

*Research Article*

# Exact Null Controllability for Fractional Nonlocal Integrodifferential Equations via Implicit Evolution System

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We introduce a new concept called implicit evolution system to establish the existence results of mild and strong solutions of a class of fractional nonlocal nonlinear integrodifferential system, then we prove the exact null controllability result of a class of fractional evolution nonlocal integrodifferential control system in Banach space. As an application that illustrates the abstract results, two examples are provided.

## 1. Introduction

In this paper, we study the fractional nonlocal integrodifferential system of the form

$$\frac{d^\alpha u(t)}{dt^\alpha} + A(t, B_1 u(t))u(t) = f(t, B_2 u(t)) + \int_0^t g(t, s, B_3 u(s))ds, \quad (1.1)$$

$$B_4(u(0) - u_0) = \int_0^a h(u(t))dt, \quad (1.2)$$

where  $0 < \alpha \leq 1$ ,  $t \in [0, a]$ . Let  $-A$  be the infinitesimal generator of a  $C_0$ -semigroup in a Banach space  $X$ , and  $\{B_i(t) : i = 1, 2, 3, 4\}$  is a family of linear closed operators defined on dense sets  $S_i \supset D(A)$ ,  $i = 1, 2, 3, 4$ , respectively, in  $X$  into  $X$ . It is assumed that  $u_0 \in X$ ,

$f : I \times X \rightarrow X$ ,  $g : \Delta \times X \rightarrow X$  and  $h : C(I : X) \rightarrow X$  are given abstract functions. Here,  $I = [0, a]$  and  $\Delta = \{(s, t) : 0 \leq s \leq t \leq a\}$ .

Basic researches in differential equations have showed that many phenomena in nature are modeled more accurately using fractional derivatives and integrals; for more detail, we can refer to [1–13] and the references therein. There are many applications where the fractional calculus can be used, for example, viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos, and fractals [14].

Controllability is a fundamental concept in mathematical control theory and plays an important role in both finite and infinite dimensional spaces, that is, systems represented by ordinary differential equations and partial differential equations, respectively. So it is natural to extend this concept to dynamical systems represented by fractional differential equations. Several fractional partial differential equations and integrodifferential equations can be expressed abstractly in some Banach spaces, in many cases, the accurate analysis, design and assessment of systems subjected to realistic environments must take into account the potential of random loads and randomness in the system properties. Randomness is intrinsic to the mathematical formulation of many phenomena such as fluctuations in the stock market or noise in communication networks. Fu studied the controllability results of some kinds of neutral functional differential systems, see [15, 16]. In our previous work [17], we established the controllability of fractional evolution nonlocal impulsive quasilinear delay integrodifferential systems.

The existence results to evolution equations with nonlocal conditions in Banach space were studied first by Byszewski [18, 19]; subsequently, many authors have been studied the same question, see for instance [20–23].

Deng [24] indicated that, using the nonlocal condition  $u(0) + h(u) = u_0$  to describe for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy problem  $u(0) = u_0$ . Let us observe also that, since Deng's papers, the function  $h$  is considered

$$h(u) = \sum_{k=1}^p c_k u(t_k), \tag{1.3}$$

where  $c_k, k = 1, 2, \dots, p$  are given constants and  $0 \leq t_1 < \dots < t_p \leq a$ .

In this paper, we introduce a new concept in the theory of Semigroup named "implicit evolution system" to show the reader "what is the main difference between the solutions of fractional ( $0 < \alpha < 1$ ) and classical (first order) homogeneous evolution equation?" which is based on the work [17] and Pazy [25]. A new form of nonlocal condition is also presented.

Our paper is organized as follows. Section 2 is devoted to a review of some essential results which will be used in this work to obtain our main results. In Section 3, we use the theory of semigroups [25] in order to introduce our new concept that is called implicit evolution system. In Section 4, we establish the existence, uniqueness, and regularity of mild solutions of a class of fractional evolution nonlinear integrodifferential systems with nonlocal conditions in Banach space. In Section 5, we prove the exact null controllability of a class of fractional evolution nonlocal integrodifferential control systems; the last section deals to give examples that provide the abstract results.

## 2. Preliminary Results

*Definition 2.1.* The fractional integral of order  $\alpha$  with the lower limit zero for a function  $f \in C([0, \infty))$  is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, 0 < \alpha < 1, \quad (2.1)$$

provided the right side is pointwise defined on  $[0, \infty)$ , where  $\Gamma$  is the gamma function.

Riemann-Liouville derivative of order  $\alpha$  with the lower limit zero for a function  $f \in C([0, \infty))$  can be written as

$${}^L D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\alpha} ds, \quad t > 0, 0 < \alpha < 1. \quad (2.2)$$

The Caputo derivative of order  $\alpha$  for a function  $f \in C([0, \infty))$  can be written as

$${}^C D^\alpha f(t) = {}^L D^\alpha (f(t) - f(0)), \quad t > 0, 0 < \alpha < 1. \quad (2.3)$$

*Remark 2.2.* (1) If  $f \in C^1([0, \infty))$ , then

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds = I^{1-\alpha} f'(t), \quad t > 0, 0 < \alpha < 1. \quad (2.4)$$

(2) The Caputo derivative of a constant is equal to zero.

(3) If  $f$  is an abstract function with values in  $X$ , then integrals which appear in Definition 2.1 are taken in Bochner's sense.

*Definition 2.3.* By a strong solution of the nonlocal Cauchy problem (1.1), (1.2), we mean a function  $u$  with values in  $X$  such that

- (i)  $u$  is a continuous function in  $t \in I$  and  $u(t) \in D(A)$ ,
- (ii)  $d^\alpha u/dt^\alpha$  exists and continuous on  $(0, a]$ ,  $0 < \alpha < 1$ , and  $u$  satisfies (1.1) on  $(0, a]$  and (1.2).

It is suitable to rewrite (1.1), (1.2) in the form

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} \times \left[ -A(\eta, B_1 u(\eta))u(\eta) + f(\eta, B_2 u(\eta)) + \int_0^\eta g(\eta, s, B_3 u(s)) ds \right] d\eta, \quad (2.5)$$

see also [26, 27].

Let  $X$  and  $Y$  be two Banach spaces such that  $Y$  is densely and continuously embedded in  $X$ . We denote by  $Z$  every Banach space  $Z(I, X)$  endowed with the usual norm, which is given by  $\|u\|_Z = \sup_{t \in I} \|u(t)\|$ , for  $u \in Z$ . The space of all bounded linear operators from  $X$  to  $Y$  is denoted by  $B(X, Y)$ . We recall some definitions and known facts from Pazy [25].

*Definition 2.4.* Let  $S$  be a linear operator in  $X$ , and let  $Y$  be a subspace of  $X$ . The operator  $\tilde{S}$  defined by  $D(\tilde{S}) = \{x \in D(S) \cap Y : Sx \in Y\}$  and  $\tilde{S}x = Sx$  for  $x \in D(\tilde{S})$  is called the part of  $S$  in  $Y$ .

*Definition 2.5.* Let  $\Omega$  be a subset of  $X$  and for every  $t \in I$  and  $B_1u \in \Omega$ , and let  $-A(t, B_1u)$  be the infinitesimal generator of a  $C_0$ -semigroup  $S_{t, B_1u}(s)$ ,  $s \geq 0$ , on  $X$ . The family of operators  $\{A(t, B_1u), (t, B_1u) \in I \times \Omega\}$  is stable if there are constants  $M \geq 1$  and  $\omega$  such that

$$\rho(A(t, B_1u)) \supset (\omega, \infty)$$

$$\left\| \prod_{j=1}^k R(\lambda : A(t_j, B_1u)) \right\| \leq M(\lambda - \omega)^{-k} \quad (2.6)$$

for  $\lambda > \omega$  every finite sequences  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq a$ ,  $1 \leq j \leq k$ .

The stability of  $\{A(t, B_1u), (t, B_1u) \in I \times \Omega\}$  implies that

$$\left\| \prod_{j=1}^k S_{t_j, B_1u}(s_j) \right\| \leq M \exp\left(\omega \sum_{j=1}^k s_j\right), \quad s_j \geq 0, \quad (2.7)$$

and any finite sequences  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq a$ ,  $1 \leq j \leq k$ ,  $k = 1, 2, \dots$

*Definition 2.6.* Let  $S_{t, B_1u}(s)$ ,  $s \geq 0$  be the  $C_0$ -semigroup generated by  $A(t, B_1u)$ ,  $(t, B_1u) \in I \times \Omega$ . A subspace  $Y$  of  $X$  is called  $A(t, B_1u)$ -admissible if  $Y$  is invariant subspace of  $S_{t, B_1u}(s)$ , and the restriction of  $S_{t, B_1u}(s)$  to  $Y$  is a  $C_0$ -semigroup in  $Y$ .

Let  $\Omega \subset X$  be a subset of  $X$  such that for every  $(t, B_1u) \in I \times \Omega$ ,  $A(t, B_1u)$  is the infinitesimal generator of a  $C_0$ -semigroup  $S_{t, B_1u}(s)$ ,  $s \geq 0$  on  $X$ . We make the following assumptions.

(H<sub>1</sub>) The family  $\{A(t, B_1u), (t, B_1u) \in I \times \Omega\}$  is stable.

(H<sub>2</sub>)  $Y$  is  $A(t, B_1u)$ -admissible for  $(t, B_1u) \in I \times \Omega$ , and the family  $\{\tilde{A}(t, B_1u), (t, B_1u) \in I \times \Omega\}$  of parts  $\tilde{A}(t, B_1u)$  of  $A(t, B_1u)$  in  $Y$  is stable in  $Y$ .

(H<sub>3</sub>) For  $(t, B_1u) \in I \times \Omega$ ,  $D(A(t, B_1u)) \supset Y$ ,  $A(t, B_1u)$  is a bounded linear operator from  $Y$  to  $X$  and  $t \rightarrow A(t, B_1u)$  is continuous in the  $B(Y, X)$  norm  $\|\cdot\|$ .

(H<sub>4</sub>) There is a constant  $L > 0$  such that

$$\|A(t, B_1u) - A(t, B_1v)\|_{Y \rightarrow X} \leq L\|u - v\|_X \quad (2.8)$$

holds for every  $B_1u, B_1v \in \Omega$ , and  $t \in I$ .

In the next section, we will introduce a new concept in the theory of semigroups.

### 3. Implicit Evolution System

Let  $\Omega$  be a subset of  $X$  and  $\{A(t, B_1u), (t, B_1u) \in I \times \Omega\}$  a family of operators satisfying the conditions (H<sub>1</sub>)–(H<sub>4</sub>). If  $u \in C(I : X)$  has values in  $\Omega$ , then there is a unique evolution system  $U_\alpha(t, s; B_1u)$ ,  $0 < \alpha \leq 1$ ,  $0 \leq s \leq t \leq a$ , in  $X$  satisfying

- (i)  $\|U_\alpha(t, s; B_1 u)\| \leq M e^{\omega(t-s)}$  for  $0 \leq s \leq t \leq a$ , where  $M$  and  $\omega$  are stability constants,
- (ii)  $(\partial^\alpha / \partial t^\alpha) U_\alpha(t, s; B_1 u) y = A(s, B_1 u(s)) U_\alpha(t, s; B_1 u) y$  for  $y \in Y$  and  $0 \leq s \leq t \leq a$ ,
- (iii)  $(\partial^\alpha / \partial s^\alpha) U_\alpha(t, s; B_1 u) y = -U_\alpha(t, s; B_1 u) A(s, B_1 u(s)) y$  for  $y \in Y$  and  $0 \leq s \leq t \leq a$ .

*Remark 3.1.* (1) If  $B_1$  is the identity and  $\alpha = 1$ , then  $U(t, s; u)$  is the explicit evolution system given in Pazy [25] and in Zaidman [28].

(2) Since, in our case,  $U(t, s; u)$  is dependent of  $\alpha$  and  $B_1$ , so we call it an implicit evolution system generated by  $-A(t, B_1 u)$ .

(3) For nonautonomous differential equations in a Banach space, the implicit evolution system is similar to our concept  $(\alpha, u)$ -resolvent family.

(4) We can deduce that (1.1)-(1.2) is well posed if and only if  $-A(t, B_1 u)$  is the generator of the implicit evolution system  $U(t, s; u)$ .

Further, we assume the following.

(H<sub>5</sub>) For every  $u \in C(I : X)$  satisfying  $u(t) \in \Omega$  for  $0 \leq t \leq a$ , we have

$$U(t, s; u)Y \subset Y, \quad 0 \leq s \leq t \leq a \quad (3.1)$$

and  $U(t, s; u)$  is strongly continuous in  $Y$  for  $0 \leq s \leq t \leq a$ .

(H<sub>6</sub>)  $Y$  is reflexive.

(H<sub>7</sub>) For every  $(t, B_2 u) \in I \times \Omega$ ,  $f(t, B_2 u) \in Y$ .

(H<sub>8</sub>) The operator  $[B_4(t) + \lambda^\alpha I]^{-1}$  exists in  $B(X)$  for any  $\lambda$  with  $\text{Re } \lambda \leq 0$  and

$$\|[B_4(t) + \lambda^\alpha I]^{-1}\| \leq \frac{C_\alpha}{|\lambda| + 1}, \quad t \in I, \quad (3.2)$$

where  $C_\alpha$  is a positive constant independent of both  $t$  and  $\lambda$ .

(H<sub>9</sub>)  $h : C(I : \Omega) \rightarrow Y$  is Lipschitz continuous in  $X$  and bounded in  $Y$ , that is, there exist constants  $k_1 > 0$  and  $k_2 > 0$  such that

$$\begin{aligned} \|h(u)\|_Y &\leq k_1, \\ \|h(u) - h(v)\|_Y &\leq k_2 \max_{t \in I} \|u - v\|_X. \end{aligned} \quad (3.3)$$

For the conditions (H<sub>9</sub>) and (H<sub>10</sub>), let  $Z$  be taken as both  $X$  and  $Y$ .

(H<sub>10</sub>)  $g : \Delta \times Z \rightarrow Z$  is continuous, and there exist constants  $k_3 > 0$  and  $k_4 > 0$  such that

$$\begin{aligned} \int_0^t \|g(t, s, B_3 u) - g(t, s, B_3 v)\|_Z ds &\leq k_3 \|u - v\|_Z, \quad u, v \in X, \\ k_4 &= \max \left\{ \int_0^t \|g(t, s, 0)\|_Z ds : (t, s) \in \Delta \right\}. \end{aligned} \quad (3.4)$$

(H<sub>11</sub>)  $f : I \times Z \rightarrow Z$  is continuous, and there exist constants  $k_5 > 0$  and  $k_6 > 0$  such that

$$\begin{aligned} \|f(t, B_2u) - f(t, B_2v)\|_Z &\leq k_5\|u - v\|_Z, \quad u, v \in X, \\ k_6 &= \max_{t \in I} \|f(t, 0)\|_Z. \end{aligned} \quad (3.5)$$

Let us take  $M_0 = \max \|U(t, s; u)\|_{\Omega(Z)}$ ,  $0 \leq s \leq t \leq a$ ,  $u \in \Omega$ .

(H<sub>12</sub>) There exist positive constants  $r > 0$  and  $0 < \lambda < 1$  such that

$$\begin{aligned} M_0\{\|u_0\| + aC_\alpha k_1 + a[r(k_3 + k_5) + k_4 + k_6]\} &\leq r, \\ \lambda = Ka\|u_0\|_\gamma + a^2C_\alpha k_1K + aM_0C_\alpha k_2 + a\{K[(k_3 + k_5)r + k_4 + k_6]a + M_0(k_3 + k_5)\}. \end{aligned} \quad (3.6)$$

By a mild solution of (1.1), (1.2), we mean a function  $u \in C(I : X)$  with values in  $\Omega$  and  $u_0 \in X$  satisfying the integral equation

$$\begin{aligned} u(t) &= U(t, 0; u)u_0 + U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt \\ &+ \int_0^t U(t, s; u) \left[ f(s, B_2u(s)) + \int_0^s g(s, \eta, B_3u(\eta))d\eta \right] ds. \end{aligned} \quad (3.7)$$

(H<sub>13</sub>) Further, there exists a constant  $K > 0$  such that for every  $u, v \in C(I : X)$  with values in  $\Omega$  and every  $\omega \in Y$  we have

$$\|U(t, s; u)\omega - U(t, s; v)\omega\| \leq K\|\omega\|_\gamma \int_s^t \|u(\tau) - v(\tau)\|d\tau. \quad (3.8)$$

#### 4. Existence Results

**Theorem 4.1.** *Let  $u_0 \in Y$  and  $\Omega = \{u \in X : \|u\|_\gamma \leq r\}$ ,  $r > 0$ . If  $-A(t, B_1u)$  is the generator of an implicit evolution system  $U(t, s; u)$  and the assumptions (H<sub>5</sub>)~(H<sub>13</sub>) are satisfied, then (1.1), (1.2) has a unique mild solution on  $I$ .*

*Proof.* Let  $S$  be a nonempty closed subset of  $C(I : X)$  defined by

$$S = \{u : u \in C(I : X), \|u\|_\gamma \leq r\}, \quad t \in I. \quad (4.1)$$

Consider a mapping  $P$  on  $S$  defined by

$$\begin{aligned} (Pu)(t) &= U(t, 0; u)u_0 + U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt \\ &+ \int_0^t U(t, s; u) \left[ f(s, B_2u(s)) + \int_0^s g(s, \eta, B_3u(\eta))d\eta \right] ds. \end{aligned} \quad (4.2)$$

For  $u \in S$ , we have

$$\begin{aligned} \|Pu(t)\|_Y &\leq \|U(t, 0; u)u_0\| + \left\| U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt \right\| \\ &+ \int_0^t \|U(t, s; u)\| \left\{ \|f(s, B_2u(s)) - f(s, 0)\| + \|f(s, 0)\| \right. \\ &\quad \left. + \left\| \int_0^s [g(s, \eta, B_3u(\eta)) - g(s, \eta, 0)]d\eta \right\| + \left\| \int_0^s g(s, \eta, 0)d\eta \right\| \right\} ds. \\ &\leq M_0\|u_0\| + aM_0C_\alpha k_1 \\ &\quad + \int_0^t M_0\{k_5\|u(s)\| + k_6 + k_3\|u(s)\| + k_4\} ds \\ &\leq M_0\|u_0\| + aM_0C_\alpha k_1 + aM_0\{k_5\|u(s)\| + k_6 + k_3\|u(s)\| + k_4\} \\ &\leq M_0\{\|u_0\| + aC_\alpha k_1 + a[r(k_3 + k_5) + k_4 + k_6]\} \\ &\leq r. \end{aligned} \quad (4.3)$$

Thus,  $P$  maps  $S$  into itself. Now, we will show that  $P$  is a strict contraction on  $S$  which will ensure the existence of a unique continuous function satisfying (3.7) on  $I$ .

If  $u, v \in S$ , then

$$\begin{aligned} &\|Pu(t) - Pv(t)\| \\ &\leq \|U(t, 0; u)u_0 - U(t, 0; v)u_0\| + \left\| U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt - U(t, 0; v)B_4^{-1} \int_0^a h(v(t))dt \right\| \\ &\quad + \int_0^t \left\| U(t, s; u) \left[ f(s, B_2u(s)) + \int_0^s g(s, \eta, B_3u(\eta))d\eta \right] \right. \\ &\quad \left. - U(t, s; v) \left[ f(s, B_2v(s)) + \int_0^s g(s, \eta, B_3v(\eta))d\eta \right] \right\| ds \\ &\leq \|U(t, 0; u)u_0 - U(t, 0; v)u_0\| + \left\| U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt - U(t, 0; v)B_4^{-1} \int_0^a h(u(t))dt \right\| \\ &\quad + \left\| U(t, 0; v)B_4^{-1} \int_0^a h(u(t))dt - U(t, 0; v)B_4^{-1} \int_0^a h(v(t))dt \right\| \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \left\{ \left\| U(t, s; u) \left[ f(s, B_2 u(s)) + \int_0^s g(s, \eta, B_3 u(\eta)) d\eta \right] \right. \right. \\
& \quad \left. \left. - U(t, s; v) \left[ f(s, B_2 u(s)) + \int_0^s g(s, \eta, B_3 u(\eta)) d\eta \right] \right\| \right. \\
& \quad \left. + \left\| U(t, s; v) \left[ f(s, B_2 u(s)) + \int_0^s g(s, \eta, B_3 u(\eta)) d\eta \right] \right. \right. \\
& \quad \left. \left. - U(t, s; v) \left[ f(s, B_2 v(s)) + \int_0^s g(s, \eta, B_3 v(\eta)) d\eta \right] \right\| \right\} ds \\
& \leq K a \|u_0\|_Y \max_{\tau \in I} \|u(\tau) - v(\tau)\| + a^2 C_\alpha k_1 K \max_{\tau \in I} \|u(\tau) - v(\tau)\| \\
& \quad + a M_0 C_\alpha k_2 \max_{\tau \in I} \|u(\tau) - v(\tau)\| \\
& \quad + \int_0^t \left\{ K \left\| f(s, B_2 u(s)) + \int_0^s g(s, \eta, B_3 u(\eta)) d\eta \right\|_Y a \max_{\tau \in I} \|u(\tau) - v(\tau)\| \right. \\
& \quad \left. + M_0 \left\| f(s, B_2 u(s)) - f(s, B_2 v(s)) \right\| \right. \\
& \quad \left. + \int_0^s \left\| g(s, \eta, B_3 u(\eta)) - g(s, \eta, B_3 v(\eta)) \right\| d\eta \right\} ds \\
& \leq \left( K a \|u_0\|_Y + a^2 C_\alpha k_1 K + a M_0 C_\alpha k_2 \right) \max_{\tau \in I} \|u(\tau) - v(\tau)\| \\
& \quad + a \{ K [(k_3 + k_5)r + k_4 + k_6] a + M_0 (k_3 + k_5) \} \max_{\tau \in I} \|u(\tau) - v(\tau)\|.
\end{aligned} \tag{4.4}$$

Thus,

$$\|Pu(t) - Pv(t)\| \leq \lambda \max_{\tau \in I} \|u(\tau) - v(\tau)\|, \tag{4.5}$$

which means that  $P$  is a strict contraction map from  $S$  into  $S$ , and therefore by the Banach contraction principle there exists a unique fixed point  $u \in S$  such that  $Pu = u$ . Hence,  $u$  is a unique mild solution of (1.1), (1.2) on  $I$ .  $\square$

**Theorem 4.2.** *Assume the following.*

- (i) *Conditions  $(H_1) \sim (H_{13})$  hold.*
- (ii) *The functions  $B_2(t)\omega$  and  $B_3(t)\omega$  are uniformly Hölder continuous in  $t \in I$  for every element  $\omega$  in  $S_2 \cap S_3$ .*
- (iii) *There are numbers  $L_1, L_2 > 0$  and  $p, q \in (0, 1]$  such that*



$$\begin{aligned} \|f(t_1, B_2u) - f(t_2, B_2v)\| &\leq L_1(|t_1 - t_2|^p + \|B_2u - B_2v\|), \\ \|g(s_1, \eta, B_3u) - g(s_2, \eta, B_3v)\| &\leq L_2|s_1 - s_2|^q \end{aligned} \quad (4.6)$$

for all  $t_1, t_2 \in I$  and all  $(s_1, \eta), (s_2, \eta) \in \Delta$ .

Then, the problem (1.1), (1.2) has a unique strong solution on  $I$ .

*Proof.* Applying Theorem 4.1, the problem (1.1), (1.2) has a mild solution  $u \in S$ . Now, we will show that  $u$  is a unique strong solution of the considered problem on  $I$ .

According to (ii),  $\|B_2u - B_2v\|$  is uniformly Hölder continuous in  $t \in I$  for every element  $u$  in  $S_2 \cap S_3$ , also (iii) implies that  $t \rightarrow f(t, B_2u)$  and  $t \rightarrow \int_0^t g(t, s, B_3u)ds$  are uniformly Hölder continuous on  $I$  ([20, 26]).

Set

$$V(t) = f(t, B_2u) + \int_0^t g(t, s, B_3u)ds. \quad (4.7)$$

Clearly  $V(t)$  is uniformly Hölder continuous in  $t \in I$ .

Consider the following nonlocal Cauchy problem:

$$\begin{aligned} \frac{d^\alpha v(t)}{dt^\alpha} + A(t, B_1u(t))u(t) &= V(t), \\ B_4(u(0) - u_0) &= \int_0^a h(u(t))dt. \end{aligned} \quad (4.8)$$

From Pazy, (4.8) has a unique solution  $v$  on  $I$  given by

$$v(t) = U(t, 0; u)u_0 + U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt + \int_0^t U(t, s; u)V(s)ds. \quad (4.9)$$

Noting that [21], each term on the right hand side of (4.9) belongs to  $D(A)$ , thus  $v(t) \in D(A)$ , using the uniqueness of  $V(t)$ , we have that  $u(t) = v(t)$ . Hence,  $u$  is the unique strong solution of (1.1), (1.2) on  $I$ .  $\square$

In next section, some results are obtained from Sakthivel et al. [29, 30].

## 5. Exactly Null Controllability Results

Consider fractional nonlocal evolution integrodifferential control system of the form

$$\begin{aligned} \frac{d^\alpha u(t)}{dt^\alpha} + A(t, B_1u(t))u(t) &= (\Phi\mu)(t) + \Psi \left( t, f(t, B_2u(t)), \int_0^t g(t, s, B_3u(s))ds \right), \\ B_4(u(0) - u_0) &= \int_0^a h(u(t))dt, \end{aligned} \quad (5.1)$$

where the unknown  $u(\cdot)$  takes values in the Banach space  $X$ , the control function  $\mu$  belongs to the spaces  $L^2(I, H)$ , a Banach space of admissible control functions with  $H$ , a Banach space. Further,  $\Phi$  is a bounded linear operator from  $H$  into  $X$ , the function  $\Psi : I \times X \times X \rightarrow X$  is given, and the others terms are defined as above.

For all  $u_0 \in X$  and admissible control  $\mu \in L^2(I, H)$ , the problem (5.1) admits a mild solution given by

$$u_\mu(t) = U(t, 0; u)u_0 + U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt + \int_0^t U(t, s; u) \left[ \Phi\mu(s) + \Psi \left( s, f(s, B_2u(s)), \int_0^s g(s, \tau, B_3u(\tau))d\tau \right) \right] ds. \quad (5.2)$$

*Definition 5.1.* We will say that system (5.1) is exactly null controllable on the interval  $I$  if for all  $u_0 \in X$ , there exists a control  $\mu \in L^2(I, H)$ , such that the mild solution  $u(t)$  of (5.1) corresponding to  $\mu$  verifies  $u(0) - B_4^{-1} \int_0^a h(u(t))dt = u_0$  and  $u(a) = 0$ .

In order to prove the controllability result, in addition, we consider the following conditions.

(H<sub>14</sub>)  $\Psi : I \times X \times X \rightarrow X$  is continuous, and there exist constants  $N_1$  and  $N_2$  such that for all  $x_i, y_i \in X, i = 1, 2$ , we have

$$\begin{aligned} \|\Psi(t, x_1, y_1) - \Psi(t, x_2, y_2)\| &\leq N_1 [\|x_1 - x_2\| + \|y_1 - y_2\|], \\ N_2 &= \max_{t \in I} \|\Psi(t, 0, 0)\|. \end{aligned} \quad (5.3)$$

(H<sub>15</sub>) Let

$$\rho + aM_0M_1M_2(\rho + \sigma) + \sigma \leq r, \quad (5.4)$$

where  $\sigma = aM_0\{N_1[(k_3 + k_5)r + k_4 + k_6] + N_2\}$  and  $\rho = M_0\|u_0\| + aM_0C_\alpha k_1$ , and let

$$\lambda = \left\{ Ka\|u_0\| + a^2C_\alpha k_1K + aM_0C_\alpha k_2 \right\} + 2a\{M_0M_1M_2(\rho + ap)\} + a\{Kqa + p\}, \quad (5.5)$$

where  $p = M_0N_1(k_3 + k_5)$ ,  $q = N_1[(k_3 + k_5)r + k_4 + k_6] + N_2$ , and  $0 \leq \lambda < 1$ .

(H<sub>16</sub>) The bounded linear operator  $W : L^2(I, H) \rightarrow X$  defined by

$$W\mu = \int_0^a U(a, s; u)\Phi\mu ds \quad (5.6)$$

has an induced inverse operator  $\widetilde{W}^{-1}$  which takes values in  $L^2(I, H)/\ker W$  and there exist positive constants  $M_1, M_2$ , such that  $\|\Phi\| \leq M_1$  and  $\|\widetilde{W}^{-1}\| \leq M_2$ .

**Theorem 5.2.** *If hypotheses  $(H_1)$ – $(H_{16})$  are satisfied, then the control nonlocal fractional integrodifferential system (5.1) is exactly null controllable on  $I$ .*

*Proof.* Let  $S_r = \{u : u \in C(I : X), u(0) - B_4^{-1} \int_0^a h(u(t))dt = u_0, \|u\| \leq r, t \in I\}$ .

We define an operator  $Q : S_r \rightarrow S_r$  by

$$\begin{aligned} (Qu_\mu)(t) &= U(t, 0; u)u_0 + U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt \\ &+ \int_0^t U(t, \eta; u)\Phi\widetilde{W}^{-1} \left[ -U(a, 0; u)u_0 - U(a, 0; u)B_4^{-1} \int_0^a h(u(t))dt \right. \\ &\quad \left. - \int_0^a U(a, s; u)\Psi \left( s, f(s, B_2u(s)), \int_0^s g(s, \tau, B_3u(\tau))d\tau \right) ds \right] (\eta)d\eta \\ &+ \int_0^t U(t, s; u)\Psi \left( s, f(s, B_2u(s)), \int_0^s g(s, \tau, B_3u(\tau))d\tau \right) ds. \end{aligned} \quad (5.7)$$

Using the hypothesis  $(H_{14})$ , for an arbitrary function  $u(\cdot)$ , we define the control

$$\begin{aligned} \mu(t) &= \widetilde{W}^{-1} \left[ -U(a, 0; u)u_0 - U(a, 0; u)B_4^{-1} \int_0^a h(u(t))dt \right. \\ &\quad \left. - \int_0^a U(a, s; u)\Psi \left( s, f(s, B_2u(s)), \int_0^s g(s, \tau, B_3u(\tau))d\tau \right) ds \right] (t). \end{aligned} \quad (5.8)$$

Using this controller, we will show that the operator  $Q$  has a fixed point. This fixed point is then a solution of (5.2).

Clearly,  $Q_\mu u(a) = 0$ , which means that the control  $\mu$  steers system (5.1) from the initial state  $u_0$  to origin in time  $a$ , provided we can obtain a fixed point of the nonlinear operator  $Q$ .

Now, we show that  $Q$  maps  $S_r$  into itself.

We have

$$\begin{aligned} &\| (Qu_\mu)(t) \| \\ &\leq \left\| U(t, 0; u)u_0 + U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt \right\| \\ &\quad + \int_0^t \| U(t, \eta; u) \| \left\| \Phi\widetilde{W}^{-1} \left[ \left\| U(a, 0; u)u_0 + U(a, 0; u)B_4^{-1} \int_0^a h(u(t))dt \right\| \right. \right. \\ &\quad \left. \left. + \int_0^a \| U(a, s; u) \| \left\{ \left\| \Psi \left( s, f(s, B_2u(s)), \int_0^s g(s, \tau, B_3u(\tau))d\tau \right) \right\| \right. \right. \right. \\ &\quad \left. \left. \left. - \Psi(s, 0, 0) \right\| + \left\| \Psi(s, 0, 0) \right\| \right\} ds \right] d\eta \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \|U(t, s; u)\| \\
& \quad \times \left\{ \left\| \Psi \left( s, f(s, B_2 u(s)), \int_0^s g(s, \tau, B_3 u(\tau)) d\tau \right) - \Psi(s, 0, 0) \right\| + \|\Psi(s, 0, 0)\| \right\} ds. \\
& \leq M_0 \|u_0\| + aM_0 C_\alpha k_1 \\
& \quad + aM_0 M_1 M_2 [M_0 \|u_0\| + aM_0 C_\alpha k_1 + aM_0 \{N_1(k_5 r + k_6 + k_3 r + k_4) + N_2\}] \\
& \quad + aM_0 \{N_1(k_5 r + k_6 + k_3 r + k_4) + N_2\} \\
& \leq r.
\end{aligned} \tag{5.9}$$

Thus,  $Q$  maps  $S_r$  into itself. Now, for  $u, v \in S_r$ , we have

$$\begin{aligned}
& \|Qu_\mu(t) - Qv_\mu(t)\| \\
& \leq \|U(t, 0; u)u_0 - U(t, 0; v)u_0\| + \left\| U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt - U(t, 0; v)B_4^{-1} \int_0^a h(v(t))dt \right\| \\
& \quad + \int_0^t \left\{ \left\| U(t, \eta; u)\Phi\widetilde{W}^{-1} \left[ -U(a, 0; u)u_0 - U(a, 0; u)B_4^{-1} \int_0^a h(u(t))dt \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. - \int_0^a U(a, s; u)\Psi \left( s, f(s, B_2 u(s)), \int_0^s g(s, \tau, B_3 u(\tau))d\tau \right) ds \right] \right. \right. \\
& \quad \quad \left. \left. - U(t, \eta; v)\Phi\widetilde{W}^{-1} \left[ -U(a, 0; v)u_0 - U(a, 0; v)B_4^{-1} \int_0^a h(v(t))dt \right. \right. \right. \\
& \quad \quad \quad \left. \left. \left. - \int_0^a U(a, s; v)\Psi \left( s, f(s, B_2 v(s)), \right. \right. \right. \\
& \quad \quad \quad \quad \left. \left. \left. \int_0^s g(s, \tau, B_3 v(\tau))d\tau \right) ds \right] \right\} d\eta \\
& \quad + \int_0^t \left\| U(t, s; u)\Psi \left( s, f(s, B_2 u(s)), \int_0^s g(s, \tau, B_3 u(\tau))d\tau \right) \right. \\
& \quad \quad \left. - U(t, s; v)\Psi \left( s, f(s, B_2 v(s)), \int_0^s g(s, \tau, B_3 v(\tau))d\tau \right) \right\| ds. \\
& \leq \|U(t, 0; u)u_0 - U(t, 0; v)u_0\| + \left\| U(t, 0; u)B_4^{-1} \int_0^a h(u(t))dt - U(t, 0; v)B_4^{-1} \int_0^a h(u(t))dt \right\| \\
& \quad + \left\| U(t, 0; v)B_4^{-1} \int_0^a h(u(t))dt - U(t, 0; v)B_4^{-1} \int_0^a h(v(t))dt \right\| \\
& \quad + \int_0^t \left\{ \left\| U(t, \eta; u)\Phi\widetilde{W}^{-1} \left[ -U(a, 0; u)u_0 - U(a, 0; u)B_4^{-1} \int_0^a h(u(t))dt \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \int_0^a U(a, s; u) \Psi \left( s, f(s, B_2 u(s)), \int_0^s g(s, \tau, B_3 u(\tau)) d\tau \right) ds \Big] \\
& - U(t, \eta; v) \Phi \widetilde{W}^{-1} \left[ -U(a, 0; v) u_0 - U(a, 0; v) B_4^{-1} \int_0^a h(v(t)) dt \right. \\
& \quad \left. - \int_0^a U(a, s; v) \Psi \left( s, f(s, B_2 v(s)), \int_0^s g(s, \tau, B_3 v(\tau)) d\tau \right) ds \right] \Big] d\eta \\
& + \int_0^t \left\{ \left\| U(t, s; u) \Psi \left( s, f(s, B_2 u(s)), \int_0^s g(s, \tau, B_3 u(\tau)) d\tau \right) \right. \right. \\
& \quad \left. \left. - U(t, s; v) \Psi \left( s, f(s, B_2 u(s)), \int_0^s g(s, \tau, B_3 u(\tau)) d\tau \right) \right\| \right. \\
& \quad \left. + \left\| U(t, s; v) \Psi \left( s, f(s, B_2 u(s)), \int_0^s g(s, \tau, B_3 u(\tau)) d\tau \right) \right. \right. \\
& \quad \left. \left. - U(t, s; v) \Psi \left( s, f(s, B_2 v(s)), \int_0^s g(s, \tau, B_3 v(\tau)) d\tau \right) \right\| \right\} ds. \\
& \leq \left( K a \|u_0\| + a^2 C_\alpha k_1 K + a M_0 C_\alpha k_2 \right) \max_{\tau \in I} \|u(\tau) - v(\tau)\| \\
& \quad + 2a \{ M_0 M_1 M_2 [M_0 \|u_0\| + a M_0 C_\alpha k_1 + a M_0 N_1 (k_5 + k_3)] \} \max_{\tau \in I} \|u(\tau) - v(\tau)\| \\
& \quad + a \{ a K [N_1 (k_5 r + k_6 + k_3 r + k_4) + N_2] + M_0 N_1 (k_5 + k_3) \} \max_{\tau \in I} \|u(\tau) - v(\tau)\| \\
& \leq \lambda \max_{\tau \in I} \|u(\tau) - v(\tau)\|.
\end{aligned} \tag{5.10}$$

Therefore,  $Q$  is a contraction mapping, and hence there exists a unique fixed point  $u \in X$ , such that  $Qu(t) = u(t)$ . Any fixed point of  $Q$  is a mild solution of (5.1) on  $I$  which satisfies  $u(a) = 0$ . Thus, system (5.1) is exactly null controllable on  $I$ .  $\square$

## 6. Examples

To illustrate the abstract results, we give the following examples.

*Example 6.1.* Consider the nonlinear integropartial differential equation of fractional order

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + \sum_{|q| \leq 2m} a_q(x, t; b_q u(x, t)) D_x^q u(x, t) = F(x, t, w_1) + \int_0^t G(x, t, s, w_2(s)) ds, \tag{6.1}$$

with nonlocal condition

$$u(x, 0) + \sum_{k=1}^p c_k u(x, t_k) = g(x), \quad (6.2)$$

where  $0 < \alpha \leq 1$ ,  $0 \leq t_1 < \dots < t_p \leq a$ ,  $x \in R^n$ ,  $D_x^q = D_{x_1}^{q_1} \dots D_{x_n}^{q_n}$ ,  $D_{x_i} = \partial / \partial x_i$ ,  $q = (q_1, \dots, q_n)$  is an  $n$ -dimensional multi-index,  $|q| = q_1 + \dots + q_n$ , and  $w_i$ ,  $i = 1, 2$ , is given by

$$w_i(x, t) = \sum_{|q| \leq 2m-1} b_{q_i}(x, t) D_x^q u(x, t) + \int_{\Omega} \sum_{|q| \leq 2m-1} c_{q_i}(x, t) D_y^q u(y, t) dy. \quad (6.3)$$

Let  $L_2(R^n)$  be the set of all square integrable functions on  $R^n$ . We denote by  $C^m(R^n)$  the set of all continuous real-valued functions defined on  $R^n$  which have continuous partial derivatives of order less than or equal to  $m$ . By  $C_0^m(R^n)$ , we denote the set of all functions  $f \in C^m(R^n)$  with compact supports. Let  $H^m(R^n)$  be the completion of  $C_0^m(R^n)$  with respect to the norm

$$\|f\|_m^2 = \sum_{|q| \leq m} \int_{R^n} |D_x^q f(x)|^2 dx. \quad (6.4)$$

It is supposed that the following hold.

(i) The operator  $A = - \sum_{|q| \leq 2m} a_q(x, t; b_q u(x, t)) D_x^q$  is uniformly elliptic on  $R^n$ . In other words, all the coefficients  $a_q$ ,  $|q| = 2m$ , are continuous and bounded on  $R^n$ , and there is a positive number  $c$  such that

$$(-1)^{m+1} \sum_{|q|=2m} e_q(x, t; b_q) \xi^q \geq c |\xi|^{2m}, \quad (6.5)$$

for all  $x \in R^n$  and all  $\xi \neq 0$ ,  $\xi \in R^n$ ,  $\xi^q = \xi_1^{q_1} \dots \xi_n^{q_n}$ , and  $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$ .

(ii) All the coefficients  $a_q$ ,  $|q| = 2m$ , satisfy a uniform Hölder condition on  $R^n$ . Under these conditions, the operator  $A$  with domain of definition  $D(A) = H^{2m}(R^n)$  generates an evolution operator defined on  $L_2(R^n)$ , and it is well known that  $H^{2m}(R^n)$  is dense in  $X = L_2(R^n)$  and the initial function  $g(x)$  is an element in Hilbert space  $H^{2m}(R^n)$ , see [26, page 438]. Applying Theorem 4.1, this achieves the proof of the existence of mild solutions of the problem (6.1), (6.2). In addition,

(iii) If the coefficients  $b_q$ ,  $c_q$ ,  $|q| \leq 2m - 1$  satisfy a uniform Hölder condition on  $R^n$  and the operators  $F$  and  $G$  satisfy.

There are numbers  $L_1, L_2 \geq 0$  and  $\lambda_1, \lambda_2 \in (0, 1)$  such that

$$\begin{aligned} \sum_{|q| \leq 2m-1} \int_{R^n} |F(x, t, D_x^q w_1) - F(x, s, D_x^q w_1^*)|^2 dx &\leq L_1 (|t - s|^{\lambda_1} + |w_1 - w_1^*|^2 dx), \\ \sum_{|q| \leq 2m-1} \int_{R^n} |G(x, t, \eta, D_x^q w_2) - G(x, s, \eta, D_x^q w_2)|^2 dx &\leq L_2 |t - s|^{\lambda_2}, \end{aligned} \quad (6.6)$$

for all  $t, s \in I$ ,  $(t, \eta), (s, \eta) \in \Delta$ , and all  $x \in R^n$ . Applying Theorem 4.2, we deduce that (6.1), (6.2) has a unique strong solution.

The second example is concerned with the controllability result.

*Example 6.2.* Consider the fractional nonlocal evolution integropartial differential control system of the form

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + a(x, t, (bu)(x, t)) \frac{\partial^2 u(x, t)}{\partial x^2} &= \zeta(x, t) + \Upsilon \left( t, \mu_1(t, u(x, t)), \int_0^t \mu_2(t, s, u(x, t)) ds \right), \\ u(x, 0) + \sum_{k=1}^p c_k u(x, t_k) &= g(x), \quad x \in [0, \pi], \\ u(0, t) = u(\pi, t) &= 0, \quad t \in I, \end{aligned} \tag{6.7}$$

where  $0 < \alpha \leq 1$ ,  $0 \leq t_1 < \dots < t_p \leq a$ , and the functions  $a(x, t, \cdot)$ ,  $b(x, t)$  are continuous.

Let us take

$$X = L^2[0, \pi], \quad S_r = \{y \in L^2[0, \pi] : \|y\| \leq r\}. \tag{6.8}$$

Put  $(\Phi\mu)(x)(t) = \zeta(x, t)$ ,  $x \in [0, \pi]$  where  $\mu(t) = \zeta(\cdot, t)$  and  $\zeta : [0, \pi] \times I \rightarrow [0, \pi]$  is continuous.

We define  $A(t, \cdot) : X \rightarrow X$  by  $(A(t, \cdot)w)(x) = a(x, t, \cdot)w''$  with domain  $D(A) = \{w \in X : w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = w(\pi) = 0\}$ . Assume that  $-A(t, \cdot)$  generates an evolution system  $U(t, s, \cdot)$  such that for every positive numbers  $n_1$  and  $n_2$ ,  $\|U(t, s, \cdot)\| \leq n_1$  and  $\|U(t, s, \cdot)A(s, \cdot)\| \leq n_2$ .

Also, define  $B_1(t) : D(B_1) = S_1 \rightarrow X$  by  $(B_1(t)z)(x) = b(x, t)z$ , for all  $z \in X$  and  $x \in [0, \pi]$ .

Assume that the linear operator  $W$  that is given by

$$W\mu(x) = \int_0^a U(a, s; u)\zeta(x, s)ds, \quad x \in [0, \pi], \tag{6.9}$$

has a bounded invertible operator  $\widetilde{W}^{-1}$  in  $L^2(I, H)/\ker W$ .

Let us assume that the nonlinear functions  $\Upsilon$ ,  $\mu_1$ , and  $\mu_2$  satisfy the following Lipschitz conditions

$$\begin{aligned} \|\Upsilon(t, z_1, y_1) - \Upsilon(t, z_2, y_2)\| &\leq c_1 [\|z_1 - z_2\| + \|y_1 - y_2\|], \\ \|\mu_1(t, x_1) - \mu_1(t, x_2)\| &\leq c_2 \|x_1 - x_2\|, \\ \int_0^t \|\mu_2(t, s, v_1) - \mu_2(t, s, v_2)\| ds &\leq c_3 \|v_1 - v_2\|, \end{aligned} \tag{6.10}$$

where  $c_j > 0$ ,  $j = 1, 2, 3$ ,  $x_i, y_i, z_i, v_i \in X$ ,  $i = 1, 2$ , and  $s, t \in I$ .

All the conditions stated in Theorem 5.2 are satisfied. Hence, system (6.7) is exactly null controllable on  $I$ .

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