Research Article

Nonclassical Symmetry Analysis of Boundary Layer Equations

Rehana Naz,[1] Mohammad Danish Khan,[2] and Imran Naeem[2]

[1] *Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore 53200, Pakistan*
[2] *Department of Mathematics, School of Science and Engineering,*
  Lahore University of Management Sciences, Lahore Cantt 54792, Pakistan

Correspondence should be addressed to Imran Naeem, imran.naeem@lums.edu.pk

The nonclassical symmetries of boundary layer equations for two-dimensional and radial flows are considered. A number of exact solutions for problems under consideration were found in the literature, and here we find new similarity solution by implementing the SADE package for finding nonclassical symmetries.

1. Introduction

Prandt [1] derived the boundary layer equations by simplifying the Navier-Stokes equations. Schlichting [2, 3] showed that the two-dimensional flow is represented by the boundary layer equation:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \psi_{yyy} = 0. \qquad (1.1)$$

Here (x, y) denote the usual orthogonal cartesian coordinates parallel and perpendicular to the boundary $y = 0$, ψ denotes the stream function. The velocity components in the x and y directions, $u(x, y)$ and $v(x, y)$, are related with ψ as $u = \psi_y$ and $v = -\psi_x$. The boundary-layer equations are usually solved subject to certain conditions to describe flow in jets, films, and plates. In jet flow problems due to homogeneous boundary conditions a further condition known as conserved quantity is required. The conserved quantity is a measure of the strength of the jet. A new method of deriving conserved quantities for different types of jet flow problems was discussed by Naz et al. [4]. The liquid jet, the free jet, and the wall jet satisfy

the same partial differential equations (1.1), but the boundary conditions and conserved quantities for each jet flow problem are different. The boundary-layer equations were solved subject to certain boundary conditions and conserved quantity for two-dimensional free, wall, and liquid jets in [2, 3, 5–8].

The radial flow is represented by the boundary layer equation (see, e.g., Squire [9]):

$$\frac{1}{x}\psi_y\psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x\psi_{yy} - \psi_{yyy} = 0. \tag{1.2}$$

Cylindrical polar coordinates (x, θ, y) are used. The radial coordinate is x, the axis of symmetry is $x = 0$, and all quantities are independent of θ. The velocity components in the x and y directions, $u(x, y)$ and $v(x, y)$, are related to ψ as $u = (1/x)\psi_y$ and $v = (-1/x)\psi_x$. The boundary layer equations were solved subject to certain boundary conditions and conserved quantity for radial-free jet in works [6, 9–11], wall jet by Glauert [7], and liquid jet in [8, 10].

The classical and nonclassical symmetry methods play a vital role in deriving the exact solutions to nonlinear partial differential equations. The nonclassical method due to Bluman and Cole [12] and the direct method due to Clarkson and Kruskal [13] have been successfully applied for constructing the nonclassical symmetries and new solutions for partial differential equations. Olver [14] has shown that for a scalar equation, every reduction obtainable using the direct method is also obtainable using the nonclassical method. An algorithm for calculating the determining equations associated with the nonclassical method was introduced by Clarkson and Mansfield [15]. In the nonclassical method the invariant surface condition is augmented by the invariant surface condition. A new procedure for finding nonclassical symmetries is given in [16, 17], but this is restricted to a specific class of PDEs. Recently Filho and Figueiredo [18] developed a powerful computer package SADE for calculating the nonclassical symmetries by converting given PDE system to involutive form or without converting it to involutive form. We will use SADE to calculate the nonclassical symmetries and similarity reductions of boundary layer equations for two-dimensional as well as radial flows.

The paper is arranged in the following pattern: in Section 2 the nonclassical symmetries and similarity solution of boundary layer equations for two-dimensional flows are presented. The nonclassical symmetries and similarity solution of boundary layer equations for radial flows are given in Section 3. Finally, Conclusions are summarized in Section 4.

2. Nonclassical Symmetries and Similarity Solution of Boundary Layer Equations for Two-Dimensional Flows

The two-dimensional flow is represented by the boundary layer equation:

$$\Delta_1 = \psi_y\psi_{xy} - \psi_x\psi_{yy} - \psi_{yyy} = 0. \tag{2.1}$$

Consider the infinitesimal operator:

$$X = \xi^1(x, y, \psi)\frac{\partial}{\partial x} + \xi^2(x, y, \psi)\frac{\partial}{\partial y} + \eta(x, y, \psi)\frac{\partial}{\partial \psi}. \tag{2.2}$$

The invariant surface condition is

$$\Delta_2 = \xi^1(x, y, \psi)\psi_x + \xi^2(x, y, \psi)\psi_y - \eta(x, y, \psi) = 0. \tag{2.3}$$

The nonclassical symmetries determining equations are

$$X^{[3]}\Delta_1\Big|_{\Delta_1=0,\Delta_2=0} = 0, \qquad X^{[1]}\Delta_2\Big|_{\Delta_1=0,\Delta_2} = 0, \tag{2.4}$$

where $X^{[1]}$ and $X^{[3]}$ are the usual first and third prolongations of operator X. Two cases arise: Case 1 $\xi^1 \neq 0$ and Case 2 $\xi^1 = 0,\ \xi^2 \neq 0$.

Case 1 ($\xi^1 \neq 0$). In this case we set $\xi^1 = 1$, the SADE package yields the following six determining equations:

$$\xi_u^2 = 0,$$

$$\eta_{uu} = 0,$$

$$\eta_{yyy} - (\eta_y)^2 + \eta\eta_{yy} = 0,$$

$$3\xi_{yy}^2 - 3\eta_{yu} - \eta\xi_y^2 - \eta_u\eta - \eta_x - \eta_y\xi^2 = 0, \tag{2.5}$$

$$\xi_{yyy}^2 - 3\eta_{yu} + \eta\xi_{yy}^2 + \xi^2\eta_{yy} + \eta_{xy} - \eta\eta_{yu} + 2\eta_u\eta_y = 0,$$

$$\eta_{xu} - (\xi_y^2)^2 - \xi_{xy}^2 + \xi^2\eta_{yu} - \xi^2\xi_{yy}^2 + (\eta_u)^2 = 0.$$

The solution determining equations in (2.5) yield all classical symmetries (see [5]) and the following infinite many nonclassical symmetry generators:

$$X = \frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial y} + h(x, y)\frac{\partial}{\partial u}, \tag{2.6}$$

where $h(x, y)$ and $g(x)$ satisfy

$$h_x + g(x)h_y = 0, \qquad h_{yyy} - h_y^2 + hh_{yy} = 0. \tag{2.7}$$

Equation (2.7) yields

$$h(x, y) = \frac{6}{y - G(x)}, \qquad G'(x) = g(x), \tag{2.8}$$

and thus the nonclassical symmetry generators in (2.6) take the following form:

$$X = \frac{\partial}{\partial x} + G'(x)\frac{\partial}{\partial y} + \frac{6}{y - G(x)}\frac{\partial}{\partial u}. \tag{2.9}$$

Now, $\psi = \phi(x, y)$ is group invariant solution of (2.1) if

$$X(\psi - \phi(x, y))\big|_{\psi = \phi} = 0, \tag{2.10}$$

where the operator X is given in (2.9). The solution of (2.10) for $\psi = \phi(x, y)$ is of the form

$$\psi(x, y) = \frac{6x}{\chi} + w(\chi), \qquad \chi = y - G(x). \tag{2.11}$$

Substitution of (2.11) in (2.1) yields

$$\chi^2 \frac{d^3 w}{d\chi^3} + 6\chi \frac{d^2 w}{d\chi^2} + 6\frac{dw}{d\chi} = 0 \tag{2.12}$$

and thus

$$w(\chi) = c_1 + \frac{c_2}{\chi} + \frac{c_3}{\chi^2}. \tag{2.13}$$

The invariant solution (2.11) with the help of (2.13) takes the following form:

$$\psi(x, y) = \frac{6x}{\chi} + c_1 + \frac{c_2}{\chi} + \frac{c_3}{\chi^2}, \qquad \chi = y - G(x). \tag{2.14}$$

The invariant solution (2.14) is new solution for boundary layer equations for two-dimensional flows.

Case 2 ($\xi^1 = 0, \xi^2 \neq 0$). Results are in no-go case.

3. Nonclassical Symmetries and Similarity Solution of Boundary Layer Equations for Radial Flows

The radial flow is represented by the boundary layer equation:

$$\Delta = \frac{1}{x}\psi_y \psi_{xy} - \frac{1}{x^2}\psi_y^2 - \frac{1}{x}\psi_x \psi_{yy} - \psi_{yyy} = 0. \tag{3.1}$$

Case 1 ($\xi^1 \neq 0$). In this case we set $\xi^1 = 1$ and using SADE we have following six determining equations:

$$\xi_u^2 = 0,$$

$$\eta_{uu} = 0,$$

$$x\eta_{yyy} - (\eta_y)^2 + \eta\eta_{yy} = 0,$$

$$3x^2\xi_{yy}^2 - 3x^2\eta_{yu} - x\eta\xi_y^2 - x\eta_u\eta - x\eta_x - x\eta_y\xi^2 + \eta = 0,$$

$$x^2\xi_{yyy}^2 - 3x^2\eta_{yyu} + x\eta\xi_{yy}^2 + x\xi^2\eta_{yy} + x\eta_{xy} - x\eta\eta_{yu} + 2x\eta_u\eta_y - 3\eta_y = 0,$$

$$x^2\eta_{xu} - x^2\left(\xi_y^2\right)^2 - x^2\xi_{xy}^2 + x^2\xi^2\eta_{yu} - x^2\xi^2\xi_{yy}^2 + x^2(\eta_u)^2 - 2x\eta_u + 2 = 0. \tag{3.2}$$

The solution determining equations in (3.2) yield all classical symmetries and the following infinite many nonclassical symmetry generators:

$$X = \frac{\partial}{\partial x} + \frac{1}{x}\left[-y + xg(x)\right]\frac{\partial}{\partial y} + h(x,y)\frac{\partial}{\partial u}, \tag{3.3}$$

where $h(x,y)$ and $g(x)$ satisfy

$$xh_x - yh_y + xg(x)h_y - 2h = 0, \qquad xh_{yyy} - h_y^2 + hh_{yy} = 0. \tag{3.4}$$

The nonclassical symmetry generators (3.3) finally become

$$X = \frac{\partial}{\partial x} + \frac{1}{x}\left[-y + xG'(x) + G(x)\right]\frac{\partial}{\partial y} + \frac{6x}{y - G(x)}\frac{\partial}{\partial u}, \tag{3.5}$$

and we have used

$$h(x,y) = \frac{6x}{y - G(x)}, \qquad xG'(x) + G(x) = xg(x). \tag{3.6}$$

Now, $\psi = \phi(x,y)$ is group invariant solution of (3.1) if

$$X\left(\psi - \phi(x,y)\right)\big|_{\psi=\phi} = 0 \tag{3.7}$$

where the operator X is given in (3.5). The solution of (3.7) for $\psi = \phi(x,y)$ is of the form:

$$\psi(x,y) = \frac{2x^3}{\chi} + w(\chi), \qquad \chi = x\left[y - G(x)\right]. \tag{3.8}$$

Substitution of (3.8) in (3.1) yields

$$\chi^2 \frac{d^3 w}{d\chi^3} + 6\chi \frac{d^2 w}{d\chi^2} + 6 \frac{dw}{d\chi} = 0 \qquad (3.9)$$

and thus

$$w(\chi) = c_1 + \frac{c_2}{\chi} + \frac{c_3}{\chi^2}. \qquad (3.10)$$

Finally, we have following form invariant solution:

$$\psi(x, y) = \frac{2x^3}{\chi} + c_1 + \frac{c_2}{\chi} + \frac{c_3}{\chi^2}, \qquad \chi = x[y - G(x)], \qquad (3.11)$$

and this is new solution not obtained in the literature.

Case 2 ($\xi^1 = 0, \xi^2 \neq 0$). Results are in no-go case for radial flow also.

4. Conclusions

The nonclassical symmetries of boundary layer equations for two-dimensional and radial flows were computed by computer package SADE. A new similarity solution for two-dimensional flows was given in (2.14). For radial flows a new similarity solution (3.11) was derived. It would be of interest to identify what type of physical phenomena can be associated with the solutions derived in this paper.

References

[1] L. Prandt, "Uber Flussigkeitsbewegungen bei sehr kleiner Reibung," in *3 Internationalen Mathematischen Kongress*, pp. 484–491, Heidelberg, Germany, 1904.

[2] H. Schlichting, "Laminare strahlausbreitung," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 13, pp. 260–263, 1933.

[3] H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York, NY, USA, 1955.

[4] R. Naz, D. P. Mason, and F. M. Mahomed, "Conservation laws and conserved quantities for laminar two-dimensional and radial jets," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 5, pp. 2641–2651, 2009.

[5] D. P. Mason, "Group invariant solution and conservation law for a free laminar two-dimensional jet," *Journal of Nonlinear Mathematical Physics*, vol. 9, supplement 2, pp. 92–101, 2002.

[6] R. Naz, F. M. Mahomed, and D. P. Mason, "Conservation laws via the partial Lagrangian and group invariant solutions for radial and two-dimensional free jets," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 6, pp. 3457–3465, 2009.

[7] M. B. Glauert, "The wall jet," *Journal of Fluid Mechanics*, vol. 1, pp. 625–643, 1956.

[8] E. J. Watson, "The radial spread of a liquid jet over a horizontal plane," *Journal of Fluid Mechanics*, vol. 20, pp. 481–499, 1964.

[9] H. B. Squire, *50 Jahre Grenzschichtforschung. Eine Festschrift in Originalbeiträgen*, Friedrich Vieweg & Sohn, Braunschweig, Germany, 1955, Edited by H. Gorter, W. Tollmien.

[10] N. Riley, "Radial jets with swirl. I. Incompressible flow," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 15, pp. 435–458, 1962.

[11] W. H. Schwarz, "The radial free jet," *Chemical Engineering Science*, vol. 18, pp. 779–786, 1963.

[12] G. W. Bluman and J. D. Cole, "The general similarity solution of the heat equation," vol. 18, pp. 1025–1042, 1968.

[13] P. A. Clarkson and M. D. Kruskal, "New similarity reductions of the Boussinesq equation," *Journal of Mathematical Physics*, vol. 30, no. 10, pp. 2201–2213, 1989.

[14] P. J. Olver, "Direct reduction and differential constraints," *Proceedings of the Royal Society. London A*, vol. 444, no. 1922, pp. 509–523, 1994.

[15] P. A. Clarkson and E. L. Mansfield, "Algorithms for the nonclassical method of symmetry reductions," *SIAM Journal on Applied Mathematics*, vol. 54, no. 6, pp. 1693–1719, 1994.

[16] N. Bîlă and J. Niesen, "On a new procedure for finding nonclassical symmetries," *Journal of Symbolic Computation*, vol. 38, no. 6, pp. 1523–1533, 2004.

[17] M. S. Bruzón and M. L. Gandarias, "Applying a new algorithm to derive nonclassical symmetries," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 3, pp. 517–523, 2008.

[18] T. M. R. Filho and A. Figueiredo, "[SADE] a Maple package for the symmetry analysis of differential equations," *Computer Physics Communications*, vol. 182, pp. 467–476, 2011.