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Research Article

Higher-Order Dynamic Delay Differential Equations on Time Scales

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We study the existence of positive solutions for the nonlinear four-point singular boundary value problem with higher-order p-Laplacian dynamic delay differential equations on time scales, subject to some boundary conditions. By using the fixed-point index theory, the existence of positive solution and many positive solutions for nonlinear four-point singular boundary value problem with p-Laplacian operator are obtained.

1. Introduction

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [1] and is a new area of still fairly theoretical exploration in mathematics. Boundary value problems for delay differential equations arise in a variety of areas of applied mathematics, physics, and variational problems of control theory (see [2, 3]). In recent years, many authors have begun to pay attention to the study of boundary value problems or with p-Laplacian equations or with p-Laplacian dynamic equations on time scales (see [4–19] and the references therein).

In [7], Sun and Li considered the existence of positive solution of the following dynamic equations on time scales:

$$\begin{aligned} &u^{\Delta\nabla}(t) + a(t)f(t,u(t)) = 0, \quad t \in (0,T), \\ &\beta u(0) - \gamma u^{\Delta}(0) = 0, \qquad \alpha u(\eta) = u(T), \end{aligned} \tag{1.1}$$

where $\beta, \gamma \geq 0$, $\beta + \gamma > 0$, $\eta \in (0, \rho(T))$, $0 < \alpha < T/\eta$. They obtained the existence of single and multiple positive solutions of the problem (1.1) by using fixed point theorem and Leggett-Williams fixed point theorem, respectively.

In [10], Avery and Anderson discussed the following dynamic equation on time scales:

$$\begin{gathered} u^{\Delta\nabla}(t) + a(t) f(u(t)) = 0, \quad t \in (0, T), \\ u(0) = 0, \qquad \alpha u(\eta) = u(T). \end{gathered} \tag{1.2}$$

He obtained some results for the existence of one positive solution of the problem (1.2) based on the limits $f_0 = \lim_{u \to 0^+}(f(u)/u)$ and $f_\infty = \lim_{u \to \infty}(f(u)/u)$.

In [11], Wang et al. discussed the following dynamic equation by using Avery-Peterson fixed theorem (see [10]):

$$(\phi_p(u'))' + q(t) f(t, u(t), u(t-1), u'(t)) = 0, \quad t \in (0,1), \tag{1.3}$$

$$u(t) = \xi(t), \quad -1 \leq t \leq 0, \ u(1) = 0, \tag{1.4}$$

$$u(t) = \xi(t), \quad -1 \leq t \leq 0, \ u'(1) = 0. \tag{1.5}$$

They obtained some results for the existence, three positive solutions of the problem (1.3), (1.4) and (1.3), (1.5), respectively.

However, there are not many concerning the p-Laplacian problems on time scales. Especially, for the singular multi point boundary value problems for higher-order p-Laplacian dynamic delay differential equations on time scales, with the author's acknowledg, no one has studied the existence of positive solutions in this case.

Recently, in [16], we study the existence of positive solutions for the following nonlinear two-point singular boundary value problem with p-Laplacian operator

$$\begin{gathered} (\phi_p(u'))' + a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ \alpha\phi_p(u(0)) - \beta\phi_p(u'(0)) = 0, \qquad \gamma\phi_p(u(1)) + \delta\phi_p(u'(1)) = 0, \end{gathered} \tag{1.6}$$

by using the fixed point theorem of cone expansion and compression of norm type, the existence of positive solution and infinitely many positive solutions for nonlinear singular boundary value problem (1.6) with p-Laplacian operator are obtained.

Now, motivated by the results mentioned above, in this paper, we study the existence of positive solutions for the following nonlinear four-point singular boundary value problem with higher-order p-Laplacian dynamic delay differential equations operator on time scales (SBVP):

$$\left(\phi_p\left(u^{\Delta^{n-1}}(t)\right)\right)^{\nabla} + g(t)f\left(u(t), u(t-\tau), u^{\Delta}(t), \dots, u^{\Delta^{n-2}}(t)\right) = 0, \quad 0 < t < T, \tag{1.7}$$

$$\begin{gathered}
u(t) = \zeta(t), \quad -\tau \le t \le 0, \\
u^{\Delta^i}(0) = 0, \quad 1 \le i \le n-3, \\
\left.\begin{aligned}
\alpha\phi_p\left(u^{\Delta^{n-2}}(0)\right) - \beta\phi_p\left(u^{\Delta^{n-1}}(\xi)\right) = 0, \\
\gamma\phi_p\left(u^{\Delta^{n-2}}(T)\right) + \delta\phi_p\left(u^{\Delta^{n-1}}(\eta)\right) = 0,
\end{aligned}\right. \quad n \ge 3,
\end{gathered} \tag{1.8}$$

where $\phi_p(s)$ is p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = \phi_p^{-1}$, $1/p + 1/q = 1$; $\xi, \eta \in (0, T)$, $\tau \in [0, T]$ is prescribed and $\xi < \eta$, $g : (0, T) \to [0, \infty)$, $\alpha > 0$, $\beta \ge 0$, $\gamma > 0$, $\delta \ge 0$.

In this paper, by constructing one integral equation which is equivalent to the problem (1.7), (1.8), we research the existence of positive solutions for nonlinear singular boundary value problem (1.7), (1.8) when g and f satisfy some suitable conditions.

Our main tool of this paper is the following fixed point index theory.

Theorem 1.1 (see [18]). *Suppose E is a real Banach space, $K \subset E$ is a cone, let $\Omega_r = \{u \in K : \|u\| \le r\}$. Let operator $T : \Omega_r \to K$ be completely continuous and satisfy $Tx \ne x$, for all $x \in \partial\Omega_r$. Then*

(i) *if $\|Tx\| \le \|x\|$, for all $x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 1$;*

(ii) *if $\|Tx\| \ge \|x\|$, for all $x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 0$.*

This paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we study the existence of at least two solutions of the systems (1.7), (1.8). In Section 4, we give an examples as the application.

2. Preliminaries and Lemmas

A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of real numbers R^+. In [1, 14, 20], we can find some basic definitions about time scale. The operators σ and ρ from $\mathbf{T}$ to $\mathbf{T}$:

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T}, \qquad \rho(t) = \sup\{\tau \in \mathbf{T} \mid \tau < t\} \in \mathbf{T} \tag{2.1}$$

are called the forward jump operator and the backward jump operator, respectively.

If $\mathbf{T} = R$, then $x^{\Delta}(t) = x^{\nabla}(t) = x'(t)$. If $\mathbf{T} = Z$, then $x^{\Delta}(t) = x(t+1) - x(t)$ is the forward difference operator, while $x^{\nabla}(t) = x(t) - x(t-1)$ is the backward difference operator.

A function f is left-dense continuous (i.e., ld-continuous), if f is continuous at each left-dense point in $\mathbf{T}$ and its right-sided limit exists at each right-dense point in $\mathbf{T}$. It is well known that f is ld-continuous.

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a). \tag{2.2}$$

If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a). \tag{2.3}$$

In the rest of this paper, $\mathbf{T}$ is closed subset of R with $0 \in \mathbf{T}_k$, $T \in \mathbf{T}^k$. And let

$$B = \left\{u \in C[-\tau, 0] \cap C_{ld}^{n-2}[0,T] : u^{\Delta^i}(0) = 0,\ 0 \le i \le n-3\right\}. \tag{2.4}$$

Here,

$$C_{ld}^{n-2}[0,T] = \{u : [0,T] \to R \mid u(t) \text{ is left}-\text{dense } n-2 \text{ order continuously differentiable}\}. \tag{2.5}$$

Then B is a Banach space with the norm $\|u\| = \max_{t\in[0,T]}|u^{\Delta^{n-2}}(t)|$. And let

$$K = \left\{u \in B : u^{\Delta^{n-2}}(t) \ge 0,\ u^{\Delta^{n-2}}(t) \text{ is concave function},\ t \in [0,T]\right\}. \tag{2.6}$$

Obviously, K is a cone in B. Set $K_r = \{u \in K : \|u\| \le r\}$.

Definition 2.1. $u(t)$ is called a solution of SBVP (1.7) and (1.8) if it satisfies the following:

(1) $u \in C[-\tau, 0] \cap C_{ld}^{n-1}(0,T)$;

(2) $u(t) > 0$ for all $t \in (0,T)$ and satisfy conditions (1.8);

(3) $(\phi_p(u^{\Delta^{n-1}}(t)))^{\nabla} = -g(t) f(u(t), u(t-\tau), u^{\Delta}(t), \dots, u^{\Delta^{n-2}}(t))$ hold for $t \in (0,T)$.

In the rest of the paper, we also make the following assumptions:

(H_1) $f \in C_{ld}([0,+\infty)^n, [0,+\infty))$;

(H_2) $g(t) \in C_{ld}((0,T), [0,+\infty))$ and there exists $t_0 \in (0,T)$, such that

$$g(t_0) > 0, \qquad 0 < \int_0^T g(s)\nabla s < +\infty; \tag{2.7}$$

(H_3) $\zeta(t) \in C([-\tau,0]$, $\zeta(t) > 0$ on $[-\tau,0)$ and $\zeta(0) = 0$.

It is easy to check that condition (H_2) implies that

$$0 < \int_0^T \phi_q\left(\int_0^s g(s_1)\nabla s_1\right)\Delta s < +\infty. \tag{2.8}$$

We can easily get the following lemmas.

Lemma 2.2. *Suppose condition (H_2) holds. Then there exists a constant $\theta \in (0,1/2)$ satisfing*

$$0 < \int_\theta^{T-\theta} g(t)\nabla t < \infty. \tag{2.9}$$

Furthermore, the function

$$A(t) = \int_\theta^t \phi_q\left(\int_s^t g(s_1)\nabla s_1\right)\Delta s + \int_t^{T-\theta} \phi_q\left(\int_t^s g(s_1)\nabla s_1\right)\nabla s, \quad t \in [\theta, T-\theta], \tag{2.10}$$

is positive continuous functions on $[\theta, T-\theta]$; therefore, $A(t)$ has minimum on $[\theta, T-\theta]$. Hence we suppose, that there exists $L > 0$ such that $A(t) \geq L$, $t \in [\theta, T-\theta]$.

Proof. At first, it is easily seen that $A(t)$ is continuous on $[\theta, T-\theta]$. Next, let

$$A_1(t) = \int_\theta^t \phi_q\left(\int_s^t g(s_1)\nabla s_1\right)\Delta s, \qquad A_2(t) = \int_t^{T-\theta} \phi_q\left(\int_t^s g(s_1)\nabla s_1\right)\Delta s. \tag{2.11}$$

Then, from condition (H_2), we have that the function $A_1(t)$ is strictly monotone nondecreasing on $[\theta, T-\theta]$ and $A_1(\theta) = 0$, the function $A_2(t)$ is strictly monotone nonincreasing on $[\theta, T-\theta]$ and $A_2(T-\theta) = 0$, which implies $L = \min_{t\in[\theta,T-\theta]} A(t) > 0$. The proof is complete. □

Lemma 2.3. *Let $u \in K$ and θ of Lemma 2.2, then*

$$u(t) \geq \theta\|u\|, \quad t \in [\theta, T-\theta]. \tag{2.12}$$

The proof of the above lemma is similar to the proof in [17, Lemma 2.2], so we omit it.

Lemma 2.4. *Suppose that conditions* (H_1), (H_2), (H_3) *hold,* $u(t) \in B \cap C_{ld}^{n-1}(0,1)$ *is a solution of the following boundary value problems:*

$$\left(\phi_p\left(u^{\Delta^{n-1}}(t)\right)\right)^{\nabla} + g(t)f\left(u(t), u(t-\tau)+h(t-\tau), u^{\Delta}(t), \ldots, u^{\Delta^{n-2}}(t)\right) = 0, \quad 0 < t < T, \tag{2.13}$$

$$\begin{gathered} u(t) = 0, \quad -\tau \le t \le 0, \\ u^{\Delta^i}(0) = 0, \quad 1 \le i \le n-3, \\ \left.\begin{aligned} \alpha\phi_p\left(u^{\Delta^{n-2}}(0)\right) - \beta\phi_p\left(u^{\Delta^{n-1}}(\xi)\right) = 0, \\ \gamma\phi_p\left(u^{\Delta^{n-2}}(T)\right) + \delta\phi_p\left(u^{\Delta^{n-1}}(\eta)\right) = 0, \end{aligned}\right. \quad n \ge 3, \end{gathered} \tag{2.14}$$

where

$$h(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0, \\ 0, & 0 \le t \le T. \end{cases} \tag{2.15}$$

Then, $\overline{u}(t) = u(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.7) *and* (1.8).

Proof. It is easy to check that $\overline{u}(t)$ satisfies (1.7) and (1.8). □

So in the rest of the sections of this paper, we focus on SBVP (2.13) and (2.14).

Lemma 2.5. *Suppose that conditions* (H_1), (H_2), (H_3) *hold,* $u(t) \in B \cap C_{ld}^{n-1}(0,1)$ *is a solution of boundary value problems* (2.13), (2.14) *if and only if* $u(t) \in B$ *is a solution of the following integral equation:*

$$u(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0, \\ \displaystyle\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2}) \Delta s_{n-2} \Delta s_{n-3} \cdots \Delta s_1, & 0 \le t \le T, \end{cases} \tag{2.16}$$

where

$$w(t) = \begin{cases} \phi_q\left(\dfrac{\beta}{\alpha}\displaystyle\int_{\xi}^{\sigma} g(s) f\left(u(s), u(s-\tau)+h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\nabla s\right) \\ \quad + \displaystyle\int_0^t \phi_q\left(\int_s^{\sigma} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s, \\ \hfill 0 \le t \le \sigma, \\ \phi_q\left(\dfrac{\delta}{\gamma}\displaystyle\int_{\sigma}^{\eta} g(s) f\left(u(s), u(s-\tau)+h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\nabla s\right) \\ \quad + \displaystyle\int_t^T \phi_q\left(\int_{\sigma}^{s} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s, \\ \hfill \sigma \le t \le T. \end{cases} \tag{2.17}$$

Proof. Necessity. Obviously, for $t \in (-\tau, 0)$, we have $u(t) = \zeta(t)$. If $t \in (0,1)$, by the equation of the boundary condition, we have $u^{\Delta^{n-1}}(\xi) \geq 0$, $u^{\Delta^{n-1}}(\eta) \leq 0$, then there exists a constant $\sigma \in [\xi, \eta] \subset (0, T)$ such that $u^{\Delta^{n-1}}(\sigma) = 0$.

Firstly, by integrating the equation of the problems (2.13) on (σ, T), we have

$$\phi_p\left(u^{\Delta^{n-1}}(t)\right) = \phi_p\left(u^{\Delta^{n-1}}(\sigma)\right) - \int_\sigma^t g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s, \tag{2.18}$$

then

$$u^{\Delta^{n-1}}(t) = -\phi_q\left(\int_\sigma^t g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s\right), \tag{2.19}$$

thus

$$u^{\Delta^{n-2}}(t) = u^{\Delta^{n-2}}(\sigma) - \int_\sigma^t \phi_q\left(\int_\sigma^s g(r) f\left(u(r), u(r-\tau) + h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right) \nabla r\right) \Delta s. \tag{2.20}$$

By $u^{\Delta^{n-1}}(\sigma) = 0$ and condition (2.18), let $t = \eta$ on (2.18), we have

$$\phi_p\left(u^{\Delta^{n-1}}(\eta)\right) = -\int_\sigma^\eta g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s. \tag{2.21}$$

By the equation of the boundary condition (2.14), we have

$$\phi_p\left(u^{\Delta^{n-2}}(T)\right) = -\frac{\delta}{\gamma} \phi_p\left(u^{\Delta^{n-1}}(\eta)\right), \tag{2.22}$$

then

$$u^{\Delta^{n-2}}(T) = \phi_q\left(\frac{\delta}{\gamma} \int_\sigma^\eta g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s\right). \tag{2.23}$$

Then, by (2.20) and leting $t = T$ on (2.20), we have

$$\begin{aligned} u^{\Delta^{n-2}}(\sigma) = {} & \phi_q\left(\frac{\delta}{\gamma} \int_\sigma^\eta g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s\right) \\ & + \int_\sigma^T \phi_q\left(\int_\sigma^s g(r) f\left(u(r), u(r-\tau) + h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right) \nabla r\right) \Delta s. \end{aligned} \tag{2.24}$$

Then

$$u^{\Delta^{n-2}}(t)=\phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta g(s)f\left(u(s),u(s-\tau)+h(s-\tau),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\right)\nabla s\right)$$
$$+\int_t^T\phi_q\left(\int_\sigma^s g(r)f\left(u(r),u(r-\tau)+h(r-\tau),u^{\Delta}(r),\ldots,u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s. \tag{2.25}$$

Then, by integrating (2.25) for $n-2$ times on $(0,T)$, we have

$$u(t)=\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\phi_q\left(\frac{\sigma}{\gamma}\int_\delta^\eta g(s)f\left(u(s),u(s-\tau)\right.\right.$$
$$\left.\left.+h(s-\tau),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\right)\nabla s\right)\Delta s_{s_{n-2}}\cdots\Delta s_2\Delta s_1$$
$$+\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\left(\int_{s_{n-2}}^T\phi_q\left(\int_\sigma^s g(r)f\left(u(r),u(r-\tau)+h(r-\tau),u^{\Delta}(r),\ldots,u^{\Delta^{n-2}}(r)\right)\right.\right.$$
$$\left.\left.\times\nabla r\right)\Delta s\right)\Delta s_{s_{n-2}}\cdots\Delta s_2\Delta s_1. \tag{2.26}$$

Similarly, for $t\in(0,\sigma)$, by integrating the equation of problems (2.13) on $(0,\sigma)$, we have

$$u(t)=\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s)f\left(u(s),u(s-\tau)\right.\right.$$
$$\left.\left.+h(s-\tau),u^{\Delta}(s),\ldots,u^{\Delta^{n-2}}(s)\right)\nabla s\right)\Delta s_{s_{n-2}}\cdots\Delta s_2\Delta s_1$$
$$+\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\left(\int_0^{s_{n-2}}\phi_q\left(\int_s^\sigma g(r)f\left(u(r),u(r-\tau)+h(r-\tau),u^{\Delta}(r),\ldots,u^{\Delta^{n-2}}(r)\right)\right.\right.$$
$$\left.\left.\times\nabla r\right)\Delta s\right)\Delta s_{s_{n-2}}\cdots\Delta s_2\Delta s_1. \tag{2.27}$$

Therefore, for any $t\in[0,T]$, $u(t)$ can be expressed as equation

$$u(t)=\begin{cases}\zeta(t), & -\tau\le t\le 0,\\ \displaystyle\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}w(s_{n-2})\Delta s_{n-2}\Delta s_{n-3}\cdots\Delta s_1, & 0\le t\le T,\end{cases} \tag{2.28}$$

where $w(t)$ is expressed as (2.17). Then the results of Lemma 2.3 hold.

Sufficiency. Suppose that $u(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2}) \Delta s_{n-2} \Delta s_{n-3} \cdots \Delta s_1$, $0 \le t \le T$. Then by (2.17), we have

$$u^{\Delta^{n-1}}(t) = \begin{cases} \phi_q\left(\int_t^{\sigma} g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s\right) \ge 0, \\ \qquad\qquad 0 \le t \le \sigma, \\ -\phi_q\left(\int_{\sigma}^{t} g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s\right) \le 0, \\ \qquad\qquad \sigma \le t \le T. \end{cases} \tag{2.29}$$

So, $(\phi_p(u^{\Delta^{n-1}}))^{\nabla} + g(t) f(u(t), u(t-\tau) + h(t-\tau), u^{\Delta}(t), \ldots, u^{\Delta^{n-2}}(t)) = 0$, $0 < t < T$. These imply that (2.13) holds. Furthermore, by letting $t = 0$ and $t = T$ on (2.17) and (2.29), we can obtain the boundary value equations of (2.14). The proof is complete. □

Now, we define an operator equation T given by

$$(Tu)(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0, \\ \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2}) \Delta s_{n-2} \Delta s_{n-3} \cdots \Delta s_1, & 0 \le t \le T, \end{cases} \tag{2.30}$$

where $w(t)$ is given by (2.17).

From the definition of T and the previous discussion, we deduce that, for each $u \in K$, $Tu \in K$. Moreover, we have the following lemmas.

Lemma 2.6. *$T : K \to K$ is completely continuous.*

Proof. Because

$$(Tu)^{\Delta^{n-1}}(t) = w^{\Delta}(t) = \begin{cases} \phi_q\left(\int_t^{\sigma} g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s\right) \ge 0, \\ \qquad\qquad 0 \le t \le \sigma, \\ -\phi_q\left(\int_{\sigma}^{t} g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right) \nabla s\right) \le 0, \\ \qquad\qquad \sigma \le t \le T, \end{cases} \tag{2.31}$$

is continuous, decreasing on $[0, T]$ and satisfies $(Tu)^{\Delta^{n-1}}(\sigma) = 0$, then, $Tu \in K$ for each $u \in K$ and $(Tu)^{\Delta^{n-2}}(\sigma) = \max_{t \in [0,T]} (Tu)^{\Delta^{n-2}}(t)$. This shows that $TK \subset K$. Furthermore, it is easy to check by Arzela-ascoli Theorem that $T : K \to K$ is completely continuous. □

Lemma 2.7. *Suppose that conditions* (H_1), (H_2), (H_3) *hold, the solution* $u(t)$ *of problem* (2.13), (2.14) *satisfies*

$$\max_{0\le t\le T}|u(t-\tau)+h(t-\tau)| \le \max_{-\tau\le t\le 0}|\zeta(t)|,$$
$$u(t)\le Tu^{\Delta}(t)\le\cdots\le T^{n-3}u^{\Delta^{n-3}}(t), \quad t\in[0,T], \tag{2.32}$$

and for $\theta\in(0,T/2)$ *in Lemma 2.2, one has*

$$u^{\Delta^{n-3}}(t)\le\frac{T}{\theta}u^{\Delta^{n-2}}(t), \quad t\in[\theta,T-\theta]. \tag{2.33}$$

Proof. Firstly, we can have

$$\begin{aligned}\max_{0\le t\le T}|u(t-\tau)+h(t-\tau)| &\le \max_{0\le t\le T}|u(t-\tau)|+\max_{0\le t\le T}|h(t-\tau)|\\ &= \max_{-\tau\le t\le T-\tau}|u(t)|+\max_{-\tau\le t\le T-\tau}|h(t)|\\ &= \max_{-\tau\le t\le 0}|\zeta(t)|.\end{aligned} \tag{2.34}$$

Next, if $u(t)$ is the solution of problem (2.13), (2.14), then $u^{\Delta^{n-2}}(t)$ is concave function, and $u^{\Delta^{i}}(t)\ge 0$ $(i=0,1,\ldots,n-2)$, $t\in[0,T]$. Thus, we have

$$u^{\Delta^{i}}(t)=\int_0^t u^{\Delta^{i+1}}(s)\Delta s\le tu^{\Delta^{i+1}}(t)\le Tu^{\Delta^{i+1}}(t), \quad i=0,1,\ldots,n-4, \tag{2.35}$$

that is, $u(t)\le Tu^{\Delta}(t)\le\cdots\le T^{n-3}u^{\Delta^{n-3}}(t)$, $t\in[0,T]$.

Finally, by Lemma 2.3, for $t\in[\theta,T-\theta]$, we have $u^{\Delta^{n-2}}(t)\ge\theta\|u^{\Delta^{n-2}}\|$. By $u^{\Delta^{n-3}}(t)=\int_0^t u^{\Delta^{n-2}}(s)\Delta s\le T\|u^{\Delta^{n-2}}\|$, we have

$$u^{\Delta^{n-3}}(t)\le\frac{T}{\theta}u^{\Delta^{n-2}}(t), \quad t\in[\theta,T-\theta]. \tag{2.36}$$

The proof is complete. □

For convenience, we set

$$H=\max_{-\tau\le t\le 0}|\zeta(t)|, \qquad \theta^*=\frac{2}{L}, \qquad \theta_*=\frac{1}{(T+\phi_q(\beta/\alpha))\phi_q\left(\int_0^T g(r)\nabla r\right)},$$
$$m\in(\theta^*,\infty),\ M\in(0,\theta_*), \tag{2.37}$$

where L is the constant from Lemma 2.2. By Lemma 2.5, we can also set

$$f_0 = \lim_{u_n \to 0} \max_{(u_1,u_2,\ldots,u_n)\in\aleph} \frac{f(u_1,u_2,\ldots,u_n)}{u_n^{p-1}},$$
$$f_\infty = \lim_{u_n \to \infty} \min_{(u_1,u_2,\ldots,u_n)\in\aleph} \frac{f(u_1,u_2,\ldots,u_n)}{u_n^{p-1}}, \tag{2.38}$$

where $\aleph = \{(u_1,u_2,\ldots,u_n) \mid 0 \le u_1 \le Tu_3 \cdots \le T^{n-3}u_{n-1} \le (T^{n-2}/\theta)u_n,\ u_2 \le H\}$.

3. The Existence of Multiple Positive Solutions

In this section, we also make the following conditions:

(A_1) $f(u_1,u_2,\ldots,u_n) \ge (mr)^{p-1}$, for $\theta r \le u_n \le r$, $(u_1,u_2,\ldots,u_n) \in \aleph$;

(A_2) $f(u_1,u_2,\ldots,u_n) \le (MR)^{p-1}$, for $0 \le u_n \le R$, $(u_1,u_2,\ldots,u_n) \in \aleph$.

Next, we will discuss the existence of multiple positive solutions.

Theorem 3.1. *Suppose that conditions (H_1), (H_2), (H_3), and (A_2) hold. Assume that f also satisfies*

(A_3) $f_0 = +\infty$;

(A_4) $f_\infty = +\infty$.

Then, the SBVP (2.13), (2.14) *hase at last two solutions u_1, u_2 such that*

$$0 < \|u_1\| < R < \|u_2\|. \tag{3.1}$$

Proof. For any $u \in K$, by Lemma 2.3, we have

$$u^{\Delta^{n-2}}(t) \ge \theta\|u\|, \quad t \in [\theta, T-\theta]. \tag{3.2}$$

First, by condition (A_3), for any $N > 2/\theta L$, there exists a constant $\rho_* \in (0,R)$ such that

$$f(u_1,u_2,\ldots,u_n) \ge (Nu_n)^{p-1}, \quad 0 < u_n \le \rho_*,\ u_n \ne 0. \tag{3.3}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$. For any $u \in \partial\Omega_{\rho_*}$, by (3.2) we have

$$\rho_* = \|u\| \ge u^{\Delta^{n-2}}(t) \ge \theta\|u\| = \theta\rho_*, \quad t \in [\theta, T-\theta]. \tag{3.4}$$

For any $u \in \partial\Omega_{\rho_*}$, by (3.3) and Lemmas 2.3–2.6, we will discuss it from three perspectives.

(i) If $\sigma \in [\theta, T-\theta]$, we have

$$
\begin{aligned}
2\|Tu\| &= 2(Tu)^{\Delta^{n-2}}(\sigma)\\
&\geq \int_0^{\sigma} \phi_q\left(\int_s^{\sigma} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s\\
&\quad+\int_{\sigma}^{T} \phi_q\left(\int_{\sigma}^{s} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s\\
&\geq \int_{\theta}^{\sigma} \phi_q\left(\int_s^{\sigma} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s\\
&\quad+\int_{\sigma}^{T-\theta} \phi_q\left(\int_{\sigma}^{s} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s\\
&\geq N\theta A(\sigma)\|u\| \geq 2\|u\|.
\end{aligned}
\tag{3.5}
$$

(ii) If $\sigma \in (T-\theta, T]$, we have

$$
\begin{aligned}
\|Tu\| &= (Tu)^{\Delta^{n-2}}(\sigma)\\
&\geq \phi_q\left(\frac{\beta}{\alpha}\int_{\xi}^{\sigma} g(s) f\left(u(s), u(s-\tau)+h(s-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\nabla s\right)\\
&\quad+\int_0^{\sigma} \phi_q\left(\int_s^{\sigma} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s\\
&\geq \int_{\theta}^{T-\theta} \phi_q\left(\int_s^{T-\theta} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s\\
&\geq N\theta A(T-\theta)\|u\| > \|u\|.
\end{aligned}
\tag{3.6}
$$

(iii) If $\sigma \in (0, \theta)$, we have

$$
\begin{aligned}
\|Tu\| &= (Tu)^{\Delta^{n-2}}(\sigma)\\
&\geq \phi_q\left(\frac{\delta}{\gamma}\int_{\sigma}^{\eta} g(s) f\left(u(s), u(s-\tau)+h(r-\tau), u^{\Delta}(s), \ldots, u^{\Delta^{n-2}}(s)\right)\nabla s\right)\\
&\quad+\int_{\sigma}^{T} \phi_q\left(\int_{\sigma}^{s} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s\\
&\geq \int_{\theta}^{T-\theta} \phi_q\left(\int_{\theta}^{s} g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^{\Delta}(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s\\
&\geq N\theta A(\theta)\|u\| > \|u\|.
\end{aligned}
\tag{3.7}
$$

Therefore, no matter under which condition, we all have

$$
\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}. \tag{3.8}
$$

Then, by Theorem 1.1, we have

$$i(T, \Omega_{\rho_*}, K) = 0. \tag{3.9}$$

Next, by condition (A_4), for any $\overline{N} > 2/\theta L$, there exists a constant $\rho_0 > 0$ such that

$$f(u_1, u_2, \dots, u_n) \ge \left(\overline{N} u_n\right)^{p-1}, \quad u_n > \rho_0. \tag{3.10}$$

We choose a constant $\rho^* > \max\{R, \rho_0/\theta\}$, obviously $\rho_* < R < \rho^*$. Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$. For any $u \in \partial\Omega_{\rho^*}$, by Lemma 2.3, we have

$$u(t) \ge \theta\|u\| = \theta\rho^* > \rho_0, \quad t \in [\theta, T - \theta]. \tag{3.11}$$

Then, by (3.10), Lemmas 2.3–2.6 and also similar to the previous proof, we can also have from three perspectives that

$$\|Tu\| \ge \|u\|, \quad \forall u \in \partial\Omega_{\rho^*}. \tag{3.12}$$

Then, by Theorem 1.1, we have

$$i(T, \Omega_{\rho^*}, K) = 0. \tag{3.13}$$

Finally, set $\Omega_R = \{u \in K : \|u\| < R\}$. For any $u \in \partial\Omega_R$, we have $u(t) \le \|u\| = R$, by (A_2) we know

$$\begin{aligned}
\|Tu\| &= (Tu)^{\Delta^{n-2}}(\sigma) \\
&\le \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^\Delta(s), \dots, u^{\Delta^{n-2}}(s)\right)\nabla s\right) \\
&\quad + \int_0^T \phi_q\left(\int_s^\sigma g(r) f\left(u(r), u(r-\tau) + h(r-\tau), u^\Delta(r), \dots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s \\
&\le \left(T + \phi_q\left(\frac{\beta}{\alpha}\right)\right) MR\phi_q\left(\int_0^T g(r)\nabla r\right) \le R = \|u\|.
\end{aligned} \tag{3.14}$$

Thus,

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega_R. \tag{3.15}$$

Then, by Theorem 1.1, we have

$$i(T, \Omega_R, K) = 1. \tag{3.16}$$

Therefore, by (3.9), (3.13), (3.16), $\rho_* < R < \rho^*$ we have

$$i\left(T, \Omega_R \setminus \overline{\Omega}_{\rho_*}, K\right) = 1, \qquad i\left(T, \Omega_{\rho^*} \setminus \overline{\Omega}_R, K\right) = -1. \tag{3.17}$$

Then T has fixed point $u_1 \in \Omega_R \setminus \overline{\Omega}_{\rho_*}$ and fixed point $u_2 \in \Omega_{\rho^*} \setminus \overline{\Omega}_R$. Obviously, u_1, u_2 are all positive solutions of problem (2.13), (2.14) and $\rho_* < \|u_1\| < R < \|u_2\| < \rho^*$. Proof of Theorem 3.1 is complete. □

Theorem 3.2. *Suppose that conditions* (H_1), (H_2), (H_3), (A_1) *hold. Assume that f also satisfies*

(A_5) $f_0 = 0$;

(A_6) $f_\infty = 0$.

Then, the SBVP (2.13), (2.14) *has at last two solutions* u_1, u_2 *such that* $0 < \|u_1\| < r < \|u_2\|$.

Proof. First, by $f_0 = 0$, for $\epsilon_1 \in (0, \theta_*)$, there exists a constant $\rho_* \in (0, r)$ such that

$$f(u_1, u_2, \ldots, u_n) \le (\epsilon_1 u_n)^{p-1}, \quad 0 < u_n \le \rho_*. \tag{3.18}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (3.18), we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{\Delta^{n-2}}(\sigma) \\
&\le \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^\Delta(s), \ldots, u^{\Delta^{n-2}}(s)\right)\nabla s\right) \\
&\quad + \int_0^T \phi_q\left(\int_s^\sigma g(r) f\left(u(r), u(r-\tau) + h(r-\tau), u^\Delta(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s \\
&\le \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s) f\left(u(s), u(s-\tau) + h(s-\tau), u^\Delta(s), \ldots, u^{\Delta^{n-2}}(s)\right)\nabla s\right) \\
&\quad + T\phi_q\left(\int_0^T g(r) f\left(u(r), u(r-\tau) + h(r-\tau), u^\Delta(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right) \\
&\le \left(T + \phi_q\left(\frac{\beta}{\alpha}\right)\right)\epsilon_1 \rho_* \phi_q\left(\int_0^T g(r)\nabla r\right) \le \rho_* = \|u\|,
\end{aligned} \tag{3.19}$$

that is,

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}. \tag{3.20}$$

Then, by Theorem 1.1, we have

$$i\left(T, \Omega_{\rho_*}, K\right) = 1. \tag{3.21}$$

Next, let $f^*(x) = \max_{0 \le u_{n-1} \le x} f(u_1, u_2, \ldots, u_{n-1})$; note that $f^*(x)$ is monotone increasing with respect to $x \ge 0$. Then, from $f_\infty = 0$, it is easy to see that

$$\lim_{x \to \infty} \frac{f^*(x)}{x^{p-1}} = 0. \tag{3.22}$$

Therefore, for any $\epsilon_2 \in (0, \theta_*)$, there exists a constant $\rho^* > r$ such that

$$f^*(x) \le (\epsilon_2 x)^{p-1}, \quad x \ge \rho^*. \tag{3.23}$$

Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$, for any $u \in \partial\Omega_{\rho^*}$, by (3.23), we have

$$\begin{aligned}
\|Tu\| &= (Tu)^{\Delta^{n-2}}(\sigma) \\
&\le \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s) f\left(u(s), u(s-\tau)+h(s-\tau), u^\Delta(s), \ldots, u^{\Delta^{n-2}}(s)\right)\nabla s\right) \\
&\quad + \int_0^T \phi_q\left(\int_s^\sigma g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^\Delta(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right)\Delta s \\
&\le \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s) f\left(u(s), u(s-\tau)+h(s-\tau), u^\Delta(s), \ldots, u^{\Delta^{n-2}}(s)\right)\nabla s\right) \\
&\quad + T\phi_q\left(\int_0^T g(r) f\left(u(r), u(r-\tau)+h(r-\tau), u^\Delta(r), \ldots, u^{\Delta^{n-2}}(r)\right)\nabla r\right) \\
&\le \left(T + \phi_q\left(\frac{\beta}{\alpha}\right)\right)\phi_q\left(\int_0^T g(r) f^*(\rho^*)\nabla r\right) \\
&\le \left(T + \phi_q\left(\frac{\beta}{\alpha}\right)\right)\epsilon_2 \rho^* \phi_q\left(\int_0^T g(r)\nabla r\right) \le r^* = \|u\|,
\end{aligned} \tag{3.24}$$

that is,

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega_{\rho^*}. \tag{3.25}$$

Then, by Theorem 1.1, we have

$$i\left(T, \Omega_{\rho^*}, K\right) = 1. \tag{3.26}$$

Finally, set $\Omega_r = \{u \in K : \|u\| < r\}$. For any $u \in \partial\Omega_r$, by (A_1), Lemma 2.3 and also similar to the previous proof of Theorem 3.1, we can also have

$$\|Tu\| \ge \|u\|, \quad \forall u \in \partial\Omega_r. \tag{3.27}$$

Then, by Theorem 1.1, we have

$$i(T, \Omega_r, K) = 0. \tag{3.28}$$

Therefore, by (3.21), (3.28), (3.26), $\rho_* < r < \rho^*$, we have

$$i\left(T, \Omega_r \setminus \overline{\Omega}_{\rho_*}, K\right) = -1, \qquad i\left(T, \Omega_{\rho^*} \setminus \overline{\Omega}_r, K\right) = 1. \tag{3.29}$$

Then T has fixed point $u_1 \in \Omega_r \setminus \overline{\Omega}_{\rho_*}$ and fixed point $u_2 \in \Omega_{\rho_*} \setminus \overline{\Omega}_r$. Obviously, u_1, u_2 are all positive solutions of problem (2.13), (2.14) and $\rho_* < \|u_1\| < r < \|u_2\| < \rho^*$. The proof of Theorem 3.2 is complete. □

Similar to Theorems 3.1 and 3.2, we also obtain the following theorems.

Theorem 3.3. *Suppose that conditions (H_1), (H_2), (H_3), and (A_2) hold and*

(A_7) $f_\infty = \lambda \in ((2\theta^*/\theta)^{p-1}, \infty)$,

(A_8) $f_0 = \varphi \in ((2\theta^*/\theta)^{p-1}, \infty)$.

Then, the SBVP (2.13), (2.14) *has at last two solutions* u_1, u_2 *such that* $0 < \|u_1\| < R < \|u_2\|$.

Theorem 3.4. *Suppose that conditions (H_1), (H_2), (H_3), and (A_1) hold and*

(A_9) $f_0 = \varphi \in [0, (\theta_*/4)^{p-1})$;

(A_{10}) $f_\infty = \lambda \in [0, (\theta_*/4)^{p-1})$.

Then, the SBVP (2.13), (2.14) *has at last two solutions* u_1, u_2 *such that* $0 < \|u_1\| < r < \|u_2\|$.

4. An Example

Example 4.1. Consider the following 3-order singular boundary value problem (SBVP) with p-Laplacian:

$$\begin{gathered}\left(\phi_p\left(u^{\Delta\Delta}\right)\right)^{\nabla}(t) + \frac{1}{64\pi^4} t^{-1/2}(1-t)\left[u(t) + u(t-1) + \left(u^{\Delta}\right)^2(t) + \left(u^{\Delta}\right)^4(t)\right] = 0, \quad 0 < t < 1, \\ u(t) = -te^t, \quad -1 \le t \le 0, \\ 2\phi_p\left(u^{\Delta}(0)\right) - \phi_p\left(u^{\Delta\Delta}\left(\frac{1}{4}\right)\right) = 0, \qquad \phi_p\left(u^{\Delta}(1)\right) + \delta\phi_p\left(u^{\Delta\Delta}\left(\frac{1}{2}\right)\right) = 0,\end{gathered} \tag{4.1}$$

where

$$\begin{gathered}\beta = \gamma = 1, \qquad \alpha = 2, \qquad p = 4, \qquad \delta \ge 0, \qquad p = 4, \qquad \xi = \frac{1}{4}, \\ \eta = \frac{1}{3}, \qquad \theta = \frac{1}{4}, \qquad \tau = T = 1.\end{gathered} \tag{4.2}$$

So, by Lemma 2.4, we discuss the following SBVP:

$$\begin{gathered}\left(\phi_p\left(u^{\Delta\Delta}\right)\right)^{\nabla}(t)+\frac{1}{64\pi^4}t^{-1/2}(1-t)\Big[u(t)+[u(t-1)+h(t-1)]\\ +\left(u^{\Delta}\right)^2(t)+\left(u^{\Delta}\right)^4(t)\Big]=0,\quad 0<t<1,\\ u(t)=0,\quad -1\le t\le 0,\\ 2\phi_p\left(u^{\Delta}(0)\right)-\phi_p\left(u^{\Delta\Delta}\left(\frac{1}{4}\right)\right)=0,\qquad \phi_p\left(u^{\Delta}(1)\right)+\delta\phi_p\left(u^{\Delta\Delta}\left(\frac{1}{2}\right)\right)=0,\end{gathered}\tag{4.3}$$

where

$$\begin{gathered}h(t)=\begin{cases}\zeta(t), & -1\le t\le 0,\\ 0, & 0\le t\le 1,\end{cases}\qquad \zeta(t)=-te^{t},\\ g(t)=\frac{1}{64\pi^4}t^{-1/2}(1-t),\quad f(u_1,u_2,u_3)=u_1+u_2+u_3^2+u_3^4.\end{gathered}\tag{4.4}$$

Then, obviously,

$$q=\frac{4}{3},\qquad \int_0^1 g(t)\nabla t=\frac{1}{64\pi^3},\qquad H=\max_{-1\le t\le 0}|\zeta(t)|=e,\qquad f_\infty=+\infty,\qquad f_0=+\infty,\tag{4.5}$$

so conditions (H_1), (H_2), (H_3), (A_2), and (A_3) hold.

Next,

$$\phi_q\left(\int_0^1 a(t)\nabla t\right)=\frac{1}{4\pi},\qquad \theta_*=\frac{4\pi}{1+\sqrt[3]{4}},\tag{4.6}$$

we choose $R=3$, $M=2$ and for $\theta=1/4$, because of the monotone increasing of $f(u_1,u_2,u_3)$ on $[0,\infty)^3$, then

$$f(u_1,u_2,u_3)\le f\left(\frac{3}{4},e,3\right)=\frac{3}{4}+e+90,\quad 0\le u_3\le 3,\ 0\le u_1\le\frac{1}{4}u_3,\ 0\le u_2\le e.\tag{4.7}$$

Therefore, by

$$M\in(0,\theta_*),\qquad (MR)^{p-1}=(6)^3=216,\tag{4.8}$$

we know

$$f(u_1,u_2,u_3)\le(MR)^{p-1},\quad 0\le u_3\le 3,\ 0\le u_1\le\frac{1}{4}u_3,\ 0\le u_2\le e,\tag{4.9}$$

so condition (A_2) holds. Then, by Theorem 3.1, SBVP (4.3) has at least two positive solutions v_1, v_2 and $0 < \|v_1\| < 3 < \|v_2\|$. Then, by Lemma 2.4, $\overline{v}_1(t) = v_1(t) + h(t)$, $\overline{v}_2(t) = v_2(t) + h(t)$, $t \in (-1, 1)$ are the positive solutions of the SBVP (4.1).

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