

## Research Article

# The Dual of Generalized Fuzzy Subspaces

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We define the dual of generalized fuzzy subspaces first. These concepts generalize the dual of fuzzy subspaces. And then we investigate the double dual of generalized fuzzy subspaces.

## 1. Introduction and Preliminaries

In fuzzy algebra, fuzzy subspaces are basic concepts. They had been introduced by Katsaras and Liu [1] in 1977 as a generalization of the usual notion of vector spaces. Since then, many results of fuzzy subspaces had been obtained in the literature [1–4]. Moreover, many researches in fuzzy algebra are closely related to fuzzy subspaces, such as fuzzy subalgebras of an associative algebra [5], fuzzy Lie ideals of a Lie algebra [6], fuzzy subcoalgebras of a coalgebra [7]. Hence fuzzy subspaces play an important role in fuzzy algebra. In 1996, Abdukhalikov [8] defined the dual of fuzzy subspaces as a generalization of the dual of  $k$ -vector spaces. This notion was also studied and applied in many branches [2, 7–9], especially in the fuzzy subcoalgebras [7] and fuzzy bialgebras [9].

After the introduction of fuzzy sets by Zadeh [10], there are a number of generalizations of this fundamental concept. So it is natural to study algebraic structures connecting with them. In this paper, we aim our attention at the dual of vector space in intuitionistic fuzzy sets, interval-valued fuzzy sets, and interval-valued intuitionistic fuzzy sets for our further researches.

### 1.1. Atanassov's Intuitionistic Fuzzy Sets

In [11] intuitionistic fuzzy sets are defined as follows:

*Definition 1.1.* An intuitionistic fuzzy set (IFS, for short) on a universe  $X$  is defined as an object having the form  $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ , where the functions  $\mu_A : X \rightarrow [0, 1]$

and  $\nu_A : X \rightarrow [0, 1]$  denote the degree of membership (namely,  $\mu_A(x)$ ) and the degree of nonmembership (namely,  $\nu_A(x)$ ) of each element  $x \in X$  to the set  $A$ , respectively, and  $0 \leq \mu_A(x) + \nu_A(x) \leq 1$  for each  $x \in X$ . For the sake of simplicity, we shall use the symbol  $A = (\mu_A, \nu_A)$  for the intuitionistic fuzzy set  $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X\}$ . The class of IFSs on a universe  $X$  is denoted by  $\text{IFS}(X)$ .

In 1993, Gau and Buehrer [12] defined vague sets. Later, Bustince and Burillo [13] proved that the notion of vague sets is as same as that of intuitionistic fuzzy sets.

With the definition of intuitionistic fuzzy sets, we can give the following definition.

*Definition 1.2.* Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy set of  $k$ -vector space  $V$ . For any  $x, y \in V$  and  $\alpha, \beta \in k$ , if it satisfies  $\mu_A(\alpha x + \beta y) \geq \min\{\mu_A(x), \mu_A(y)\}$  and  $\nu_A(\alpha x + \beta y) \leq \max\{\nu_A(x), \nu_A(y)\}$ , then  $A = (\mu_A, \nu_A)$  is called an *intuitionistic fuzzy subspace* of  $V$ .

## 1.2. Interval-Valued Fuzzy Sets

The notion of interval-valued fuzzy sets was first introduced by Zadeh [14] as an extension of fuzzy sets in which the values of the membership degrees are intervals of numbers instead of the numbers.

*Definition 1.3.* An interval-valued fuzzy set  $A = (A^L, A^U)$  on a universe  $X$  (IVFS, for short) is a mapping  $X \rightarrow \text{Int}([0, 1])$ , where  $\text{Int}([0, 1])$  stands for the set of all closed subintervals of  $[0, 1]$ . The class of all IVFSs on a universe  $X$  is denoted by  $\text{IVFS}(X)$ .

In [15] the interval-valued fuzzy sets are called grey sets.

*Notations.* For interval numbers  $D_1 = [a_1^L, b_1^U], D_2 = [a_2^L, b_2^U] \in \text{Int}([0, 1])$ . We define

$$\begin{aligned} r \min\{D_1, D_2\} &= r \min\left\{ [a_1^L, b_1^U], [a_2^L, b_2^U] \right\} = \left[ \min\{a_1^L, a_2^L\}, \min\{b_1^U, b_2^U\} \right], \\ r \max\{D_1, D_2\} &= r \max\left\{ [a_1^L, b_1^U], [a_2^L, b_2^U] \right\} = \left[ \max\{a_1^L, a_2^L\}, \max\{b_1^U, b_2^U\} \right], \end{aligned} \quad (1.1)$$

and put

- (a)  $D_1 \leq D_2 \Leftrightarrow a_1^L \leq a_2^L$  and  $b_1^U \leq b_2^U$ ,
- (b)  $D_1 = D_2 \Leftrightarrow a_1^L = a_2^L$  and  $b_1^U = b_2^U$ ,
- (c)  $D_1 < D_2 \Leftrightarrow D_1 \leq D_2$  and  $D_1 \neq D_2$ .

In [16], Deschrijver and Kerre presented that the mapping between the lattices  $\text{IVFS}(X)$  and  $\text{IFS}(X)$  is an isomorphism. Thus intuitionistic fuzzy sets and interval-valued fuzzy sets are same from mathematical viewpoints.

Similarly, we can define the following.

*Definition 1.4.* Let  $A = (A^L, A^U)$  be an interval-valued fuzzy set of  $k$ -vector space  $V$ . For any  $x, y \in V$  and  $\alpha, \beta \in k$ , if it satisfies  $A(\alpha x + \beta y) \geq r \min\{A(x), A(y)\}$ , then  $A = (A^L, A^U)$  is called an *interval-valued fuzzy subspace* of  $V$ .

### 1.3. Interval-Valued Intuitionistic Fuzzy Sets

The following definition generalizes the definitions of IFS and IVFS.

*Definition 1.5* (see [17]). An interval-valued intuitionistic fuzzy set on a universe  $X$  (IVIFS, for short) is an object of the form  $A = \{(x, M_A(x), N_A(x)) \mid x \in X\}$ , where  $M_A : X \rightarrow \text{Int}([0, 1])$  and  $N_A : X \rightarrow \text{Int}([0, 1])$  satisfy  $\sup M_A(x) + \sup N_A(x) \leq 1$  for any  $x \in X$ . The class of all IVIFSs on a universe  $X$  is denoted by  $\text{IVIFS}(X)$ .

*Definition 1.6.* Let  $A = (M_A, N_A)$  be an interval-valued intuitionistic fuzzy set of  $k$ -vector space  $V$ . For any  $x, y \in V$  and  $\alpha, \beta \in k$ , if it satisfies  $M_A(\alpha x + \beta y) \geq r \min \{M_A(x), M_A(y)\}$  and  $N_A(\alpha x + \beta y) \leq r \max \{N_A(x), N_A(y)\}$ , then  $A = (M_A, N_A)$  is called an *interval-valued intuitionistic fuzzy subspace* of  $V$ .

In this paper, intuitionistic fuzzy subspaces, interval-valued fuzzy subspaces and interval-valued intuitionistic fuzzy subspaces are called generalized fuzzy subspaces. In Section 2, we study the dual of generalized fuzzy subspaces. At first, we give the definitions of the dual of generalized fuzzy subspaces, and then discuss their properties and the relationship between them. In Section 3, we investigate the double dual of intuitionistic fuzzy subspaces. Other cases can be investigated similarly. At last a conclusion is presented.

## 2. The Dual of Generalized Fuzzy Subspaces

In this paper,  $V^*$  is denoted the dual space of  $V$ , that is, the vector space of all linear maps from  $V$  to  $k$ . We recall the following.

*Definition 2.1* (see [8]). Let  $\mu$  be a fuzzy subspace of  $k$ -vector space  $V$ .

Define  $\mu^* : V^* \rightarrow [0, 1]$  by

$$\mu^*(f) = \begin{cases} 1 - \sup\{\mu(x) \mid x \in V, f(x) \neq 0\} & \text{if } f \neq 0 \\ 1 - \inf\{\mu(x) \mid x \in V\} & \text{if } f = 0, \end{cases} \quad (2.1)$$

then  $\mu^*$  is called the dual of fuzzy subspace  $\mu$ .

Now, we study the dual of generalized fuzzy subspaces.

### 2.1. The Dual of Intuitionistic Fuzzy Subspaces

Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy subspace of  $k$ -vector space  $V$ . Then  $0 \leq \mu_A(x) + \nu_A(x) \leq 1$  for any  $x \in V$ , which implies  $0 \leq \sup_{x \in V} \{\mu_A(x)\} + \inf_{x \in V} \{\nu_A(x)\} \leq 1$ . Hence we give the definition of the dual of intuitionistic fuzzy subspaces as follows.

*Definition 2.2.* Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy subspace of  $k$ -vector space  $V$ . Define  $A^* = (\mu_{A^*}, \nu_{A^*})$ , where

$$\begin{aligned} \mu_{A^*} : V^* \longrightarrow [0, 1] \text{ by } \mu_{A^*}(f) &= \begin{cases} \inf\{\nu_A(x) \mid x \in V, f(x) \neq 0\} & \text{if } f \neq 0 \\ \sup\{\nu_A(x) \mid x \in V\} & \text{if } f = 0, \end{cases} \\ \nu_{A^*} : V^* \longrightarrow [0, 1] \text{ by } \nu_{A^*}(f) &= \begin{cases} \sup\{\mu_A(x) \mid x \in V, f(x) \neq 0\} & \text{if } f \neq 0 \\ \inf\{\mu_A(x) \mid x \in V\} & \text{if } f = 0. \end{cases} \end{aligned} \quad (2.2)$$

Obviously,  $A^* = (\mu_{A^*}, \nu_{A^*})$  is an intuitionistic fuzzy set of  $V^*$  and is called the dual of intuitionistic fuzzy subspace. The class of all the dual of intuitionistic fuzzy subspaces of  $V^*$  is denoted by  $\text{DIFS}(V^*)$ .

In [11], Atanassov defined two operators  $\square$  and  $\diamond$ . Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy set. Then  $\square A = (\mu_A, \mu_A^c)$  and  $\diamond A = (\nu_A^c, \nu_A)$ . Using the notations in the dual of intuitionistic fuzzy subspaces, we have the following.

*Remark 2.3.* (1) If  $A = \square A$ , then  $A^* = \diamond A^*$ .

(2) If  $A = \diamond A$ , then  $A^* = \square A^*$ .

Following the above remark, Definition 2.1 is the special case of Definition 2.2. And we can give a characterization of the intuitionistic fuzzy subspaces.

**Theorem 2.4.**  $A^* = (\mu_{A^*}, \nu_{A^*})$  is an intuitionistic fuzzy subspace of  $V^*$  if and only if  $\square A^*$  and  $\diamond A^*$  are intuitionistic fuzzy subspaces of  $V^*$ .

In the end, we give an explain of Definition 2.2 intuitively.

*Remark 2.5.* Let 10 experts vote to  $A$  and  $B$  and require per expert to vote at most one. The result is that  $A$  is 5 and  $B$  is 3. That is, the number of the supporters of  $A$  is 5 and the number of the supporters of  $B$  is 3.

In the voting model,  $A$  and  $B$  can be regarded as the dual objects. The analysis indicates that the number of supporters of  $A$  is equal to the number of objectors of  $B$  and the number of supporters of  $B$  is equal to the number of objectors of  $A$ . So if  $B$  is the dual of  $A$  and the numbers of supporters and objectors of  $A$  are known, then we can calculate the supporters of  $B$  by the objectors of  $A$  and the objectors of  $B$  by the supporters of  $A$ .

This is the idea that the  $\mu_{A^*}$  is defined by  $\nu_A$  and the  $\nu_{A^*}$  is defined by  $\mu_A$  in Definition 2.2.

## 2.2. The Dual of Interval-Valued Fuzzy Subspaces

*Definition 2.6.* Let  $A = (A^L, A^U)$  be an interval-valued fuzzy subspace of  $k$ -vector space  $V$ . Define  $A^* = (A^{*L}, A^{*U})$ , where

$$A^* : V^* \longrightarrow \text{Int}([0, 1]) \text{ by } A^*(f) = \begin{cases} [1, 1] - \sup\{A(x) \mid x \in V, f(x) \neq 0\} & \text{if } f \neq 0 \\ [1, 1] - \inf\{A(x) \mid x \in V\} & \text{if } f = 0. \end{cases} \quad (2.3)$$

Then  $A^* = (A^{*L}, A^{*U})$  is called the dual of an interval-valued fuzzy subspace. The class of all the dual of interval-valued fuzzy subspace of  $V^*$  is denoted by  $DIVFS(V^*)$ .

*Remark 2.7.* In Definition 2.6, the definition of  $A^* = (A^{*L}, A^{*U})$  can be described in detail as follows:

$$\begin{aligned} A^{*L} : V^* \longrightarrow [0,1] \text{ by } A^{*L}(f) &= \begin{cases} 1 - \sup\{A^L(x) \mid x \in V, f(x) \neq 0\} & \text{if } f \neq 0 \\ 1 - \inf\{A^L(x) \mid x \in V\} & \text{if } f = 0, \end{cases} \\ A^{*U} : V^* \longrightarrow [0,1] \text{ by } A^{*U}(f) &= \begin{cases} 1 - \sup\{A^U(x) \mid x \in V, f(x) \neq 0\} & \text{if } f \neq 0 \\ 1 - \inf\{A^U(x) \mid x \in V\} & \text{if } f = 0. \end{cases} \end{aligned} \quad (2.4)$$

The following theorem indicates that the dual of intuitionistic fuzzy subspaces and the dual of interval-valued fuzzy subspaces are same upon the lattice isomorphism.

**Theorem 2.8.** *The mapping  $\varphi : DIVFS(V^*) \rightarrow DIFS(V^*)$  by  $\mu \mapsto A$  is an isomorphism between the lattices  $DIVFS(V^*)$  and  $DIFS(V^*)$ , where  $\mu = (\mu^L, \mu^U) : V^* \rightarrow \text{Int}([0,1])$  by  $f \mapsto (\mu^L(f), \mu^U(f))$  and  $A = (\mu_A, \nu_A) = (\mu_A = \mu^U, \nu_A = 1 - \mu^L)$ .*

### 2.3. The Dual of Interval-Valued Intuitionistic Fuzzy Subspaces

*Definition 2.9.* Let  $A = (M_A, N_A)$  be an interval-valued intuitionistic fuzzy subspace of  $k$ -vector space  $V$ .

Define  $A^* = (M_{A^*}, N_{A^*})$ , where

$$\begin{aligned} M_{A^*} : V^* \longrightarrow \text{Int}([0,1]) \text{ by } M_{A^*}(f) &= \begin{cases} \inf\{N_A(x) \mid x \in V, f(x) \neq 0\} & \text{if } f \neq 0 \\ \sup\{N_A(x) \mid x \in V\} & \text{if } f = 0, \end{cases} \\ N_{A^*} : V^* \longrightarrow \text{Int}([0,1]) \text{ by } N_{A^*}(f) &= \begin{cases} \sup\{M_A(x) \mid x \in V, f(x) \neq 0\} & \text{if } f \neq 0 \\ \inf\{M_A(x) \mid x \in V\} & \text{if } f = 0. \end{cases} \end{aligned} \quad (2.5)$$

Then  $A^* = (M_{A^*}, N_{A^*})$  is called the dual of interval-valued intuitionistic fuzzy subspace.

**Theorem 2.10.** *The interval-valued intuitionistic fuzzy set  $A^* = (M_{A^*}, N_{A^*})$  is an interval-valued intuitionistic fuzzy subspace of  $V^*$ .*

*Proof.* Since  $M_{A^*}(0)$  is the upper bound of  $M_{A^*}(V^*)$  and  $N_{A^*}(0)$  is the lower bound of  $N_{A^*}(V^*)$ , it suffices to show that the nonempty sets  $U(M_{A^*}, [\delta_1, \delta_2]) = \{f \in V^* \mid M_{A^*}(f) \geq [\delta_1, \delta_2]\}$  and  $L(N_{A^*}, [\xi_1, \xi_2]) = \{f \in V^* \mid N_{A^*}(f) \leq [\xi_1, \xi_2]\}$  are subspaces of  $V^*$  for all  $[\delta_1, \delta_2], [\xi_1, \xi_2] \in [N_{A^*}(0), M_{A^*}(0)]$ . The remainder proof can be imitated by Theorem 3.2 of [8].  $\square$

*Remark 2.11.* The result is also true for intuitionistic fuzzy subspaces and interval-valued fuzzy subspaces.

*Example 2.12.* Let  $V = \{x = (0, a, b) \mid a, b \in R\}$  be a two dimensional vector space and  $V^*$  be its dual space. We define  $A = (M_A, N_A)$ , where for  $k \in R$  and  $x \in V$

$$M_A(x) = \begin{cases} a_1 & \text{if } x = (0, k, 0) \\ b_1 & \text{if } x = (0, 0, k) \\ c_1 & \text{if } x = (0, 0, 0) \\ d_1 & \text{otherwise,} \end{cases} \quad N_A(x) = \begin{cases} a_2 & \text{if } x = (0, k, 0) \\ b_2 & \text{if } x = (0, 0, k) \\ c_2 & \text{if } x = (0, 0, 0) \\ d_2 & \text{otherwise,} \end{cases} \quad (2.6)$$

If  $a_1 = [0.3, 0.4]$ ,  $b_1 = [0.2, 0.5]$ ,  $c_1 = [1, 1]$ ,  $d_1 = [0.2, 0.4]$  and  $a_2 = [0.5, 0.6]$ ,  $b_2 = [0.4, 0.5]$ ,  $c_2 = [0, 0]$ ,  $d_2 = [0.5, 0.6]$ , then  $A$  is an interval-valued intuitionistic fuzzy subspace. By Definition 2.9,  $A^* = (M_{A^*}, N_{A^*})$ , where if  $f \neq 0$ ,  $M_{A^*}(f) = [0.4, 0.5]$  and  $N_{A^*}(f) = [0.3, 0.5]$ , if  $f = 0$ ,  $M_{A^*}(0) = [0.5, 0.6]$  and  $N_{A^*}(0) = [0.2, 0.4]$ . Then  $A^* = (M_{A^*}, N_{A^*})$  is an interval-valued intuitionistic fuzzy subspace of  $V^*$ .

If  $a_1 = 0.3$ ,  $b_1 = 0.4$ ,  $c_1 = 1$ ,  $d_1 = 0.3$  and  $a_2 = 0.5$ ,  $b_2 = 0.2$ ,  $c_2 = 0$ ,  $d_2 = 0.5$ , then  $A$  is an intuitionistic fuzzy subspace. By Definition 2.6,  $A^* = (M_{A^*}, N_{A^*})$ , where if  $f \neq 0$ ,  $M_{A^*}(f) = 0.2$  and  $N_{A^*}(f) = 0.4$ , if  $f = 0$ ,  $M_{A^*}(0) = 0.5$  and  $N_{A^*}(0) = 0.3$ . Then  $A^* = (M_{A^*}, N_{A^*})$  is an intuitionistic fuzzy subspace of  $V^*$ .

### 3. The Double Dual of Generalized Fuzzy Subspaces

In this section, we mainly study the double dual of intuitionistic fuzzy subspaces. The double dual of interval-valued fuzzy subspaces and the double dual of interval-valued intuitionistic fuzzy subspaces can be investigated similarly.

Let  $V$  be a  $k$ -vector space and  $V^*$  be the space of all linear maps from  $V$  to  $k$ . Then  $V^{**} = (V^*)^*$  is the space of all linear maps from  $V^*$  to  $k$ , which is the double dual space for  $V$ . There exists a canonical injection  $i : V \rightarrow V^{**}$  by  $i(x)(f) = f(x)$ . If  $\dim V < \infty$ , then the injection  $i$  is an isomorphism.

Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy subspace of  $k$ -vector space  $V$ . Then the intuitionistic fuzzy subspace  $A^{**} = (\mu_{A^{**}}, \nu_{A^{**}}) : V^{**} \rightarrow [0, 1]$  is defined by

$$\mu_{A^{**}}(x) = \begin{cases} \inf\{\nu_{A^*}(f) \mid f \in V^*, x(f) \neq 0\} & \text{if } x \neq 0 \\ \sup\{\nu_{A^*}(f) \mid f \in V^*\} & \text{if } x = 0, \end{cases} \quad (3.1)$$

$$\nu_{A^{**}}(x) = \begin{cases} \sup\{\mu_{A^*}(f) \mid f \in V^*, x(f) \neq 0\} & \text{if } x \neq 0 \\ \inf\{\mu_{A^*}(f) \mid f \in V^*\} & \text{if } x = 0. \end{cases}$$

**Theorem 3.1.** Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy subspace of  $k$ -vector space  $V$  and  $i : V \rightarrow V^{**}$  be the canonical injection. Then  $A^{**}(i(x)) = A(x)$  for all  $x \in V \setminus \{0\}$ .

*Proof.* Let  $0 \neq x \in V$ . Then let

$$\begin{aligned}\mu_{A^{**}}(i(x)) &= \inf\{\nu_{A^*}(f) \mid f \in V^*, i(x)(f) \neq 0\} \\ &= \inf_{f \in V^*, i(x)(f) = f(x) \neq 0} \{\sup\{\mu_A(y) \mid y \in V, f(y) \neq 0\}\}, \\ \nu_{A^{**}}(i(x)) &= \sup\{\mu_{A^*}(f) \mid f \in V^*, i(x)(f) \neq 0\} \\ &= \sup_{f \in V^*, i(x)(f) = f(x) \neq 0} \{\inf\{\nu_A(y) \mid y \in V, f(y) \neq 0\}\}.\end{aligned}\tag{3.2}$$

Denote  $S = \{f \in V^* \mid f(x) \neq 0\}$  and let  $\mu_A(x) = t_1, \nu_A(x) = t_2$ .

Let  $f \in S$ . Then

if  $y \in U^{t_1}$ , we have  $f(y) \neq 0$  and  $\sup_{y \in U^{t_1}, f(y) \neq 0} \{\mu_A(y)\} \geq t_1$ ;

if  $y \notin U^{t_1}$ , we have  $f(y) \neq 0$  and  $\mu_A(y) < t_1$ , so  $\sup_{y \in V \setminus U^{t_1}, f(y) \neq 0} \{\mu_A(y)\} < t_1$ .

Hence for  $y \in V \setminus \{0\}$ , then  $\sup_{f(y) \neq 0} \{\mu_A(y)\} \geq t_1$ . Since  $x \in V \setminus \{0\}$  and  $f(x) \neq 0$ , we have

$$\inf_{f \in V^*, i(x)(f) = f(x) \neq 0} \{\sup\{\mu_A(y) \mid y \in V, f(y) \neq 0\}\} = t_1 = \mu_A(x).\tag{3.3}$$

On the other hand, let  $f \in S$ . Then

if  $y \in L^{t_2}$ , we have  $f(y) \neq 0$  and  $\inf_{y \in L^{t_2}, f(y) \neq 0} \{\nu_A(y)\} \leq t_2$ ;

if  $y \notin L^{t_2}$ , we have  $f(y) \neq 0$  and  $\nu_A(y) > t_2$ , so  $\inf_{y \in V \setminus L^{t_2}, f(y) \neq 0} \{\nu_A(y)\} > t_2$ .

Hence for  $y \in V \setminus \{0\}$ , then  $\inf_{f(y) \neq 0} \{\nu_A(y)\} \leq t_2$ . Since  $x \in V \setminus \{0\}$  and  $f(x) \neq 0$ , we have

$$\sup_{f \in V^*, i(x)(f) = f(x) \neq 0} \{\inf\{\nu_A(y) \mid y \in V, f(y) \neq 0\}\} = t_2 = \nu_A(x).\tag{3.4}$$

So  $\mu_{A^{**}}(i(x)) = \mu_A(x)$  and  $\nu_{A^{**}}(i(x)) = \nu_A(x)$ . □

**Theorem 3.2.** Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy subspace of  $k$ -vector space  $V$ . Then  $A^{**}(0) = (\mu_{A^{**}}(0), \nu_{A^{**}}(0))$ , where  $\mu_{A^{**}}(0) = \sup\{\mu_A(x) \mid x \in V \setminus \{0\}\}$  and  $\nu_{A^{**}}(0) = \inf\{\nu_A(x) \mid x \in V \setminus \{0\}\}$ .

*Proof.* We have

$$\begin{aligned}
 \mu_{A^{**}}(0) &= \sup\{\nu_{A^*}(f) \mid f \in V^*\} \\
 &= \sup\{\nu_{A^*}(f) \mid 0 \neq f \in V^*\} \\
 &= \sup_{0 \neq f \in V^*} \{\sup\{\mu_A(x) \mid x \in V, f(x) \neq 0\}\} \\
 &= \sup\{\mu_A(x) \mid x \in V \setminus \{0\}\}, \\
 \nu_{A^{**}}(0) &= \inf\{\mu_{A^*}(f) \mid f \in V^*\} \\
 &= \inf\{\mu_{A^*}(f) \mid 0 \neq f \in V^*\} \\
 &= \inf_{0 \neq f \in V^*} \{\inf\{\nu_A(x) \mid x \in V, f(x) \neq 0\}\} \\
 &= \inf\{\nu_A(x) \mid x \in V \setminus \{0\}\}.
 \end{aligned} \tag{3.5}$$

□

**Theorem 3.3.** *Let  $A = (\mu_A, \nu_A)$  be an intuitionistic fuzzy subspace of finite dimensional  $k$ -vector space  $V$  and  $\mu_A(0) = \sup\{\mu_A(V \setminus \{0\})\}$  and  $\nu_A(0) = \inf\{\nu_A(V \setminus \{0\})\}$ . Then the canonical map  $i : V \rightarrow V^{**}$  is an isomorphism between the intuitionistic fuzzy subspace  $A$  and  $A^{**}$ .*

*Proof.* Follows from Theorems 3.1 and 3.2. □

## 4. Conclusions

In this paper, we study the contents of the dual of generalized fuzzy subspaces. Generalized fuzzy subspaces, including intuitionistic fuzzy subspaces, interval-valued fuzzy subspaces and interval-valued intuitionistic fuzzy subspaces, are the basic contents for the further study of some algebras [18, 19]. Moreover, many algebras have the dual structures. Therefore it makes sense to investigate the dual of generalized fuzzy subspaces. In the future, we will consider the applications of the dual of generalized fuzzy subspaces in coalgebras and bialgebras.

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