $c_{i1j1} \in M(\mathcal{U},\mathcal{C}) - \{c_{ij}, c_{i0j0}\}$, $c_{i1j1} \not\subseteq c_{i0j0}$, then $c'_{i0j0} = c_{i0j0} \neq \emptyset$. Since $\mathcal{C}' \cap c'_{i0j0} \neq \emptyset$, then $\mathcal{C}' \cap c_{i0j0} \neq \emptyset$; thus, $\mathcal{C}' \cap c_{ij} \neq \emptyset$. If $c_{i0j0} \not\subseteq c_{ij}$ for any nonempty element $c_{i0j0} \in M(\mathcal{U},\mathcal{C}) - \{c_{ij}\}$, then $c'_{ij} = c_{ij}$. Since $\mathcal{C}' \cap c'_{ij} \neq \emptyset$, then $\mathcal{C}' \cap c_{ij} \neq \emptyset$. Thus, $\mathcal{C}' \cap c_{ij} \neq \emptyset$ for every nonempty $c_{ij} \in M(\mathcal{U},\mathcal{C})$. □

Proposition 4.3. *Suppose that $\mathcal{C}' \subseteq \mathcal{C}$, then $\mathcal{C}' \in N\mathrm{red}(\mathcal{C})$ if and only if $\mathcal{C}'$ is a minimal set satisfying $\mathcal{C}' \cap c'_{ij} \neq \emptyset$ for every $c'_{ij} \neq \emptyset$ and $c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})$.*

Proposition 4.4. *Consider that $\cup\{c'_{ij} \mid c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})\} = \cup N\mathrm{red}(\mathcal{C})$.*

Proof. Suppose that $C \in \cup\{c'_{ij} \mid c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})\}$, then there is $c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})$ such that $C \in c'_{ij}$ and $c'_{ij} \cap NI(\mathcal{C}) = \emptyset$. For any $c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})$, if $C \in c'_{ij}$, let $c^1_{ij} = \{C\}$. Otherwise, $c^1_{ij} = \{C_{ij}\}$, where $C_{ij} \in c'_{ij}$. Suppose that $M_1(\mathcal{U},\mathcal{C}) = (c^1_{ij})_{n\times n}$; it is easy to prove that $C \in \cup\{c^1_{ij} \mid c^1_{ij} \in M_1(\mathcal{U},\mathcal{C})\} \in N\mathrm{red}(\mathcal{C})$. Thus, $C \in \cup N\mathrm{red}(\mathcal{C})$.

Suppose that $C \in \cup N\mathrm{red}(\mathcal{C})$, then there is $\mathcal{C}_k \in N\mathrm{red}(\mathcal{C})$ such that $C \in \mathcal{C}_k$. From Proposition 4.3, we know that $\mathcal{C}_k$ is a minimal set satisfying $\mathcal{C}_k \cap c'_{ij} \neq \emptyset$ for every $c'_{ij} \neq \emptyset$ and $c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})$. So there is a $c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})$ such that $C \in c'_{ij}$, or else C is redundant in $\mathcal{C}_k$. Thus, $C \in \cup\{c'_{ij} \mid c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})\}$.

In summary, $\cup\{c'_{ij} \mid c'_{ij} \in \mathrm{SIM}(\mathcal{U},\mathcal{C})\} = \cup N\mathrm{red}(\mathcal{C})$. □

Proposition 4.5. *Let $\mathrm{SIM}(\mathcal{U},\mathcal{C}) = (c'_{ij})_{n\times n}$ be the simplified discernibility matrix of $(\mathcal{U},\mathcal{C})$, then $\mathrm{SIM}(\mathcal{U},\mathcal{C})$ is the minimal matrix to compute all granular reducts of $\mathcal{C}$, that is, for any matrix $M_0(\mathcal{U},\mathcal{C}) = (d_{ij})_{n\times n}$ where $d_{ij} \subseteq c'_{ij}$, $M_0(\mathcal{U},\mathcal{C})$ can compute all granular reducts of $\mathcal{C}$ if and only if $d_{ij} = c'_{ij}$ for $1 \le i, j \le n$.*

Proof. If $d_{ij} = c'_{ij}$ for $1 \le i, j \le n$, then $M_0(\mathcal{U},\mathcal{C}) = \mathrm{SIM}(\mathcal{U},\mathcal{C})$, and $M_0(\mathcal{U},\mathcal{C})$ can compute all granular reducts of $\mathcal{C}$.

Suppose that there is a nonempty $c'_{i0j0} \in \mathrm{SIM}(U, \mathcal{C})$ such that $d_{i0j0} \subset c'_{i0j0}$. If $|c'_{i0j0}| = 1$, suppose that $c'_{i0j0} = \{C_0\}$, then $d_{i0j0} = \varnothing$. From the definition of the simplification discernibility matrix, we know that $C_0 \notin c'_{ij}$ for any $c'_{ij} \in \mathrm{SIM}(U, \mathcal{C}) - \{c'_{i0j0}\}$, then $C_0 \notin d_{ij}$ for any $d_{ij} \in M_0(U, \mathcal{C})$. So $M_0(U, \mathcal{C})$ cannot compute any granular reducts of $\mathcal{C}$. If $|c'_{i0j0}| \geq 2$, we suppose that $d_{i0j0} \neq \varnothing$. Then there is a $C \in (c'_{i0j0} - d_{i0j0})$, and let $c^1_{i0j0} = \{C\}$. For any $c'_{ij} \in \mathrm{SIM}(U, \mathcal{C}) - \{c'_{i0j0}\}$, if $C \in c'_{ij}$, let $c^1_{ij} = \varnothing$. Otherwise, let $c^1_{ij} = \{C_{ij}\}$ where $C_{ij} \in c'_{ij} - c'_{i0j0}$. Let $M_1(U, \mathcal{C}) = (c^1_{ij})_{n\times n}$ and $\mathcal{C}' = \cup\{c^1_{ij} \mid c^1_{ij} \in M_1(U, \mathcal{C})\}$, and it is easy to prove that $\mathcal{C}' \in N\mathrm{red}(\mathcal{C})$. However, $\mathcal{C}' \cap d_{i0j0} = \varnothing$, that is, $M_0(U, \mathcal{C})$ cannot compute all granular reducts of $\mathcal{C}$. Thus, if $M_0(U, \mathcal{C})$ can compute all granular reducts of $\mathcal{C}$, then $d_{ij} = c'_{ij}$ for $1 \leq i, j \leq n$. □

From the above propositions, we know that the simplified discernibility matrix is the minimal discernibility matrix which can compute the same reducts as the original one. Hereafter, we only examine simplified discernibility matrixes instead of general discernibility matrixes. The following example is used to illustrate our idea.

Example 4.6. The discernibility matrix of $(U, \mathcal{C})$ in Example 3.10 is as follows:

$$\begin{pmatrix} \varnothing & \{C_5\} & \varnothing & \{C_1\} & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \{C_3\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \{C_4, C_7\} & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix},$$

$$\begin{aligned} f(U,\Delta)\left(\overline{C_1},\overline{C_2},\overline{C_3},\overline{C_4},\overline{C_5},\overline{C_6},\overline{C_7}\right) &= \wedge\left\{\vee\left(c'_{ij}\right) \mid i,j = 1,2,\ldots,6, c_{ij} \neq \varnothing\right\} \\ &= C_5 \wedge C_1 \wedge C_3 \wedge (C_4 \vee C_7) \\ &= (C_5 \wedge C_1 \wedge C_3 \wedge C_4) \vee (C_5 \wedge C_1 \wedge C_3 \wedge C_7). \end{aligned} \tag{4.1}$$

So $N\mathrm{red}(\mathcal{C}) = \{\{C_1, C_3, C_4, C_5\}, \{C_1, C_3, C_5, C_7\}\}$, $NI(\mathcal{C}) = \{C_1, C_3, C_5\}$.

From the above example, it is easy to see that simplified discernibility matrix can simplify the computing processes remarkably. Especially when $\mathcal{C}$ is a consistent covering proposed in [30], that is, $N\mathrm{red}(\mathcal{C}) = \{NI(\mathcal{C})\}$, the unique reduct $N\mathrm{red}(\mathcal{C}) = \{\cup\{c'_{ij} \mid c'_{ij} \in \mathrm{SIM}(U, \mathcal{C})\}\}$.

Unfortunately, although the simplified discernibility matrixes are more simple, the processes of computing reducts by discernibility function are still NP hard. Accordingly, we develop a heuristic algorithm to obtain a reduct from a discernibility matrix directly.

Let $M(U, \mathcal{C}) = (c_{ij})_{n\times n}$ be a discernibility matrix. We denote the number of the elements in c_{ij} by $|c_{ij}|$. For any $C \in \mathcal{C}$, $||C||$ denotes the number of c_{ij} which contain C. Let $c_{ij} \in M(U, \mathcal{C})$, if for any $C \in NI(\mathcal{C})$, $C \notin c_{ij}$, then $c'_{ij} = c_{ij}$. Since $\cup\{c'_{ij} \mid |c'_{ij}| \geq 2\} = \cup N\mathrm{red}(\mathcal{C}) - NI(\mathcal{C})$, if $|c'_{ij}| \geq 2$, then the elements in c'_{ij} may either be deleted from $\mathcal{C}$ or be preserved. Suppose that $C_0 \in \cup\{c'_{ij} \mid |c'_{ij}| \geq 2\}$, if $||C_0|| \geq ||C||$ for any $C \in \cup\{c'_{ij} \mid |c'_{ij}| \geq 2\}$, C_0 is called the maximal element with respect to the simplified discernibility matrix $\mathrm{SIM}(U, \mathcal{C})$. The heuristic algorithm to get a reduct from a discernibility matrix directly proceeds as follows.

Algorithm 4.7. Consider the following:

input: $\langle U, \mathcal{C} \rangle$,

output: granular reducts red

Step 1: $M(U, \mathcal{C})=(c_{ij})_{n\times n}$, for each c_{ij}, let $c_{ij} = \varnothing$.

Step 2: for each $x_i \in U$, compute $N(x_i) = \cap\{C \in \mathcal{C} x_i \in C\}$.

If $x_j \notin N(x_i)$,

$c_{ij} = \{C \in \mathcal{C} \mid x_i \in C, x_j \notin C\}$//get the discernibility matrix.

Step 3: for each $c_{ij} \in M(U, \mathcal{C})$,

if there is a nonempty element $c_{i0j0} \in M(U, \mathcal{C}) - \{c_{ij}\}$ such that

$c_{i0j0} \subseteq c_{ij}$, let $c_{ij} = \varnothing$ // get the simplified discernibility matrix.

Step 4: for each $C_i \in \cup M(U, \mathcal{C})$, compute $\|C_i\|$ and select the maximal element C_0 of SIM$(U, \mathcal{C})$.

For each $c_{ij} \in M(U, \mathcal{C})$,

if $C_0 \in c_{ij}$,

let $c_{ij} = \{C_0\}$.

Step 5: if there is $c_{ij} \in M(U, \mathcal{C})$ such that $|c_{ij}| \geqslant 2$,

return to Step 3;

else

output red $= \cup M(U, \mathcal{C})$.

Step 5: end.

Example 4.8. The simplified discernibility matrix of $(U, \mathcal{C})$ in Example 3.10 is as follows:

$$\begin{pmatrix} \varnothing & \{C_5\} & \varnothing & \{C_1\} & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \{C_3\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \{C_4, C_7\} & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}. \tag{4.2}$$

For a maximal element C_4 of SIM$(U, \mathcal{C})$, let $c^1_{53} = \{C_4\}$, then we get $M_1(U, \mathcal{C})$ as follows:

$$\begin{pmatrix} \varnothing & \{C_5\} & \varnothing & \{C_1\} & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \{C_3\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \{C_4\} & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}. \tag{4.3}$$

Thus, $\{C_1, C_3, C_4, C_5\} = \cup\{c^1_{ij} \mid c^1_{ij} \in M_1(U, \mathcal{C})\}$ is a granular reduct of $\mathcal{C}$.

For a maximal element C_7 of $\mathrm{SIM}(\mathcal{U},\mathcal{C})$, let $c^1_{53} = \{C_7\}$, then we get $M_2(\mathcal{U},\mathcal{C})$ as follows:

$$\begin{pmatrix} \varnothing & \{C_5\} & \varnothing & \{C_1\} & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \{C_3\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \{C_7\} & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}. \tag{4.4}$$

Thus, $\{C_1, C_3, C_5, C_7\} = \cup\{c^2_{ij} \mid c^2_{ij} \in M_2(\mathcal{U},\mathcal{C})\}$ is also a granular reduct of $\mathcal{C}$.

From the above example, we show that the heuristic algorithm can avoid the NP hard problem and generate a granular reduct from the simplified discernability matrix directly. With the heuristic algorithm, the granular reduction theory based on discernability matrix is no longer limited to the theoretic level but applicable in practical usage.

5. Conclusion

In this paper, we develop an algorithm by discernability matrixes to compute all the granular reducts with covering rough sets initially. A simplification of discernibility matrix is proposed for the first time. Moreover, a heuristic algorithm to compute a granular reduct is presented to avoid the NP hard problem in granular reduction such that a granular reduct is generated rapidly.

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