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Research Article

Some Oscillation Results for Linear Hamiltonian Systems

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The purpose of this paper is to develop a generalized matrix Riccati technique for the selfadjoint matrix Hamiltonian system $U' = A(t)U+B(t)V$, $V' = C(t)U-A^*(t)V$. By using the standard integral averaging technique and positive functionals, new oscillation and interval oscillation criteria are established for the system. These criteria extend and improve some results that have been required before. An interesting example is included to illustrate the importance of our results.

1. Introduction

In this paper, we consider oscillatory properties for the linear Hamiltonian system

$$\begin{aligned} U' &= A(t)U + B(t)V, \\ V' &= C(t)U - A^*(t)V, \quad t \geq t_0, \end{aligned} \tag{1.1}$$

where $A(t), B(t)$, and $C(t)$ are real $n \times n$ matrix-valued functions, B, C are Hermitian, and B is positive definite. By M^*, we mean the conjugate transpose of the matrix M, for any $n \times n$ Hermitian matrix M.

For any two solutions $(U_1(t), V_1(t))$ and $(U_2(t), V_2(t))$ of system (1.1), the Wronski matrix $U_1^*(t)V_2(t) - V_1^*(t)U_2(t)$ is a constant matrix. In particular, for any solution $(U(t), V(t))$ of system (1.1), $U^*(t)\mathrm{V}(t) - V^*(t)U(t)$ is a constant matrix.

A solution $(U(t), V(t))$ of system (1.1) is said to be nontrivial if $\det U(t) \neq 0$ is fulfilled for at least one $t \geq t_0$. A nontrivial solution $(U(t), V(t))$ of system (1.1) is said to be conjoined (prepared) if $U^*(t)V(t) - V^*(t)U(t) \equiv 0, t \geq t_0$. A conjoined solution $(U(t), V(t))$ of system

(1.1) is said to be a conjoined basis of system (1.1) if the rank of the $2n \times n$ matrix $(U(t), V(t))^T$ is n.

In 2000, Kumari and Umamaheswaram [1], Yang and Cheng [2], and Wang [3] used the substitution

$$W_1(x) = a(x)\left[V(x)U^{-1}(x) + f(x)E_n\right], \quad a(x) = \exp\left\{-2\int_{x_0}^{x} f(s)ds\right\}, \tag{1.2}$$

to study the oscillation of system (1.1). One of the main results in [1] is as follows.

Theorem A. *Let* $D = \{(x,s) \mid x_0 \le s \le x\}$ *and* $D_0 = \{(x,s) \mid x_0 \le s < x\}$. *Let the functions* $H \in C(D,\mathbb{R})$ *and* $h \in C(D_0,\mathbb{R})$ *satisfy the following three conditions:*

(i) $H(x,x) = 0$, *for* $x \ge x_0$, $H(x,s) > 0$ *on* D_0;

(ii) H *has a continuous and nonpositive partial derivative on* D_0 *with respect to the second variable;*

(iii) $-(\partial/\partial s)H(x,s) = h(x,s)\sqrt{H(x,s)}$, *for all* $(x,s) \in D_0$.

If there exists a function $f \in C^1[x_0,\infty)$ *such that*

$$\limsup_{x\to\infty} \frac{1}{H(x,x_0)}\lambda_1\left[\int_{x_0}^{x}\{H(x,s)T(s) + F(x,s)\}ds\right] = \infty, \tag{1.3}$$

where $T(x) = a(x)[-C - f(A + A^*) + f^2B - f'E_n](x)$, $a(x) = \exp\{-2\int_{x_0}^{x} f(s)ds\}$, E_n *is the* $n \times n$ *identity matrix, and*

$$\begin{aligned} F(x,s) = {} & H(x,s)\left[af(A + A^*) - aA^*B^{-1}A\right](s) \\ & - a(s)\left[\frac{1}{2}h(x,s)\sqrt{H(x,s)} + f(s)H(x,s)\right]\left[A^*B^{-1} + B^{-1}A\right](s) \\ & - a(s)\left[\left(\frac{1}{2}h(x,s) + f(s)\sqrt{H(x,s)}\right)B^{-1/2}(s) - f(s)\sqrt{H(x,s)B(s)}\right]^2, \end{aligned} \tag{1.4}$$

then, system (1.1) *is oscillatory.*

In 2003, Meng and Mingarelli [4], Wang [3], and Zheng and Zhu [5] studied the oscillation of system (1.1) by using the substitution

$$W_2(x) = a(x)\left[V(x)U^{-1}(x) + f(x)B^{-1}(x)\right], \quad a(x) = \exp\left\{-2\int_{x_0}^{x} f(s)ds\right\}. \tag{1.5}$$

One of the main results in [4] is as follows.

Theorem B. *Let the functions $H \in C(D, \mathbb{R})$ and $h \in C(D_0, \mathbb{R})$ satisfy (i)–(iii) in Theorem A and, for all sufficiently large $s \in \mathbb{R}, \liminf_{x\to\infty} H(x,s) \geq 1$. Assume that there exist a function $f \in C^1[x_0,\infty)$ and a monotone subhomogeneous functional q of degree c on $\mathcal{S}$ such that*

$$\limsup_{x\to\infty} \frac{1}{H(x,x_0)^c} q\left[\int_{x_0}^{x} \left\{ H(x,s)R_1(s) + \frac{1}{4}a(s)h^2(x,s)B_1^{-1}(s) \right\} ds\right] = \infty, \tag{1.6}$$

where $R_1(x) = \phi^*(x)R(x)\phi(x), B_1(x) = \phi^{-1}(x)B(x)[\phi^*(x)]^{-1}$, $\phi(x)$ is a fundamental matrix of the linear equation $v' = A(x)v$, and

$$R(x) = a(x)\left[-C - f\left(A^*B^{-1} + B^{-1}A\right) + f^2B^{-1} - \left(fB^{-1}\right)'\right](x). \tag{1.7}$$

Then, system (1.1) *is oscillatory.*

In 2004, Sun and Meng [6] also studied the oscillation of system (1.1). One of the main results in [6] is as follows.

Theorem C. *Let H, h be as in Theorem A, and suppose that*

$$0 < \inf_{s\geq t_0}\left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq +\infty. \tag{1.8}$$

If there exist a function $f \in C^1[t_0,\infty)$ and a positive linear functional g on $\mathbb{R}$ such that

$$\begin{gathered}\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} g\left[-H(t,s)\left(C_1 + A^*B_1^{-1}A + \left(B_1^{-1}A\right)'\right)(s)\right] ds > -\infty, \\ \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} h^2(t,s) g\left[B_1^{-1}(s)\right] ds < \infty,\end{gathered} \tag{1.9}$$

and suppose also that there exists a function $m \in C[t_0,\infty)$ such that

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} g\left[H(t,s)\left(C_1 + A^*B_1^{-1}A + \left(B_1^{-1}A\right)'\right)(s) - \frac{1}{4}h^2(t,s)B_1^{-1}(s)\right] ds, \\ &\quad \geq m(T),\end{aligned} \tag{1.10}$$

for all $T \geq t_0$ and

$$\int_{t_0}^{\infty} \frac{m_+^2(t)}{g\left[B_1^{-1}(t)\right]} dt = +\infty, \tag{1.11}$$

where $m_+(t) = \max\{m(t), 0\}$ and $B_1(t), C_1(t)$ are the same as in Theorem A, then, the system (1.1) *is oscillatory.*

Recently, Li et al. [7] also studied the oscillation of system (1.1) by using the standard integral averaging technique and the substitution

$$W_3(t) = -a(t)\left[Y(t)X^{-1}(t) + f(t)B^{-1}(t)\right], \quad t \geq t_0, \tag{1.12}$$

where $a(t)$ is as in (1.5). One of the main results in [7] is as follows.

Theorem D. *Let H, h be as in Theorem A, and suppose that there exist a function $f \in C^1[t_0, \infty)$ and a positive linear functional g on $\mathbb{R}$, for some $\beta \geq 1$, such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} g\left[-H(t,s)\left(C_1 + A^* B_1^{-1} A + \left(B_1^{-1}A\right)'\right)(s) - \frac{\beta}{4} h^2(t,s) B_1^{-1}(s)\right] ds = \infty, \tag{1.13}$$

where

$$\begin{gathered} B_1(t) = a^{-1}(t)B(t), \quad a(t) = \exp\left\{-2\int^t f(s)ds\right\}, \\ C_1(t) = a(t)\left\{C(t) + f(t)\left[B^{-1}A + A^*B^{-1}\right](t) + \left[f(t)B^{-1}(t)\right]' - f^2(t)B^{-1}(t)\right\}. \end{gathered} \tag{1.14}$$

Then, system (1.1) *is oscillatory.*

The purpose of this paper is further to improve Theorems A, B, C, and D as well as other related results regarding the oscillation of the system (1.1), by refining the standard integral averaging technique and Riccati transformation.

Now we use the general weighted functions from the class $\mathscr{H}$. Let $D = \{(t,s) \mid t_0 < s \leq t < +\infty\}$ and $D_0 = \{(t,s) \mid t_0 < s < t < +\infty\}$. We say that a continuous function $H(t,s) : D \to \mathbb{R}_+$ belongs to the class $\mathscr{H}$ if

(i) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on D_0,

(ii) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable,

(iii) $-(\partial/\partial s)(H(t,s)k(s)) = h(t,s)\sqrt{H(t,s)k(s)}$, for all $(t,s) \in D_0$, where $k(t) \in C^1([t_0,+\infty),(0,+\infty))$.

We now follow [8] in defining the space $\mathcal{S}$ as the real linear spare of all real symmetric $n \times n$ matrices. Let g be a linear functional on $\mathbb{R}$, g is said to be positive if $g(A) > 0$ whenever $A \in \mathcal{S}$ and $A > 0$.

2. Main Results

In this paper, we need the following lemma.

Lemma 2.1 (see [6]). *If g is a positive linear functional on $\mathbb{R}$, then, for all $A, B \in \mathbb{R}$, one has*

$$\left|g[A^*B]\right|^2 \leq g[A^*A]g[B^*B]. \tag{2.1}$$

Theorem 2.2. *Let $H(t,s) \in \mathcal{H}$. If there exist a function $b \in C^1([t_0,+\infty),\mathbb{R}_+)$, a matrix function $\varphi \in C^1([t_0,+\infty),\mathcal{S})$, and a positive linear functional g on $\mathbb{R}$, for some $\alpha \geq 1$, such that*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} g\left\{-H(t,s)k(s)T_1(s) - \frac{\alpha}{4}B_1^{-1}(s)T_2(t,s)^2\right\}ds = \infty, \tag{2.2}$$

where $B_1(t) = (1/b(t))B(t), D(t) = A(t) - b(t)B_1(t)\varphi(t), F_1(s) = b(s)[C + A^*\varphi + \varphi A - \varphi B\varphi + \varphi'](s), T_1(s) = [F_1 + (B_1^{-1}D)' + D^*B_1^{-1}D - (b'/b)B_1^{-1}D](s)$, and $T_2(t,s) = h(t,s) - \sqrt{H(t,s)k(s)}(b'(s)/b(s))$, then, system (1.1) is oscillatory.

Proof. Assume to the contrary that system (1.1) is nonoscillatory. Then, there exists a nontrivial prepared solution of $(U(t),V(t))$ such that $U(t)$ is nonsingular for all sufficiently large t. Without loss of generality, we assume that $\det U(t) \neq 0$ for all $t \geq t_0$. This allows us to make a Riccati transformation

$$W(t) = -b(t)\left[V(t)U^{-1}(t) + \varphi(t)\right], \tag{2.3}$$

for all $t \geq t_0$. Then, $W(t)$ is well defined, Hermitian, and solves the Riccati equation

$$W'(t) - \frac{b'(t)}{b(t)}W(t) + W^*(t)(A - B\varphi)(t) + (A - B\varphi)^*(t)W(t) - \frac{1}{b(t)}W^*(t)B(t)W(t) + F_1(t) = 0, \tag{2.4}$$

on $[t_0,\infty)$.

Let $B_1(t) = (1/b(t))B(t), D(t) = A(t) - b(t)B_1(t)\varphi(t)$. So, from (2.4), we have

$$W'(t) - \frac{b'(t)}{b(t)}W(t) + W^*(t)D(t) + D^*(t)W(t) - W^*(t)B_1(t)W(t) + F_1(t) = 0. \tag{2.5}$$

Now by the substitution $P(t) = W(t) - B_1^{-1}(t)D(t)$ in (2.5), we obtain

$$P'(t) - \frac{b'(t)}{b(t)}P(t) - P^*(t)B_1(t)P(t) + T_1(t) = 0. \tag{2.6}$$

By rearranging the terms, we get

$$T_1(t) = -P'(t) + \frac{b'(t)}{b(t)}P(t) + P^*(t)B_1(t)P(t). \tag{2.7}$$

Multiplying (2.7), with t replaced by s, by $H(t,s)k(s)$ and integrating from t_0 and t, we obtain

$$\begin{aligned}
&-\int_{t_0}^{t} H(t,s)k(s)T_1(s)ds \\
&\quad= \int_{t_0}^{t} H(t,s)k(s)\left[P'(s) - \frac{b'(s)}{b(s)}P(s) - P^{*}(s)B_1(s)P(s)\right]ds \\
&\quad= -H(t,t_0)k(t_0)P(t_0) + \int_{t_0}^{t} P(s)\left[h(t,s)\sqrt{H(t,s)k(s)} - H(t,s)k(s)\frac{b'(s)}{b(s)}\right]ds \\
&\qquad- \int_{t_0}^{t} H(t,s)k(s)P^{*}(s)B_1(s)P(s)ds.
\end{aligned} \tag{2.8}$$

Taking the linear functional g on both sides of the above equation, we have, for some $\alpha \geq 1$,

$$\begin{aligned}
&\int_{t_0}^{t} g\{-H(t,s)k(s)T_1(s)\}ds \\
&\quad= -H(t,t_0)k(t_0)g[P(t_0)] + \int_{t_0}^{t} g[P(s)]\left[h(t,s)\sqrt{H(t,s)k(s)} - H(t,s)k(s)\frac{b'(s)}{b(s)}\right]ds \\
&\qquad- \int_{t_0}^{t} H(t,s)k(s)g[P^{*}(s)B_1(s)P(s)]ds \\
&\quad\leq -H(t,t_0)k(t_0)g[P(t_0)] + \int_{t_0}^{t} g[P(s)]\left[h(t,s)\sqrt{H(t,s)k(s)} - H(t,s)k(s)\frac{b'(s)}{b(s)}\right]ds \\
&\qquad- \int_{t_0}^{t} H(t,s)k(s)\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds \\
&\quad= -H(t,t_0)k(t_0)g[P(t_0)] \\
&\qquad- \int_{t_0}^{t}\left[\frac{\sqrt{H(t,s)k(s)}}{\sqrt{\alpha g[B_1^{-1}(s)]}}g[P(s)] - \frac{\sqrt{\alpha g[B_1^{-1}(s)]}}{2}\left(h(t,s) - \sqrt{H(t,s)k(s)}\frac{b'(s)}{b(s)}\right)\right]^2 ds \\
&\qquad+ \frac{\alpha}{4}\int_{t_0}^{t} g\left[B_1^{-1}(s)\right]\left(h(t,s) - \sqrt{H(t,s)k(s)}\frac{b'(s)}{b(s)}\right)^2 ds \\
&\qquad- \frac{\alpha-1}{\alpha}\int_{t_0}^{t} H(t,s)k(s)\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds \\
&\quad\leq -H(t,t_0)k(t_0)g[P(t_0)] + \frac{\alpha}{4}\int_{t_0}^{t} g\left[B_1^{-1}(s)\right]T_2(t,s)^2 ds.
\end{aligned} \tag{2.9}$$

So,

$$\int_{t_0}^{t} g\left\{-H(t,s)k(s)T_1(s) - \frac{\alpha}{4}B_1^{-1}(s)T_2(t,s)^2\right\}ds \leq -H(t,t_0)k(t_0)g[P(t_0)]. \tag{2.10}$$

Taking the upper limit in both sides of (2.10) as $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} g\left\{-H(t,s)k(s)T_1(s) - \frac{\alpha}{4} B_1^{-1}(s) T_2(t,s)^2\right\} ds \le -k(t_0) g[P(t_0)], \tag{2.11}$$

which contradicts (2.2). This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let the functions H, h and b, g be as in Theorem 2.2, and suppose that*

$$0 < \inf_{s\ge t_0}\left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \le +\infty. \tag{2.12}$$

If there exists a function $\phi \in C([t_0,\infty))$, such that, for all $t \ge T \ge t_0$, and for some $\alpha \ge 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} g\left\{-H(t,s)k(s)T_1(s) - \frac{\alpha}{4} B_1^{-1}(s) T_2(t,s)^2\right\} ds \ge \phi(T), \tag{2.13}$$

$$\int_{t_0}^{\infty} \frac{\phi_+^2(t)}{g\left[B_1^{-1}(t)\right] k^2(t)} dt = +\infty, \tag{2.14}$$

where $\phi_+(t) = \max\{\phi(t), 0\}, B_1(t), F_1(t), D(t), T_1(t)$, and $T_2(t,s)$ are the same as in Theorem 2.2, then, system (1.1) *is oscillatory.*

Proof. Assume to the contrary that system (1.1) is nonoscillatory. Similar to the proof of Theorem 2.2, we can obtain, for all $t \ge T \ge t_0$, and for some $\alpha \ge 1$,

$$\begin{aligned} &\frac{1}{H(t,T)} \int_{T}^{t} g\left\{-H(t,s)k(s)T_1(s) - \frac{\alpha}{4} B_1^{-1}(s) T_2(t,s)^2\right\} ds \\ &\quad \le -k(T) g[P(T)] - \frac{\alpha-1}{\alpha} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)k(s) \left\{ g\left[B_1^{-1}(s)\right]\right\}^{-1} \{g[P(s)]\}^2 ds. \end{aligned} \tag{2.15}$$

Taking the upper limit of the above inequation as $t \to \infty$,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} g\left\{-H(t,s)k(s)T_1(s) - \frac{\alpha}{4} B_1^{-1}(s) T_2(t,s)^2\right\} ds \\ &\quad \le -k(T) g[P(T)] - \frac{\alpha-1}{\alpha} \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)k(s) \left\{ g\left[B_1^{-1}(s)\right]\right\}^{-1} \{g[P(s)]\}^2 ds. \end{aligned} \tag{2.16}$$

By (2.13), we obtain

$$-k(T) g[P(T)] \ge \phi(T) + \frac{\alpha-1}{\alpha} \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)k(s) \left\{ g\left[B_1^{-1}(s)\right]\right\}^{-1} \{g[P(s)]\}^2 ds, \tag{2.17}$$

$$-k(T)g[P(T)] \geq \phi(T). \tag{2.18}$$

Besides, we have

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)k(s)\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds \\
&\quad\leq -\frac{\alpha}{\alpha-1}\left[\phi(t_0)+k(t_0)g[P(t_0)]\right] < \infty.
\end{aligned} \tag{2.19}$$

Now, we claim that

$$\int_{t_0}^{+\infty}\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds < \infty. \tag{2.20}$$

Suppose to the contrary that

$$\int_{t_0}^{+\infty}\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds = +\infty. \tag{2.21}$$

By (2.12), there exists a positive constant ε satisfying

$$\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\} > \varepsilon > 0. \tag{2.22}$$

And according to the above ε, there exists $t_1 \geq t_0$ such that

$$\int_{t_0}^{t}\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds > \frac{1}{\varepsilon^2}, \quad t \geq t_1. \tag{2.23}$$

Thus,

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)k(s)\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds \\
&\quad= \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)k(s)d\left(\int_{t_0}^{s}\left\{g\left[B_1^{-1}(\xi)\right]\right\}^{-1}\{g[P(\xi)]\}^2 d\xi\right) \\
&\quad= -\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\partial(H(t,s)k(s))}{\partial s}\int_{t_0}^{s}\left\{g\left[B_1^{-1}(\xi)\right]\right\}^{-1}\{g[P(\xi)]\}^2 d\xi\, ds \\
&\quad> \frac{1}{\varepsilon^2}\frac{1}{H(t,t_0)}\int_{t_1}^{t}-\frac{\partial(H(t,s)k(s))}{\partial s}ds \\
&\quad= \frac{k(t_1)}{\varepsilon^2}\frac{H(t,t_1)}{H(t,t_0)}.
\end{aligned} \tag{2.24}$$

From (2.22), there exists a $t_2 \geq t_1$ such that, for all $t \geq t_2$,

$$\frac{H(t,t_1)}{H(t,t_0)} > \varepsilon. \tag{2.25}$$

So,

$$\frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)k(s)\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds > \frac{k(t_1)}{\varepsilon}. \tag{2.26}$$

Since ε is arbitrary, we get

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)k(s)\left\{g\left[B_1^{-1}(s)\right]\right\}^{-1}\{g[P(s)]\}^2 ds = \infty, \tag{2.27}$$

which contradicts (2.19). So, (2.20) holds; then, by (2.18) and (2.20), we can obtain

$$\int_{t_0}^{\infty} \frac{\phi_+^2(t)}{g[B_1^{-1}(t)]k^2(t)} dt \leq \int_{t_0}^{\infty} \frac{g[P(t)]^2}{g[B_1^{-1}(t)]} dt < \infty, \tag{2.28}$$

which contradicts (2.14). This completes our proof of Theorem 2.3. □

Example 2.4. Consider the linear Hamiltonian system (1.1), where $B(t) = tI_2, C(t) = -((1/t)\cos t + (3/4t^3))I_2, A(t) = \begin{pmatrix} 0 & 1/t \\ -1/t & 0 \end{pmatrix}$ are 2×2-matrices and B, C are Hermitian.

Let $H(t,s) = (t-s)^2, h(t,s) = 2, b(t) = t, \varphi(t) = -(1/2t^2)I_2$, and $g[A] = a_{11}$, where $A = (a_{ij})$ is a 2×2-matrix. Then, $\lim_{t\to\infty}(H(t,s)/H(t,t_0)) = (t-s)^2/(t-t_0)^2 = 1, B_1(t) = I_2, D(t) = \begin{pmatrix} 1/2t & 1/t \\ -1/t & 1/2t \end{pmatrix}, F_1(t) = -\cos t I_2, T(t) = \begin{pmatrix} 1/4t^2-\cos t & -2/t^2 \\ 2/t^2 & 1/4t^2-\cos t \end{pmatrix}$, $\limsup_{t\to\infty}(1/t^2)\int_T^t g\{-(t-s)^2 T(s) - (\alpha/4)B_1^{-1}(s)[2-(t-s)(1/s)]^2\}ds > 1/\sqrt{T} \doteq \phi(T)$, and $\int_{t_0}^{\infty}(\phi_+^2(t)/g[B_1^{-1}(t)]k^2(t))dt = \int_{t_0}^{\infty}(1/t)dt = \infty$. According to Theorem 2.3, we get that this linear system is oscillatory.

Remark 2.5. In Theorem 2.2, let $b(t) = \exp\{-2\int^t f(s)ds\}, \varphi(t) = f(t)B^{-1}(t), k(t) = 1$. Theorem 2.2 reduces to Theorem D. In Theorem 2.3, we obtain the same result in which we remove the two assumptions (1.9) in Theorem C. Therefore, Theorems 2.2 and 2.3 are generalizations and improvements of [7, Theorem 2.1] and [6, Theorem 3].

Remark 2.6. The above theorems give rather wide possibilities of deriving different explicit oscillation criteria for system (1.1) with appropriate choices of the functions $H(t,s), k(s)$, and $f(s)$. For example, we can obtain some useful oscillation criteria if we choose $H(t,s) = (x-s)^m, [\ln(x/s)]^m, [\int_s^x dz/\theta(z)]^m$, or $\rho(x-s)$, and so forth.

3. Interval Oscillation Criteria

Now we establish interval oscillation criteria of system (1.1), that is, criteria given by the behavior of system (1.1) only on a sequence of subinterval of $[t_0, \infty)$. We assume that a function $H = H(t,s)$ satisfying (i). Further, we assume that $k(t) = 1$ and $H(t,s)$ has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial}{\partial t}H(t,s) = h_1(t,s)\sqrt{H(t,s)}, \tag{3.1}$$

$$\frac{\partial}{\partial s}H(t,s) = -h_2(t,s)\sqrt{H(t,s)}, \tag{3.2}$$

where $h_1, h_2 \in L_{\text{loc}}(D,\mathbb{R})$.

We first prove two lemmas.

Lemma 3.1. *Suppose that $(U(t),V(t))$ is a nontrivial prepared solution of system* (1.1) *such that* $\det U(t) \neq 0$ *on $(a_1,a_2] \subset [t_0,\infty)$. Then, for any $b(t) \in C^1([t_0,\infty),\mathbb{R}_+)$, matrix function $\varphi \in C^1([t_0,\infty),\mathcal{S})$, H satisfies (i),* (3.1) *and* (3.2)*, and a positive linear functional g on $\mathbb{R}$, one has, for some $\alpha \geq 1$,*

$$\begin{aligned}
&\frac{1}{H(a_2,a_1)}\int_{a_1}^{a_2} g\left\{-H(t,a_1)T_1(s) - \frac{\alpha}{4}B_1^{-1}(t)\left(h_1(t,a_1) + \sqrt{H(t,a_1)}\frac{b'(t)}{b(t)}\right)^2\right\}dt \\
&\quad \leq g[P(a_2)],
\end{aligned} \tag{3.3}$$

where $W(t)$ is defined by (2.3) *on $(a_1,a_2]$, $B_1(t), D(t), F_1(s)$, and $T_1(s)$ are the same as in Theorem 2.2.*

Proof. Since $(U(t),V(t))$ is a nontrivial prepared solution of system (1.1) such that $U(t)$ is nonsingular on $(a_1,a_2]$, then, $W(t)$ by (2.3) is well defined and solves the Riccati equation (2.7) on $(a_1,a_2]$.

On multiplying (2.7) by $H(t,s)$ and integrating with respect to t from s to a_2 for $s \in (a_1,a_2]$, we can find

$$\begin{aligned}
&-\int_s^{a_2} H(t,s)T_1(t)dt \\
&\quad = \int_s^{a_2} H(t,s)P'(t)dt - \int_s^{a_2} H(t,s)\frac{b'(t)}{b(t)}P(t)dt - \int_s^{a_2} H(t,s)P^*(t)B_1(t)P(t)dt \\
&\quad = H(a_2,s)P(a_2) - \int_s^{a_2} P(t)\left(h_1(t,s)\sqrt{H(t,s)} + H(t,s)\frac{b'(t)}{b(t)}\right)dt \\
&\qquad - \int_s^{a_2} H(t,s)P^*(t)B_1(t)P(t)dt.
\end{aligned} \tag{3.4}$$

Taking the linear functional g on both sides of the above equation, we have, for some $\alpha \geq 1$,

$$\begin{aligned}
&\int_s^{a_2} g\{-H(t,s)T_1(t)\}dt \\
&\quad = H(a_2,s)g[P(a_2)] - \int_s^{a_2} g[P(t)]\left(h_1(t,s)\sqrt{H(t,s)} + H(t,s)\frac{b'(t)}{b(t)}\right)dt \\
&\qquad - \int_s^{a_2} H(t,s)g[P^*(t)B_1(t)P(t)]dt \\
&\quad \leq H(a_2,s)g[P(a_2)] - \int_s^{a_2} g[P(t)]\left(h_1(t,s)\sqrt{H(t,s)} + H(t,s)\frac{b'(t)}{b(t)}\right)dt \\
&\qquad - \int_s^{a_2} H(t,s)\left\{g\left[B_1^{-1}(t)\right]\right\}^{-1}\{g[P(t)]\}^2 dt \\
&\quad = H(a_2,s)g[P(a_2)] \\
&\qquad - \int_s^{a_2}\left[\frac{\sqrt{H(t,s)}}{\sqrt{\alpha g[B_1^{-1}(t)]}}g[P(t)] + \frac{\sqrt{\alpha g[B_1^{-1}(t)]}}{2}\left(h_1(t,s) + \sqrt{H(t,s)}\frac{b'(t)}{b(t)}\right)\right]^2 dt \\
&\qquad + \frac{\alpha}{4}\int_s^{a_2} g\left[B_1^{-1}(t)\right]\left(h_1(t,s) + \sqrt{H(t,s)}\frac{b'(t)}{b(t)}\right)^2 dt \\
&\qquad - \frac{\alpha-1}{\alpha}\int_s^{a_2} H(t,s)\left\{g\left[B_1^{-1}(t)\right]\right\}^{-1}\{g[P(t)]\}^2 dt \\
&\quad \leq H(a_2,s)g[P(a_2)] + \frac{\alpha}{4}\int_s^{a_2} g\left[B_1^{-1}(t)\right]\left(h_1(t,s) + \sqrt{H(t,s)}\frac{b'(t)}{b(t)}\right)^2 dt.
\end{aligned} \tag{3.5}$$

That is,

$$\begin{aligned}
&\frac{1}{H(a_2,s)}\int_s^{a_2} g\left\{-H(t,s)T_1(t) - \frac{\alpha}{4}B_1^{-1}(t)\left(h_1(t,s) + \sqrt{H(t,s)}\frac{b'(t)}{b(t)}\right)^2\right\}dt \\
&\qquad \leq g[P(a_2)].
\end{aligned} \tag{3.6}$$

Let $s \to a_1$,

$$\begin{aligned}
&\frac{1}{H(a_2,a_1)}\int_{a_1}^{a_2} g\left\{-H(t,a_1)T_1(t) - \frac{\alpha}{4}B_1^{-1}(t)\left(h_1(t,a_1) + \sqrt{H(t,a_1)}\frac{b'(t)}{b(t)}\right)^2\right\}dt \\
&\qquad \leq g[P(a_2)].
\end{aligned} \tag{3.7}$$

□

Lemma 3.2. *Suppose that* $(U(t), V(t))$ *is a nontrivial prepared solution of system* (1.1) *such that* $\det U(t) \neq 0$ *on* $(a_2, a_3] \subset [t_0, \infty)$. *Then, for any* $b(t) \in C^1([t_0,\infty), \mathbb{R}_+)$, *matrix function*

$\varphi \in C^1([t_0,\infty),\mathcal{S})$, H satisfies (i), (3.1) *and* (3.2)*, and a positive linear functional g on $\mathbb{R}$, one has, for some $\alpha \geq 1$,*

$$\begin{aligned}&\frac{1}{H(a_3,a_2)}\int_{a_2}^{a_3} g\left\{-H(a_3,s)T_1(s)-\frac{\alpha}{4}B_1^{-1}(s)\left(h_1(a_3,s)+\sqrt{H(a_3,s)}\frac{b'(s)}{b(s)}\right)^2\right\}ds\\&\quad\leq -g[P(a_2)],\end{aligned}\tag{3.8}$$

where $W(t)$ is defined by (2.3) *on $(a_2,a_3]$, $B_1(t), D(t), F_1(s)$, and $T_1(s)$ are the same as in Theorem 2.2.*

Proof. Since $(U(t),V(t))$ is a nontrivial prepared solution of system (1.1) such that $U(t)$ is nonsingular on $(a_2,a_3]$, then, $W(t)$ by (2.3) is well defined and solves the Riccati equation (2.7) on $(a_2,a_3]$.

On multiplying (2.7) by $H(t,s)$, integrating with respect to s from a_2 to t for $t\in(a_2,a_3]$, and following the proof of Lemma 3.1, we can find

$$\begin{aligned}&\frac{1}{H(t,a_2)}\int_{a_2}^{t} g\left\{-H(t,s)T_1(s)-\frac{\alpha}{4}B_1^{-1}(s)\left(h_1(t,s)+\sqrt{H(t,s)}\frac{b'(s)}{b(s)}\right)^2\right\}ds\\&\quad\leq -g[P(a_2)]<+\infty.\end{aligned}\tag{3.9}$$

Let $t\rightarrow a_3$,

$$\begin{aligned}&\frac{1}{H(a_3,a_2)}\int_{a_2}^{a_3} g\left\{-H(a_3,s)T_1(s)-\frac{\alpha}{4}B_1^{-1}(s)\left(h_1(a_3,s)+\sqrt{H(a_3,s)}\frac{b'(s)}{b(s)}\right)^2\right\}ds\\&\quad\leq -g[P(a_2)].\end{aligned}\tag{3.10}$$

□

Theorem 3.3. *Suppose that there exist some $a_2\in(a_1,a_3)\subset[t_0,\infty)$, $b(t)\in C^1([t_0,\infty),\mathbb{R}_+)$, matrix function $\varphi\in C^1([t_0,\infty),\mathcal{S})$, H satisfies (i),* (3.1) *and* (3.2)*, and a positive linear functional g on $\mathbb{R}$ such that, for some $\alpha\geq 1$,*

$$\begin{aligned}&g\left\{\frac{1}{H(a_2,a_1)}\int_{a_1}^{a_2}-H(t,a_1)T_1(t)-\frac{\alpha}{4}B_1^{-1}(t)\left(h_1(t,a_1)+\sqrt{H(t,a_1)}\frac{b'(t)}{b(t)}\right)^2dt\right.\\&\left.\quad+\frac{1}{H(a_3,a_2)}\int_{a_2}^{a_3}-H(a_3,s)T_1(s)-\frac{\alpha}{4}B_1^{-1}(s)\left(h_1(a_3,s)+\sqrt{H(a_3,s)}\frac{b'(s)}{b(s)}\right)^2ds\right\}>0,\end{aligned}\tag{3.11}$$

where $B_1(t), D(t), F_1(s)$, and $T_1(s)$ are defined as in Theorem 2.2. Then, for any nontrivial prepared solution $(U(t),V(t))$ of system (1.1)*, $\det U(t)$ has at least one zero in (a_1,a_3).*

Theorem 3.4. *If, for each $T\geq t_0$, there exist $b(t)\in C^1([t_0,\infty),\mathbb{R}_+)$, matrix function $\varphi\in C^1([t_0,\infty),\mathcal{S})$, H satisfies (i),* (3.1), (3.2)*, a positive linear functional g on $\mathbb{R}$ and $a_1,a_2,a_3\in\mathbb{R}$, such that $T\leq a_1<a_2<a_3$ and condition* (3.1) *holds, where $B_1(t), D(t), F_1(s)$, and $T_1(t)$ are defined as in Theorem 2.2, then, system* (1.1) *is oscillatory.*

In conclusion, we note that the results given here can extend, improve and complement Theorems A–D, and deal with some cases not covered by known criteria by choosing the functions H, b, ϕ, and g. From our results, we can derive a number of easily verifiable oscillation criteria.

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