

Research Article

A Posteriori Error Estimates for the Generalized Overlapping Domain Decomposition Methods

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A posteriori error estimates for the generalized overlapping domain decomposition method (GODDM) (i.e., with Robin boundary conditions on the interfaces), for second order boundary value problems, are derived. We show that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the interfaces. After discretization of the domain by finite elements we use the techniques of the residual *a posteriori* error analysis to get an *a posteriori* error estimate for the discrete solutions on subdomains. The results of some numerical experiments are presented to support the theory.

1. Introduction

We consider the generalized overlapping domain decomposition method that is, with Robin transmission conditions on the interfaces [1, 2] for second order boundary value problems on a bounded domain Ω with Dirichlet boundary conditions. The a priori estimate of the error is given in several papers, see for instance Lions [3] in which a variational formulation of the classical Schwarz method is derived. In Chan et al. [4] a geometry related convergence results are obtained. Douglas and Huang [1] studied the accelerated version of the GODDM, Engquist and Zhao [2] studied the convergence for simple (rectangular or circular) geometries; however, these authors did not give a criterion to stop the iterative process. All these results can also be found in the recent books on domain decomposition methods of Quarteroni and Valli [5], Toselli and Widlund [6]. Recently Maday and Magoulès [7, 8] presented an improved version of the Schwarz method for highly heterogeneous media. This method uses new optimized interface conditions specially designed to take into account the heterogeneity between the subdomains on the interfaces. A recent overview of the current state of the art on domain decomposition methods can be found in two special issues of the

computer methods in applied mechanics and engineering journal, edited by Farhat and Le Tallec [9], Magoulès and Rixen [10] and in Nataf [11].

In general, the *a priori* estimate is not suitable for assessing the quality of the approximate solution on subdomains since it depends mainly on the exact solution itself which is unknown. The alternative approach is to use the approximate solution itself in order to find such an estimate. This approach, known as a posteriori estimate, became very popular in the nineties of the last century with finite element methods, see the monographs [12, 13] and the references therein. In their paper Otto and Lube [14] gave an a posteriori estimate for a nonoverlapping domain decomposition algorithm that said that “the better the local solutions fit together at the interface the better the errors of the subdomain solutions will be.” This error estimate enables us to know with certainty when one must stop the iterative process as soon as the required global precision is reached. A posteriori error analysis was also used by Bernardi et al. [15] to determine an optimal value of the penalty parameter for penalty domain decomposition methods to construct fast solvers.

In various situations it is better to use overlapping decompositions for faster convergence rate, for example, the case of decomposition into simple subdomains where uniform discretizations are possible. So the aim of this paper is to show that a similar result to that in [14], in the case of the GODDM, holds. It is obtained via the introduction of two auxiliary problems defined each over two nonoverlapping subdomains. These auxiliary problems are needed for the analysis and not for the computation.

The paper is organized as follows. In Section 1, we introduce some necessary notations, then we give a variational formulation of our model problem. We establish, in Section 2, a stopping criterion for the iterative process in the continuous case. In Section 3, an a posteriori error estimate is proposed for the convergence of the discretized solution using the finite element method on subdomains. We conclude this section by an adaptation of the techniques of the residual a posteriori error analysis to give an a posteriori estimate in the discrete case.

1.1. Problem Formulation and Domain Decomposition Method

Let Ω be a bounded domain in \mathbb{R}^2 with a piecewise $C^{1,1}$ boundary $\partial\Omega$. We denote $(\cdot, \cdot)_{\Omega}$ and $\|\cdot\|_{0,\Omega}$ the usual inner product and norm of $L^2(\Omega)$. Let $H^1(\Omega)$ be the usual Sobolev space with norm $\|\cdot\|_{1,\Omega}$ and seminorm $|\cdot|_{1,\Omega}$; $H_0^1(\Omega)$ is the subspace of functions of $H^1(\Omega)$ vanishing on $\partial\Omega$.

Let Ω_1 and Ω_2 be two subdomains of Ω (to be defined later), when $\Gamma_i \subset \partial\Omega_i$ we need the space $W_i = H_{00}^{1/2}(\Gamma_i) = \{v|_{\Gamma_i} : v \in H^1(\Omega_i), v = 0 \text{ on } \partial\Omega_i \setminus \Gamma_i\}$ which is a subspace of $H^{1/2}(\Gamma_i)$; $i = 1, 2$ equipped with the norm

$$\|\varphi\|_{W_i} = \inf_{v \in H^1(\Omega_i), v=\varphi \text{ on } \Gamma_i} |v|_{1,\Omega_i}. \quad (1.1)$$

On the subdomain Ω_i we use the space $V_i = H_0^1(\Omega) \cap H^1(\Omega_i)$ which has as a trace space $W_i = V_i|_{\Gamma_i} = H_{00}^{1/2}(\Gamma_i)$, $i = 1, 2$. By $\langle \cdot, \cdot \rangle_{\Gamma_i}$ we denote the inner product of $L^2(\Gamma_i)$.

Let us consider the following model problem:

$$\begin{aligned} -\Delta u &= f \in L^2(\Omega), \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1.2)$$

The variational formulation of this problem is given by

$$\text{Find } u \in V \text{ solution of } a(u, v) = (f, v)_\Omega, \quad \forall v \in V, \quad (1.3)$$

where $a(u, v) = \int_\Omega \nabla u \nabla v \, dx$ and $(f, v)_\Omega = \int_\Omega f v \, dx$ for all $v \in V = H_0^1(\Omega)$. We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that

$$\Omega_1 \cap \Omega_2 = \Omega_{12}, \quad \partial\Omega_i \cap \Omega_j = \Gamma_i, \quad i \neq j \quad (i, j = 1, 2). \quad (1.4)$$

The generalized overlapping domain decomposition method is written as follows: set u_i^0 in Ω_i and construct the sequence u_i^{n+1} , $i = 1, 2$ in parallel for $n = 0, 1, 2, \dots$

$$\begin{aligned} -\Delta u_1^{n+1} &= f \quad \text{in } \Omega_1, \\ u_1^{n+1} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \end{aligned} \quad (1.5)$$

$$\frac{\partial u_1^{n+1}}{\partial \eta_1} + \alpha_1 u_1^{n+1} = \frac{\partial u_2^n}{\partial \eta_1} + \alpha_1 u_2^n \quad \text{on } \Gamma_1,$$

$$\begin{aligned} -\Delta u_2^{n+1} &= f \quad \text{in } \Omega_2, \\ u_2^{n+1} &= 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \end{aligned} \quad (1.6)$$

$$\frac{\partial u_2^{n+1}}{\partial \eta_2} + \alpha_2 u_2^{n+1} = \frac{\partial u_1^n}{\partial \eta_2} + \alpha_2 u_1^n \quad \text{on } \Gamma_2,$$

where η_i is the exterior normal to Ω_i and α_i is a real parameter, $i = 1, 2$.

The convergence of the GODDM (1.5) and (1.6) is proved in [2, 5]. Our main interest is to obtain an a posteriori error estimate we need for stopping the iterative process as soon as the required global precision is reached.

The weak formulations of problems (1.5), (1.6) are given, respectively, by

$$u_1^{n+1} \in V_1 : a_1(u_1^{n+1}, v_1) + \langle \alpha_1 u_1^{n+1}, v_1 \rangle_{\Gamma_1} = (f, v_1) + \left\langle \frac{\partial u_2^n}{\partial \eta_1} + \alpha_1 u_2^n, v_1 \right\rangle_{\Gamma_1}, \quad \forall v_1 \in V_1, \quad (1.7)$$

$$u_2^{n+1} \in V_2 : a_2(u_2^{n+1}, v_2) + \langle \alpha_2 u_2^{n+1}, v_2 \rangle_{\Gamma_2} = (f, v_2) + \left\langle \frac{\partial u_1^n}{\partial \eta_2} + \alpha_2 u_1^n, v_2 \right\rangle_{\Gamma_2}, \quad \forall v_2 \in V_2, \quad (1.8)$$

where V_i is defined above. It can also be written as

$$V_i = \left\{ v \in H^1(\Omega_i) : v|_{\partial\Omega_i \cap \partial\Omega} = 0 \right\}, \quad a_i(u, v) = (\nabla u, \nabla v)_{\Omega_i}. \quad (1.9)$$

2. A Posteriori Error Estimate in the Continuous Case

Since it is numerically easier to compare the subdomain solutions on the interfaces Γ_1 and Γ_2 rather than on the overlap Ω_{12} , thus we need to introduce two auxiliary problems defined on

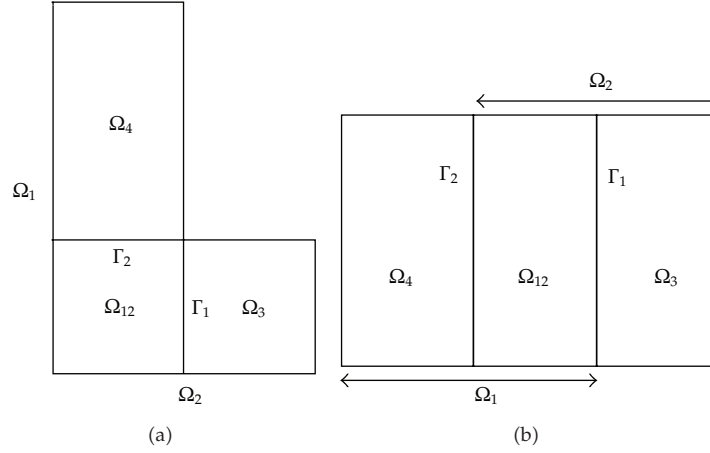


Figure 1: Two examples of domain decompositions (Schur and Schwarz).

nonoverlapping subdomains of Ω . This idea allows us to obtain the a posteriori error estimate by following the steps of Otto and Lube [14]. We get these auxiliary problems by coupling each one of the problems (1.5) and (1.6) with another problem in a nonoverlapping way over Ω . These auxiliary problems are needed for the analysis and not for the computation, to get the estimate. We see that we can go from the GODDM to a couple of nonoverlapping domain decomposition methods with Robin boundary conditions on the interfaces Γ_1 and Γ_2 . To define these auxiliary problems we need to split the domain Ω into two sets of disjoint subdomains: (Ω_1, Ω_3) and (Ω_2, Ω_4) , see Figure 1, such that

$$\Omega = \Omega_1 \cup \Omega_3, \quad \text{with } \Omega_1 \cap \Omega_3 = \emptyset, \quad \Omega = \Omega_2 \cup \Omega_4 \quad \text{with } \Omega_2 \cap \Omega_4 = \emptyset. \quad (2.1)$$

Let (u_1^{n+1}, u_2^{n+1}) be the solutions of problems (1.5), (1.6). We define the couple (u_1^{n+1}, u_3^{n+1}) over (Ω_1, Ω_3) to be the solution of the following nonoverlapping problems:

$$\begin{aligned} -\Delta u_1^{n+1} &= f \quad \text{in } \Omega_1, \\ u_1^{n+1} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \end{aligned} \quad (2.2)$$

$$\begin{aligned} \frac{\partial u_1^{n+1}}{\partial \eta_1} + \alpha_1 u_1^{n+1} &= \frac{\partial u_2^n}{\partial \eta_1} + \alpha_1 u_2^n \quad \text{on } \Gamma_1, \\ -\Delta u_3^{n+1} &= f \quad \text{in } \Omega_3, \\ u_3^{n+1} &= 0 \quad \text{in } \partial\Omega_3 \cap \partial\Omega, \end{aligned} \quad (2.3)$$

$$\frac{\partial u_3^{n+1}}{\partial \eta_3} + \alpha_3 u_3^{n+1} = \frac{\partial u_1^n}{\partial \eta_3} + \alpha_3 u_1^n \quad \text{on } \Gamma_1.$$

We can write $u_2^n = u_3^n + \epsilon_1^n$ on Γ_1 (in the 3rd equation in (2.2)) that is, ϵ_1^n is the difference between the overlapping and the nonoverlapping solutions u_2^n and u_3^n (in problems (1.5)-(1.6))

and (2.2)-(2.3) resp.,) in Ω_3 . Because both overlapping and nonoverlapping problems converge, see [2, 5] that is, u_2^n and u_3^n tend to $u_2 = u|_{\Omega_2}$, ϵ_1^n should tend to naught an n tends to infinity in V_2 .

By multiplying the first equation in (2.2) by $v_1 \in V_1$ and integrating by parts we obtain

$$\int_{\Omega_1} \nabla u_1^{n+1} \nabla v_1 dx + \int_{\Gamma_1} \alpha_1 u_1^{n+1} v_1 dx = \int_{\Omega_1} f v_1 dx + \int_{\Gamma_1} \Lambda_3^n v_1 d\sigma, \quad (2.4)$$

where Λ_3^n is given by

$$\Lambda_3^n = \frac{\partial u_3^n}{\partial \eta_1} + \alpha_1 u_3^n + \frac{\partial \epsilon_1^n}{\partial \eta_1} + \alpha_1 \epsilon_1^n. \quad (2.5)$$

On the other hand, using the last equation in (2.3) and putting

$$\theta_1^n = \frac{\partial \epsilon_1^n}{\partial \eta_1} + \alpha_1 \epsilon_1^n, \quad (2.6)$$

we get

$$\begin{aligned} \Lambda_3^{n+1} &= \frac{\partial u_3^{n+1}}{\partial \eta_1} + \alpha_1 u_3^{n+1} + \theta_1^{n+1} = -\frac{\partial u_3^{n+1}}{\partial \eta_3} + \alpha_1 u_3^{n+1} + \theta_1^{n+1} \\ &= \alpha_3 u_3^{n+1} - \frac{\partial u_1^n}{\partial \eta_3} - \alpha_3 u_1^n + \alpha_1 u_3^{n+1} + \theta_1^{n+1} \\ &= (\alpha_1 + \alpha_3) u_3^{n+1} - \Lambda_1^n + \zeta^n, \end{aligned} \quad (2.7)$$

where $\zeta^n = \theta_1^{n+1} + \theta_3^n$. From this result we can write the following algorithm which is equivalent to the auxiliary nonoverlapping problem (2.2)-(2.3). We need this algorithm and two lemmas for obtaining an a posteriori error estimate for this problem.

Algorithm 2.1. The sequences $(u_1^{n+1}, u_3^{n+1})_{n \in \mathbb{N}}$ solutions of problems (2.2), (2.3) satisfy the following domain decomposition algorithm denoted ALG.G.D.D.M.

- (1) Let $\Lambda_i^0 \in W_1^*$ be an initial value, $i = 1, 3$ (W_1^* is the dual of W_1).
- (2) Given $\Lambda_j^n \in W_1^*$ solve for $i = 1, 3, i \neq j$: Find $u_i^{n+1} \in V_i$ solutions of

$$a_i(u_i^{n+1}, v_i) + \langle \alpha_i u_i^{n+1}, v_i \rangle_{\Gamma_1} = F(v_i) + \langle \Lambda_j^n, v_i \rangle_{\Gamma_1}, \quad \forall v_i \in V_i. \quad (2.8)$$

- (3) Compute new data $\Lambda_i^{n+1} \in W_1^*$, $i = 1, 3$ from

$$\langle \Lambda_i^{n+1}, \varphi \rangle_{\Gamma_1} = \langle (\alpha_i + \alpha_j) u_i^{n+1}, \varphi \rangle_{\Gamma_1} - \langle \Lambda_j^n, \varphi \rangle_{\Gamma_1} + \langle \zeta^n, \varphi \rangle_{\Gamma_1}, \quad \forall \varphi \in W_1, j \neq i. \quad (2.9)$$

- (4) Set $n = n + 1$, go to Step 2.

Lemma 2.2. Let $u_i = u|_{\Omega_i}$, $e_i^{n+1} = u_i^{n+1} - u_i$, $\eta_i^n = \Lambda_i^n - \Lambda_i$ and $\xi_1^n = \zeta^n - \zeta$. Then for $i = 1, 3, i \neq j$, the following relations hold

$$a_i(e_i^{n+1}, v_i) + \langle \alpha_i e_i^{n+1}, v_i \rangle_{\Gamma_i} = \langle \eta_j^n, v_i \rangle_{\Gamma_i}, \quad \forall v_i \in V_i, \quad (2.10)$$

$$\langle \eta_i^{n+1}, \varphi \rangle_{\Gamma_1} = \langle (\alpha_i + \alpha_j) e_i^{n+1}, \varphi \rangle_{\Gamma_1} - \langle \eta_j^n, \varphi \rangle_{\Gamma_1} + \langle \xi_1^n, \varphi \rangle_{\Gamma_1}, \quad \forall \varphi \in W_1. \quad (2.11)$$

The proof follows directly from steps 2 and 3 of ALG.G.D.D.M.

Lemma 2.3. By letting C be a generic constant which has different values at different places one get for $i = 1, 3, i \neq j$

$$\begin{aligned} \langle \eta_j^{n-1} - \alpha_i e_i^n, w \rangle_{\Gamma_1} &\leq C \|e_i^n\|_{1, \Omega_i} \|w\|_{W_1}, \\ \langle \alpha_i w + \xi_1^n, e_i^{n+1} \rangle_{\Gamma_1} &\leq C \|e_i^{n+1}\|_{1, \Omega_i} \|w\|_{W_1}. \end{aligned} \quad (2.12)$$

Proof. Using (2.10) and the fact that the trace mapping $\text{Tr}_i : V_i \rightarrow W_i$ and its inverse are continuous, we obtain

$$\begin{aligned} \langle \eta_j^{n-1} - \alpha_i e_i^n, w \rangle_{\Gamma_1} &= a_i(e_i^n, \text{Tr}_i^{-1} w) = (\nabla e_i^n, \nabla \text{Tr}_i^{-1} w) \\ &\leq |e_i^n|_{1, \Omega_i} |\text{Tr}_i^{-1} w|_{1, \Omega_i} \\ &\leq C_{\text{Tr}_i^{-1}} \|e_i^n\|_{1, \Omega_i} \|w\|_{W_1}. \end{aligned} \quad (2.13)$$

On the otherhand we have

$$\begin{aligned} \langle \alpha_i w + \xi_1^n, e_i^{n+1} \rangle_{\Gamma_1} &= \int_{\Gamma_1} (\alpha_i w + \xi_1^n) e_i^{n+1} ds \\ &\leq [|\alpha_i| \|w\|_{0, \Gamma_1} + |\xi_1^n|_{0, \Gamma_1}] \|e_i^{n+1}\|_{0, \Gamma_1} \\ &\leq \max(|\alpha_i|, |\xi_1^n|_{0, \Gamma_1}) \|w\|_{0, \Gamma_1} \|e_i^{n+1}\|_{0, \Gamma_1} \\ &\leq C \|e_i^{n+1}\|_{1, \Omega_i} \|w\|_{W_1}. \end{aligned} \quad (2.14)$$

This ends the proof. \square

Based on the previous two lemmas, we can obtain the following *a posteriori* error estimate for the nonoverlapping domain decomposition problem (2.2)-(2.3).

Proposition 2.4. For the sequences $(u_1^{n+1}, u_3^{n+1})_{n \in \mathbb{N}}$ solutions of problems (2.2), (2.3) one has the following a posteriori error estimation

$$\|e_1^{n+1}\|_{1,\Omega_1} + \|e_3^n\|_{1,\Omega_3} \leq C \|u_1^{n+1} - u_3^n\|_{H_{00}^{1/2}(\Gamma_1)}. \quad (2.15)$$

Proof. From (2.10) and (2.11) we have

$$\begin{aligned} a_1(e_1^{n+1}, v_1) + a_3(e_3^n, v_3) &= \langle \eta_3^n - \alpha_1 e_1^{n+1}, v_1 \rangle_{\Gamma_1} + \langle \eta_1^{n-1} - \alpha_3 e_3^n, v_3 \rangle_{\Gamma_1} \\ &= \langle \alpha_1 (e_3^n - e_1^{n+1}) + \xi_1^n, v_1 \rangle_{\Gamma_1} + \langle \eta_1^{n-1} - \alpha_3 e_3^n, v_3 - v_1 \rangle_{\Gamma_1}. \end{aligned} \quad (2.16)$$

Taking $v_1 = e_1^{n+1}$ and $v_2 = e_2^n$, then using Lemma 2.3 we get

$$\begin{aligned} \frac{1}{2} \left(\|e_1^{n+1}\|_{1,\Omega_1} + \|e_3^n\|_{1,\Omega_3} \right)^2 &\leq \|e_1^{n+1}\|_{1,\Omega_1}^2 + \|e_3^n\|_{1,\Omega_3}^2 = a_1(e_1^{n+1}, e_1^{n+1}) + a_3(e_3^n, e_3^n) \\ &\leq \langle \alpha_1 (e_3^n - e_1^{n+1}) + \xi_1^n, e_1^{n+1} \rangle_{\Gamma_1} + \langle \eta_1^{n-1} - \alpha_3 e_3^n, e_3^n - e_1^{n+1} \rangle_{\Gamma_1} \\ &\leq C_1 \|e_1^{n+1}\|_{1,\Omega_1} \|e_3^n - e_1^{n+1}\|_{H_{00}^{1/2}(\Gamma_1)} + C_2 \|e_3^n\|_{1,\Omega_3} \|e_3^n - e_1^{n+1}\|_{H_{00}^{1/2}(\Gamma_1)} \\ &\leq C \left\{ \|e_1^{n+1}\|_{1,\Omega_1} + \|e_3^n\|_{1,\Omega_3} \right\} \|u_1^{n+1} - u_3^n\|_{H_{00}^{1/2}(\Gamma_1)}, \end{aligned} \quad (2.17)$$

which is the required result. \square

In the similar way, we define another nonoverlapping auxiliary problem over (Ω_2, Ω_4) . The sequences $(u_2^{n+1}, u_4^{n+1})_{n \in \mathbb{N}}$ are solutions of

$$\begin{aligned} -\Delta u_2^{n+1} &= f \quad \text{in } \Omega_2, \\ u_2^{n+1} &= 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \end{aligned} \quad (2.18)$$

$$\frac{\partial u_2^{n+1}}{\partial \eta_2} + \alpha_2 u_2^{n+1} = \frac{\partial u_1^n}{\partial \eta_2} + \alpha_2 u_1^n \quad \text{on } \Gamma_2,$$

$$\begin{aligned} -\Delta u_4^{n+1} &= f \quad \text{in } \Omega_4, \\ u_4^{n+1} &= 0 \quad \text{on } \partial\Omega_4 \cap \partial\Omega, \end{aligned} \quad (2.19)$$

$$\frac{\partial u_4^{n+1}}{\partial \eta_4} + \alpha_4 u_4^{n+1} = \frac{\partial u_2^n}{\partial \eta_4} + \alpha_4 u_2^n \quad \text{on } \Gamma_2,$$

where we can write $u_1^n = u_4^n + \epsilon_2^n$ on Γ_2 (in the 3rd equation in (2.18)) that is, ϵ_2^n is the difference between the overlapping and the nonoverlapping solutions u_1^n and u_4^n (in problems (1.5)-(1.6))

and (2.18)-(2.19) resp.) in Ω_4 . As both sequences u_1^n and u_4^n converge towards u_1 , ϵ_2^n should tend to naught as n tends to infinity. Using Lemmas 2.2 and 2.3 over (Ω_2, Ω_4) we obtain the *a posteriori* error estimate for this auxiliary problem.

Proposition 2.5. *For the sequences $(u_2^{n+1}, u_4^{n+1})_{n \in \mathbb{N}}$ solutions of problems (2.18), (2.19) one has the following *a posteriori* error estimate*

$$\|u_2^{n+1} - u_2\|_{1, \Omega_2} + \|u_4^n - u_4\|_{1, \Omega_4} \leq C \|u_2^{n+1} - u_4^n\|_{H_0^{1/2}(\Gamma_2)}. \quad (2.20)$$

Remark 2.6. We remark from Propositions 2.4 and 2.5 that we can go from sequences of solutions to problems (1.5) and (1.6) to sequences of solutions to problems generated by nonoverlapping domain decomposition problems of Robin type transmission conditions across the interfaces Γ_1 and Γ_2 . This result enable us to compare the subdomain iterations of the GODDM on the interfaces Γ_1 and Γ_2 rather than on the overlap region Ω_{12} .

In the following theorem we are going to use this fact to obtain an *a posteriori* error estimate for the continuous generalized overlapping domain decomposition method (1.5)-(1.6).

Theorem 2.7. *Let $u_i = u|_{\Omega_i}$ where u is the solution of problem (1.2). For the sequences $(u_1^{n+1}, u_2^{n+1})_{n \in \mathbb{N}}$ solutions of problems (1.5) and (1.6), one has the following *a posteriori* error estimate*

$$\|u_1^{n+1} - u_1\|_{1, \Omega_1} + \|u_2^n - u_2\|_{1, \Omega_2} \leq C \left\{ \begin{aligned} &\|u_1^{n+1} - u_2^n\|_{H_0^{1/2}(\Gamma_1)} + \|u_2^n - u_1^{n-1}\|_{H_0^{1/2}(\Gamma_2)} \\ &+ \|\epsilon_1^n\|_{H_0^{1/2}(\Gamma_1)} + \|\epsilon_2^{n-1}\|_{H_0^{1/2}(\Gamma_2)} \end{aligned} \right\}. \quad (2.21)$$

Proof. We use two auxiliary problems defined each over two nonoverlapping subdomains of Ω . These two problems are defined over (Ω_1, Ω_3) and (Ω_2, Ω_4) , respectively. From the previous two propositions we have

$$\begin{aligned} \|u_1^{n+1} - u_1\|_{1, \Omega_1} + \|u_2^n - u_2\|_{2, \Omega_2} &\leq \|u_1^{n+1} - u_1\|_{1, \Omega_1} + \|u_3^n - u_3\|_{1, \Omega_3} \\ &\quad + \|u_2^n - u_2\|_{1, \Omega_2} + \|u_4^{n-1} - u_4\|_{1, \Omega_4} \\ &\leq C \|u_1^{n+1} - u_3^n\|_{H_0^{1/2}(\Gamma_1)} + C \|u_2^n - u_4^{n-1}\|_{H_0^{1/2}(\Gamma_2)}. \end{aligned} \quad (2.22)$$

From the definitions of ϵ_1^n and ϵ_2^n , we have

$$\begin{aligned} \|u_1^{n+1} - u_1\|_{1, \Omega_1} + \|u_2^n - u_2\|_{1, \Omega_2} &\leq C \|u_2^n - u_1^{n-1} - \epsilon_2^{n-1}\|_{H_0^{1/2}(\Gamma_2)} + C \|u_1^{n+1} - u_2^n - \epsilon_1^n\|_{H_0^{1/2}(\Gamma_1)} \\ &\leq C \left\{ \|u_1^{n+1} - u_2^n\|_{H_0^{1/2}(\Gamma_1)} + \|u_2^n - u_1^{n-1}\|_{H_0^{1/2}(\Gamma_2)} \right. \\ &\quad \left. + \|\epsilon_1^n\|_{H_0^{1/2}(\Gamma_1)} + \|\epsilon_2^{n-1}\|_{H_0^{1/2}(\Gamma_2)} \right\}, \end{aligned} \quad (2.23)$$

which is the required result. \square

3. A Posteriori Error Estimate in the Discrete Case

In this section, we consider the discretization of the variational problems (1.7) and (1.8). Let τ_h be a triangulation of Ω compatible with the decomposition and $V_h \subset V$ is the subspace of continuous functions (polynomials of degree k , $k \geq 1$) which vanish over $\partial\Omega$. We have

$$V_{i,h} = V_h|_{\Omega_i}, \quad W_{i,h} = V_h|_{\Gamma_i}. \quad (3.1)$$

$W_{i,h}$ is a subspace of $H_{00}^{1/2}(\Gamma_i)$ which consists of continuous piecewise polynomial functions on Γ_i which vanish at the end points of Γ_i ($i = 1, 2$).

Let $u_h \in V_h$ be the solution of the discrete problem associated with (1.3), $u_{i,h} = u_h|_{\Omega_i}$. We construct the sequences $(u_{i,h}^{n+1})_{i=1,2}$, $u_{i,h}^{n+1} \in V_{i,h}$ solutions of the discrete versions of (1.5) and (1.6) at the $(n+1)$ th iteration, that is,

$$\begin{aligned} -\Delta u_{1,h}^{n+1} &= f \quad \text{on } \Omega_1, \\ u_{1,h}^{n+1} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \end{aligned} \quad (3.2)$$

$$\frac{\partial u_{1,h}^{n+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{n+1} = \frac{\partial u_{2,h}^n}{\partial \eta_1} + \alpha_1 u_{2,h}^n \quad \text{on } \Gamma_1,$$

$$\begin{aligned} -\Delta u_{2,h}^{n+1} &= f \quad \text{on } \Omega_2, \\ u_{2,h}^{n+1} &= 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \end{aligned} \quad (3.3)$$

$$\frac{\partial u_{2,h}^{n+1}}{\partial \eta_2} + \alpha_2 u_{2,h}^{n+1} = \frac{\partial u_{1,h}^n}{\partial \eta_2} + \alpha_2 u_{1,h}^n \quad \text{on } \Gamma_2.$$

In a similar manner to that of Section 2, we introduce two discrete auxiliary problems, one over (Ω_1, Ω_3) , the other over (Ω_2, Ω_4) . We let

$$\begin{aligned} -\Delta u_{1,h}^{n+1} &= f \quad \text{in } \Omega_1, \\ u_{1,h}^{n+1} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \end{aligned} \quad (3.4)$$

$$\frac{\partial u_{1,h}^{n+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{n+1} = \frac{\partial u_{2,h}^n}{\partial \eta_1} + \alpha_1 u_{2,h}^n \quad \text{on } \Gamma_1,$$

$$\begin{aligned} -\Delta u_{3,h}^{n+1} &= f \quad \text{in } \Omega_3, \\ u_{3,h}^{n+1} &= 0 \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \end{aligned} \quad (3.5)$$

$$\frac{\partial u_{3,h}^{n+1}}{\partial \eta_3} + \alpha_3 u_{3,h}^{n+1} = \frac{\partial u_{1,h}^n}{\partial \eta_3} + \alpha_3 u_{1,h}^n \quad \text{on } \Gamma_1.$$

We can write $u_{2,h}^n = u_{3,h}^n + e_{1,h}^n$ on Γ_1 , (in the 3rd equation in (3.4)), that is, $e_{1,h}^n$ is the difference between the discrete overlapping and the nonoverlapping solutions $u_{2,h}^n$ and $u_{3,h}^n$ (in problems

(3.2)-(3.3) and (3.4)-(3.5), resp.) in Ω_3 . Because both $u_{2,h}^n$ and $u_{3,h}^n$ converges to u_2 , $e_{1,h}^n$ should tend to naught as n tends to infinity.

Similarly, over (Ω_2, Ω_4) we consider the sequences $(u_{2,h}^{n+1}, u_{4,h}^{n+1})_{n \in \mathbb{N}}$ solutions of the following discrete problems:

$$\begin{aligned} -\Delta u_{2,h}^{n+1} &= f \quad \text{in } \Omega_2, \\ u_{2,h}^{n+1} &= 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \frac{\partial u_{2,h}^{n+1}}{\partial \eta_2} + \alpha_2 u_{2,h}^{n+1} &= \frac{\partial u_{1,h}^n}{\partial \eta_2} + \alpha_2 u_{1,h}^n \quad \text{on } \Gamma_2, \\ -\Delta u_{4,h}^{n+1} &= f \quad \text{in } \Omega_4, \\ u_{4,h}^{n+1} &= 0 \quad \text{on } \partial\Omega_4 \cap \partial\Omega, \end{aligned} \quad (3.7)$$

$$\frac{\partial u_{4,h}^{n+1}}{\partial \eta_4} + \alpha_4 u_{4,h}^{n+1} = \frac{\partial u_{2,h}^n}{\partial \eta_4} + \alpha_4 u_{2,h}^n \quad \text{on } \Gamma_2,$$

where we can also write $u_{1,h}^n = u_{4,h}^n + e_{2,h}^n$ on Γ_2 , $e_{2,h}^n$ tends to naught as n tends to infinity as explained for the previous problems. We can obtain the discrete counterparts of Propositions 2.4 and 2.5 by doing almost the same analysis as in Section 2 (i.e., passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore

$$\begin{aligned} \|u_{1,h}^{n+1} - u_{1,h}\|_{1,\Omega_1} + \|u_{3,h}^n - u_{3,h}\|_{1,\Omega_3} &\leq C \|u_{1,h}^{n+1} - u_{3,h}^n\|_{W_{1,h}}, \\ \|u_{2,h}^{n+1} - u_{2,h}\|_{1,\Omega_2} + \|u_{4,h}^n - u_{4,h}\|_{1,\Omega_4} &\leq C \|u_{2,h}^{n+1} - u_{4,h}^n\|_{W_{2,h}}, \end{aligned} \quad (3.8)$$

where the discrete trace subspace norm $\|\cdot\|_{W_{i,h}}$ defined over Γ_i ($i = 1, 2$) is taken to be

$$\|w_h\|_{W_{i,h}}^2 = \int_{\Gamma_i} w_h^2 dx + \iint_{\Gamma_i} \frac{(w_h(x) - w_h(y))^2}{|x - y|^2} dx dy. \quad (3.9)$$

Hence, using the previous two inequalities, we have

$$\|u_{1,h}^{n+1} - u_{1,h}\|_{1,\Omega_1} + \|u_{2,h}^n - u_{2,h}\|_{1,\Omega_2} \leq C \left\{ \begin{aligned} &\|u_{2,h}^n - u_{1,h}^{n-1}\|_{W_{2,h}} + \|u_{1,h}^{n+1} - u_{2,h}^n\|_{W_{1,h}} \\ &+ \|e_{1,h}^n\|_{W_{1,h}} + \|e_{2,h}^{n-1}\|_{W_{2,h}} \end{aligned} \right\}. \quad (3.10)$$

We remark that (3.10) is the discrete version of (2.21).

Remark 3.1. Let us note that (3.10) is an estimate of the error between the approximate solution $u_{i,h}$ and the approximate solution of the discretized overlapping domain decomposition algorithm $u_{i,h}^{n+1}$, $i = 1, 2$.

Next we will obtain an estimate of the error between the approximate solution $u_{i,h}^{n+1}$ and the exact solution u . We introduce some necessary notations. We denote by

$$\varepsilon_h = \{E \text{ a side of } \tau_h; E \notin \partial\Omega\}. \quad (3.11)$$

For every $T \in \tau_h$ and $E \in \varepsilon_h$, we define

$$\omega_T = \cup\{T' \in \tau_h; T' \cap T \neq \emptyset\}, \quad \omega_E = \cup\{T' \in \tau_h; T' \cap E \neq \emptyset\}. \quad (3.12)$$

For a function f which is not necessarily continuous across two neighboring elements of τ_h having E as a common side, $[f]$ denotes the jump of f across E and η_E the normal vector of E .

We have the following theorem which gives the *a posteriori* error estimate for the discrete GODDM.

Theorem 3.2. *Let $u_i = u|_{\Omega_i}$ where u is the solution of problem (1.2), the sequences $(u_{1,h}^{n+1}, u_{2,h}^{n+1})_{n \in \mathbb{N}}$ are solutions of problems (3.2)-(3.3). Then there exists a constant C independent of h such that*

$$\|u_{1,h}^{n+1} - u_1\|_{1,\Omega_1} + \|u_{2,h}^n - u_2\|_{1,\Omega_2} \leq C \left\{ \sum_{i=1}^2 \left(\sum_{T \in \tau_h} (\eta_i^T)^2 + h_T^2 \|f - f_h\|_{0,T}^2 \right)^{1/2} + \sum_{i=1}^2 \eta_{\Gamma_i} \right\}, \quad (3.13)$$

where

$$\begin{aligned} \eta_{\Gamma_i} &= \|u_{i,h}^* - u_{j,h}^{*-1}\|_{W_{i,h}} + \|\epsilon_{i,h}^*\|_{W_{i,h}}, \quad i, j = 1, 2, i \neq j, \\ \eta_i^T &= h_T \|f_h + \Delta u_{i,h}^*\|_{0,T} + \sum_{E \in \varepsilon_h} h_E^{1/2} \left\| \left[\frac{\partial u_{i,h}^*}{\partial \eta_E} \right] \right\|_{0,E}. \end{aligned} \quad (3.14)$$

The symbol $*$ corresponds to $n+1$ when $i=1$ and to n when $i=2$.

Proof. The proof is based on the technique of the residual *a posteriori* estimation (see [13]) and Theorem 2.7. We give the main steps. By the triangle inequality we have

$$\sum_{i=1}^2 \|u_i - u_{i,h}^*\|_{1,\Omega_i} \leq \sum_{i=1}^2 \|u_i - u_{i,h}\|_{1,\Omega_i} + \sum_{i=1}^2 \|u_{i,h} - u_{i,h}^*\|_{1,\Omega_i}. \quad (3.15)$$

The second term on the right-hand side of (3.15) is bounded as in (3.10) by

$$\sum_{i=1}^2 \|u_{i,h} - u_{i,h}^*\|_{1,\Omega_i} \leq C \sum_{i=1}^2 \left(\|u_i^* - u_j^{*-1}\|_{W_{i,h}} + \|\epsilon_{i,h}^*\|_{W_{i,h}} \right). \quad (3.16)$$

To bound the first term on the r.h.s of (3.15) we use the residual equation and apply the technique of the residual *a posteriori* error estimation, see [13], to get for $v_h \in V_h$ and $v_{i,h} = v_h|_{\Omega_i}$

$$\begin{aligned} a_i(u_i - u_{i,h}, v) &= a_i(u_i - u_{i,h}, v - v_h) \\ &= \sum_{T \subset \Omega_i} \int (f_h + \Delta u_{i,h})(v_i - v_{i,h}) - \sum_{E \subset \Omega_i} \int_E \left[\frac{\partial u_{i,h}}{\partial \eta_E} \right] (v_i - v_{i,h}) ds \\ &\quad - \sum_{E \subset \Gamma_i} \int_E \frac{\partial u_{i,h}}{\partial \eta_E} (v_i - v_{i,h}) ds + \sum_{T \subset \Omega_i} (f - f_h, v_i - v_{i,h})_{\Omega_i} + \left\langle \frac{\partial u_i}{\partial \eta_i}, u - v_h \right\rangle_{\Gamma_i}, \end{aligned} \quad (3.17)$$

where f_h is any approximation of f by polynomials of degree at most k , therefore

$$\begin{aligned} \sum_{i=1}^2 a_i(u_i - u_{i,h}, v) &\leq \sum_{i=1}^2 \left\{ \begin{aligned} &\sum_{T \subset \Omega_i} \|f + \Delta u_{i,h}\|_{0,T} \|v_i - v_{i,h}\|_{0,T} \\ &+ \sum_{E \subset \Omega_i} \left\| \left[\frac{\partial u_{i,h}}{\partial \eta_E} \right] \right\|_{0,E} \|v_i - v_{i,h}\|_{0,E} \\ &+ \sum_{E \subset \Gamma_i} \left\| \frac{\partial u_{i,h}}{\partial \eta_E} \right\|_{0,E} \|v_i - v_{i,h}\|_{0,E} \\ &+ \sum_{T \subset \Omega_i} \|f - f_h\|_{0,T} \|v_i - v_{i,h}\|_{0,T} \\ &+ \left\| \frac{\partial u_i}{\partial \eta_i} \right\|_{0,\Gamma_i} \|u - v_h\|_{0,\Gamma_i} \end{aligned} \right\} \\ &\leq \sum_{i=1}^2 \left(\sum_{T \subset \Omega_i} (\eta_i^T)^2 + h_T^2 \|f - f_h\|_{0,T}^2 \right)^{1/2} \sum_{i=1}^2 |v_i|_{1,\Omega_i}. \end{aligned} \quad (3.18)$$

We use the fact that

$$\|u_i - u_{i,h}\|_{1,\Omega_i} \leq \sup_{v \in V_i} \frac{a_i(u_i - u_{i,h}, v)}{\|v\|_{1,\Omega_i}}. \quad (3.19)$$

Finally, by combining (3.15), (3.18), and (3.19) the required result follows. \square

3.1. Numerical Examples

As an illustration of the theoretical results obtained in this article and to be able to use the *a posteriori* error estimate (3.10), we consider the following problem

$$\begin{aligned} -\Delta u &= f \in L^2(\Omega), \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (3.20)$$

with $\Omega =]0, 1[\times]0, 1[$ for the first and second examples and $\Omega =]0, 1[\times]0, 1/2[\cup]0, 1/2[\times]0, 1[$ for the third example (i.e., an L -shaped region). The exact solution is taken to be

Table 1: Results on the square without *a posteriori* estimates as a stopping criteria for the first example.

h	1/8	1/16	1/32	1/64
Er ₁	1.7957693(-02)	8.1821001(-03)	4.5210766(-03)	3.1683034(-03)
Er ₂	1.0956427(-02)	9.6854433(-03)	8.7326244(-03)	6.9200024(-03)
Trer ₁	2.2513498(-02)	0.1504199(-02)	8.1897348(-03)	3.4817880(-02)
Trer ₂	1.7198801(-02)	2.1677755(-02)	9.3967836(-03)	6.9538120(-03)
Iterations	12	10	7	5
Quotient	0.7280898	0.7507522	0.7536285	0.9667203

$u(x, y) = xy(x-1)(y-1)e^{xy}$ for the first example, $u(x, y) = \sin 8\pi x \cdot \sin 8\pi y$ for the second example, and finally $u(x, y) = xy(x-1)(y-1)(x-1/2)(y-1/2)$ for the third example.

We first compute the bilinear Galerkin solution u_h over Ω and then apply the generalized overlapping domain decomposition method (1.7)-(1.8) to compute the bilinear sequences $u_{i,h}^{n+1}$ ($i = 1, 2$) to be able to look at the behavior of the constant C within (3.10) as the mesh size becomes small ($h^{-1} = 8, 16, 32, 64$). We take $\Omega_1 =]0, 3/4[\times]0, 1[$ and $\Omega_2 =]1/4, 1[\times]0, 1[$ for the first and second examples. As for the third example we have $\Omega_1 =]0, 1[\times]0, 1/2[$ and $\Omega_2 =]0, 1/2[\times]0, 1[$. We denote by

$$\begin{aligned}
 \text{Er}_i &= \left\| u_{i,h} - u_{i,h}^n \right\|_{1,\Omega_i}, \\
 \text{Trer}_1 &= \left\| u_{1,h}^{n+1} - u_{2,h}^n \right\|_{W_{1,h}}, \\
 \text{Trer}_2 &= \left\| u_{2,h}^n - u_{1,h}^{n-1} \right\|_{W_{2,h}}, \\
 \text{Quotient} &= \frac{(\text{Er}_1 + \text{Er}_2)}{(\text{Trer}_1 + \text{Trer}_2)}.
 \end{aligned} \tag{3.21}$$

The generalized overlapping domain decomposition method, with $\alpha_1 = \alpha_2 = 0.5$, converges. The iterations have been stopped when the relative error between two subsequent iterates is less than $\epsilon = 10^{-6}$.

The Quotient in Tables 1, 2, 3, 4, 5, and 6 represents an estimate of the constant C in (3.10). It happens to tend to a constant value around 1 which what we expect (i.e., we have asymptotic exactness). We present, for each example, results with and without the *a posteriori* estimate as a stopping criteria. We see that we obtain the same results but with less iterations when using the *a posteriori* estimate as a stopping criteria for a moderately small mesh size for the first and third examples. But for the second example, the difference in iteration count is clear between the two approaches.

3.2. Concluding Remarks

- (1) From Theorem 3.2 we see that the subdomain errors depend on the differences of the latter on the interface, on the data f , and on the approximate solution u^h . The first two terms of (3.13) are the contribution to the *a posteriori* error from the finite element discretization whereas the last term is the domain decomposition

Table 2: Results on the square with *a posteriori* estimates as a stopping criteria for the first example.

h	1/8	1/16	1/32	1/64
Er ₁	1.3457623(-02)	4.1221001(-03)	8.5210766(-03)	3.1683034(-03)
Er ₂	1.0556422(-02)	5.2454431(-03)	8.4322244(-03)	6.2200024(-03)
Trer ₁	2.2513498(-02)	5.1504199(-03)	2.1897348(-02)	3.4817880(-03)
Trer ₂	2.4423401(-02)	6.1677755(-03)	1.3967836(-03)	6.9538120(-03)
Iterations	3	5	6	7
Quotient	0.5116240	0.7276534	0.7277927	0.8996421

Table 3: Results on the square without *a posteriori* estimates as a stopping criteria for the second example.

h	1/8	1/16	1/32	1/64
Er ₁	1.7957693(-02)	2.5598288(-03)	4.7627655(-03)	0.8936322(-04)
Er ₂	1.0956427(-02)	4.5353208(-02)	8.1970235(-03)	6.3020810(-03)
Trer ₁	2.2513498(-02)	5.2993342(-02)	1.1897348(-02)	3.4817880(-03)
Trer ₂	1.7198801(-08)	6.0068501(-09)	1.3967836(-08)	6.9538120(-08)
Iterations	10	9	7	5
Quotient	0.7280898	0.9041330	1.089299	1.560316

Table 4: Results on the square with *a posteriori* estimates as a stopping criteria for the second example.

h	1/8	1/16	1/32	1/64
Er ₁	4.7345582(-02)	8.1821001(-04)	4.8510591(-03)	1.4817880(-02)
Er ₂	1.3897220(-02)	9.6854433(-02)	8.1924358(-03)	6.9200024(-03)
Trer ₁	9.6471474(-02)	0.1504199(-02)	1.1897348(-02)	1.4817880(-02)
Trer ₂	9.1859622(-09)	0.0000000(+00)	0.3967836(-09)	2.9538120(-08)
Iterations	8	6	5	4
Quotient	0.6348280	0.6493334	1.096336	1.018248

Table 5: Results on the *L*-shaped region without *a posteriori* estimates as a stopping criteria.

h	1/8	1/16	1/32	1/64
Er ₁	3.5915386(-02)	16.3642002(-03)	9.0421132(-03)	6.3366068(-03)
Er ₂	2.19128497(-02)	19.3708866(-03)	12.4732488(-03)	9.8440048(-03)
Trer ₁	4.50217996(-02)	0.2008398(-02)	4.3794696(-02)	2.9635760(-02)
Trer ₂	3.4397602(-02)	4.3355510(-02)	2.7935672(-03)	5.9076240(-03)
Iterations	10	8	6	6
Quotient	0.5443332	0.6562642	0.8380865	1.008344

Table 6: Results on the *L*-shaped region with *a posteriori* estimates as a stopping criteria.

h	1/8	1/16	1/32	1/64
Er ₁	2.5815386(-02)	14.3642002(-03)	9.0421132(-03)	4.3366122(-03)
Er ₂	2.19128487(-02)	12.3708866(-03)	12.4732488(-03)	6.8440223(-03)
Trer ₁	4.60217446(-02)	0.4008398(-02)	4.3794696(-02)	4.8635660(-03)
Trer ₂	3.3397402(-02)	2.3355510(-02)	2.7935672(-03)	6.8076220(-03)
Iterations	4	5	6	7
Quotient	0.6009663	0.9170200	0.8380865	0.9579689

error. Numerical results indicate that the finite element part of the error estimate dominates the domain decomposition part.

- (2) As far as the asymptotic exactness of the estimator based on (3.13) is concerned, Ainsworth and Oden in [12, Theorem 3.5] gave a necessary condition for this property to hold. This condition says that the local residuals in (3.13) should be computed exactly.
- (3) The a posteriori error analysis for a second order boundary value problem or Stokes problem is possible. Also, similar results can be obtained in the general case of more than two subdomains.

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