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Research Article

A Problem Concerning Yamabe-Type Operators of Negative Admissible Metrics

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This paper is about a problem concerning nonlinear Yamabe-type operators of negative admissible metrics. We first give a result on σ_k Yamabe problem of negative admissible metrics by virtue of the degree theory in nonlinear functional analysis and the maximum principle and then establish an existence and uniqueness theorem for the solutions to the problem.

1. Introduction

Let (M, g) be a compact closed, connected Riemannian manifold of dimension $n \geq 3$. In 2003, Gursky-Viaclovsky [1] introduced a modified Schouten tensor as follows:

$$A_g^t = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{tR_g}{2(n-1)}g\right), \quad t \leq 1, \tag{1.1}$$

where Ric_g and R_g are the Ricci tensor and the scalar curvature of g, respectively.

Define

$$\begin{gathered}\sigma_k(\lambda) = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k} \quad \text{for } \lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n, \\ \Omega_k^+ = \left\{\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n;\ \sigma_j(\lambda) > 0, 1 \leq j \leq k\right\}.\end{gathered} \tag{1.2}$$

The σ_k Yamabe problem is to find a metric $\widetilde{g}$ conformal to g, such that

$$\sigma_k\left(\lambda_{\widetilde{g}}\left(A_{\widetilde{g}}\right)\right) = 1, \qquad \lambda_{\widetilde{g}}\left(A_{\widetilde{g}}\right) \in \Omega_k^+ \quad \text{on } M, \tag{1.3}$$

where $\lambda_{\tilde{g}}(A_{\tilde{g}})$ denotes the eigenvalue of $A_{\tilde{g}}$ with respect to the metric $\tilde{g}$. This problem has attracted great interest since the work of Viaclovsky in [2] (cf., e.g., [2–7] and references therein).

Assume $\Omega_k^- = -\Omega_k^+$. Then the σ_k Yamabe problem in negative cone

$$\sigma_k\left(-\lambda_{\tilde{g}}\left(A_{\tilde{g}}\right)\right) = 1, \qquad \lambda_{\tilde{g}}\left(A_{\tilde{g}}\right) \in \Omega_k^- \quad \text{on } M, \tag{1.4}$$

is still elliptic (see [1]).

Definition 1.1. A metric $\tilde{g}$ conformal to g is called negative admissible if

$$\lambda_{\tilde{g}}\left(A_{\tilde{g}}^t\right) \in \Omega_k^- \quad \text{on } M. \tag{1.5}$$

Under the conformal relation $\tilde{g} = e^{2z}g$, the transformation law for the modified Schouten tensor above is as follows:

$$A_{\tilde{g}}^{\tau} = A_g^{\tau} - \nabla^2 z - \frac{1-\tau}{n-2}(\Delta z)g - \frac{2-\tau}{2}|\nabla z|^2 g + dz \otimes dz. \tag{1.6}$$

We consider the following nonlinear equation:

$$P(Z) := \beta\left(\lambda_g(Z)\right) = \varphi(x, z), \qquad \lambda_g(Z) \in \Omega \quad \text{on } M, \tag{1.7}$$

where

$$Z = \nabla^2 z + \frac{1-t}{n-2}(\Delta z)g + \frac{2-t}{2}|\nabla z|^2 g - dz \otimes dz - A_g^t, \tag{1.8}$$

$\beta \in C^{\infty}(\Omega^+) \cap C^0(\overline{\Omega^+})$ is a symmetric function and is homogeneous of degree one normalized, and φ is a positive C^{∞} function satisfying the monotone condition:

$$\begin{gathered}\text{there exists two constants } \underline{\gamma} < 0 < \overline{\gamma} \quad \text{with} \\ \varphi\left(x, \underline{\gamma}\right) < \beta\left(-\lambda_g\left(A_g^t\right)\right) < \varphi\left(x, \overline{\gamma}\right), \quad \forall x \in M.\end{gathered} \tag{1.9}$$

For this equation, we have the following.

Theorem 1.2. *Let (M, g) be a compact, closed, connected Riemannian manifold of dimension $n \geq 3$ and*

$$A_g^t \in \Omega^-, \quad \textit{for } t < 1. \tag{1.10}$$

Suppose that $\Omega^+, \Omega^- \subset R^n$ are open convex symmetric cones with vertex at the origin, satisfying

$$\Omega_n \subset \Omega \subset \Omega_1, \qquad \Omega^- = -\Omega^+, \tag{1.11}$$

where

$$\Omega_1 := \left\{ \lambda = (\lambda_1, \ldots, \lambda_n);\ \sum_{i=1}^{n} \lambda_i > 0 \right\},$$
$$\Omega_n := \{ \lambda = (\lambda_1, \ldots, \lambda_n);\ \lambda_i > 0 \text{ for } 1 \le i \le n \}. \tag{1.12}$$

Let β satisfy

(i) $\beta > 0$ *in* Ω^+, $\beta_i := \partial\beta/\partial\lambda_i > 0$ *on* Ω^+, *and* $\beta(e) = 1$ *on* Ω^+, *where*

$$e = (1, \ldots, 1). \tag{1.13}$$

(ii) β *is concave on* Ω^+, *and*

$$\beta(\lambda) \le \varrho\sigma_1(\lambda), \quad \forall \lambda \in \Omega^+, \tag{1.14}$$

where ϱ is a positive constant.

Moreover, assume that $\varphi(x, z)$ is a positive C^∞ satisfying condition (1.9)*. Then there exists a solution to* (1.7).

Theorem 1.3. *Let (M, g) be a compact, closed, connected Riemannian manifold of dimension $n \ge 3$ and*

$$A_g^t \in \Omega^-, \quad \text{for } t < 1. \tag{1.15}$$

Let (β, Ω^+) be those as in Theorem 1.2. Then there exist a function ϕ and a positive number λ, such that ϕ is a solution to the eigenvalue problem

$$P(U) := \beta(\lambda_g(U)) = \Lambda, \tag{1.16}$$

where

$$U = -A_{\tilde{g}}^t = \nabla^2\phi + \frac{1-t}{n-2}(\Delta\phi)g + \frac{2-t}{2}|\nabla\phi|^2 g - d\phi \otimes d\phi - A_g^t \tag{1.17}$$

for conformal metric $\tilde{g} = e^{2\phi}$ and $\lambda_g(U)$ denotes the eigenvalue of U with respect to metric g.

Remark 1.4. (1) (ϕ, Λ) is unique in Theorem 1.3 under the sense that, if there is another solution (ϕ', Λ') satisfying (1.16), then

$$\Lambda = \Lambda', \qquad \phi = \phi' + c \tag{1.18}$$

for some constant c.

(2) Λ is called the eigenvalue related to fully nonlinear Yamabe-type operators of negative admissible metrics, and ϕ is called an eigenfunction with respect to Λ.

2. Proof of Theorem 1.2

To prove Theorem 1.2, firstly, let us give the following proposition.

Proposition 2.1. *Suppose all the conditions in Theorem 1.2 are satisfied. Then every C^2 solution z to* (1.7) *with*

$$\underline{\gamma} \leq z \leq \overline{\gamma} \tag{2.1}$$

satisfies

$$\underline{\gamma} < z < \overline{\gamma}. \tag{2.2}$$

Proof. Assume z is a solution to (1.7) with $\underline{\gamma} \leq z$. Denote

$$\begin{aligned} \tilde{z} &= z - \underline{\gamma}, \\ z_s &= sz + (1-s)\gamma, \\ Z_s &= \nabla^2 z_s + \frac{1-t}{n-2}(\Delta z_s)g + \frac{2-t}{2}|\nabla z_s|^2 g - dz_s \otimes dz_s - A_g^t. \end{aligned} \tag{2.3}$$

It is easy to verify that $Z_s \in \Omega^+$.

Write

$$Q[z] = P(Z) - \varphi(x,z). \tag{2.4}$$

Then

$$Q[z] - Q\left[\underline{\gamma}\right] = 0 - P\left(-A_g^t\right) + \varphi\left(x,\underline{\gamma}\right). \tag{2.5}$$

On the other hand,

$$\begin{aligned} Q[z] - Q\left[\underline{\gamma}\right] &= \int_0^1 \frac{d}{ds} Q[z_s]ds \\ &= \int_0^1 T_{ij}(Z_s)dsD_{ij}\tilde{z} + b^i D_i \tilde{z} + c\tilde{z} \\ &= L(\tilde{z}) \end{aligned} \tag{2.6}$$

for some bound b^i and constant c, where

$$\begin{aligned} T_{ij} &= P_{ij} + \frac{1-t}{n-2}\sum_l P_{ll}\gamma_{ij} \geq 0, \\ P_{ij} &= \frac{\partial P}{\partial Z_{ij}} \geq 0 \end{aligned} \tag{2.7}$$

by condition (ii).

Therefore, we know that L is an elliptic operator, and

$$L(\tilde{z}) < 0 \quad \text{with } \tilde{z} \geq 0. \tag{2.8}$$

By the maximum principle, we get $\tilde{z} > 0$. That is,

$$z > \underline{\gamma}. \tag{2.9}$$

Similarly, we can derive

$$z < \overline{\gamma}, \tag{2.10}$$

for solution z with $z \leq \overline{\gamma}$. □

Thus, we have the following Gradient and Hessian estimates for solutions to (1.7).

Lemma 2.2. *Let z be a C^3 solution to* (1.7) *for some $t < 1$ satisfying $\underline{\gamma} < z < \overline{\gamma}$. Then*

$$\|\nabla z\|_{L^\infty} < C_1, \tag{2.11}$$

where C_1 depends only upon $\underline{\gamma}, \overline{\gamma}, g, t, \varphi$.

Moreover,

$$\left\|\nabla^2 z\right\|_{L^\infty} < C_2, \tag{2.12}$$

where C_2 depends only upon $\underline{\gamma}, \overline{\gamma}, g, t, \varphi, C_1$.

Proof of Theorem 1.2. We now prove Theorem 1.2 using a priori estimates in Lemma 2.2, the maximum principle in Proposition 2.1, and the degree theory in nonlinear functional analysis (cf., e.g., [8]).

For each $0 \leq \tau \leq 1$, let

$$\beta_\tau(\lambda) := \beta(\tau\lambda + (1-\tau)\sigma_1(\lambda)e), \tag{2.13}$$

(here $e = (1, \ldots, 1)$ as in Section 1) which is defined on

$$\Omega_\tau^+ = \{\lambda \in \mathbb{R}^n;\ \tau\lambda + (1-\tau)\sigma_1(\lambda)e \in \Omega^+\}. \tag{2.14}$$

We consider the problem

$$P(\tau Z + (1-\tau)\sigma_1(Z)e) = \tau\varphi(x,z) + (1-\tau)\sigma_1\left(-A_g^t\right)e^{2z} \tag{2.15}$$

on M, where

$$Z = \nabla^2 z + \frac{1-t}{n-2}(\Delta z)g + \frac{2-t}{2}|\nabla z|^2 g - dz \otimes dz - A_g^t. \tag{2.16}$$

Since $A_g^t \in \Omega^-$, we have

$$\sigma_1\left(-A_g^t\right) > 0 \tag{2.17}$$

by condition (ii). Hence for $\tau = 0$, it follows from the maximum principle that $z = 0$ is the unique solution.

In view of Proposition 2.1, we see that, for each $\tau \in [0,1]$, every C^2 solution z^τ to (2.15) with $\underline{\gamma} \le z^\tau \le \overline{\gamma}$ satisfies

$$\underline{\gamma} < z^\tau < \overline{\gamma}. \tag{2.18}$$

This, together with Lemma 2.2, shows that for each $\tau \in [0,1]$ and solution z^τ to (2.15) with $\underline{\gamma} \le z^\tau \le \overline{\gamma}$, the following estimate holds

$$\|z^\tau\|_{C^2} < C, \tag{2.19}$$

for some constant C independent of τ.

This estimate yields uniform ellipticity, and by virtue of the concavity condition (ii), the well-known theory of Evans-Krylov, and the standard Schauder estimate (cf. [9]), we know that there exists a constant K independent of τ such that

$$\|z^\tau\|_{C^{4,\alpha}} < K, \tag{2.20}$$

where z^τ is a C^2 solution to (2.15) with $\underline{\gamma} \le z^\tau \le \overline{\gamma}$.

Set

$$S_\tau := \left\{\underline{\gamma} < z^\tau < \overline{\gamma}\right\} \cap \{\|z^\tau\|_{C^{4,\alpha}} < K\} \cap \{Z \in \Omega_\tau^+\}, \tag{2.21}$$

and define $T_\tau : C^{4,\alpha} \to C^{2,\alpha}$ by

$$T_\tau(z) = P(\tau Z + (1-\tau)\sigma_1(Z)e) - \tau\varphi(x,z) - (1-\tau)\sigma_1\left(-A_g^t\right)e^{2z}. \tag{2.22}$$

Then, by (2.19), we see that there is no solution to the equation

$$T_\tau(z) = 0 \quad \text{on } \partial S_\tau. \tag{2.23}$$

So the degree of T_τ is well defined and independent of τ. As mentioned above, there is a unique solution at $\tau = 0$. Therefore

$$\deg(T_0, S_0, 0) \neq 0. \tag{2.24}$$

Since the degree is homotopy invariant, we have

$$\deg(T_1, S_1, 0) \neq 0. \tag{2.25}$$

Thus, we conclude that (1.7) has a solution in S_1.

The proof of Theorem 1.2 is completed. □

3. Proof of Theorem 1.3

Proof of Theorem 1.3. Take a look at the following equation:

$$\widetilde{P}(u) = P\left(\nabla^2 u + \frac{1-t}{n-2}(\Delta u)g + \frac{2-t}{2}|\nabla u|^2 g - du \otimes du - A_g^t\right) - e^u = \lambda. \tag{3.1}$$

We will prove that, for small $\lambda > 0$, (3.1) has a unique smooth solution.

Since $\partial\widetilde{P}/\partial u < 0$, the uniqueness of the solution to (3.1) follows from the maximum principle.

Next, we show the existence of the solution to (3.1) by using Theorem 1.2.

It follows from

$$A_g^t \in \Omega^- \tag{3.2}$$

that, for $\lambda > 0$ small enough, we can find two constants $\underline{\gamma} < 0 < \overline{\gamma}$, such that

$$e^{\underline{\gamma}} + \lambda < P\left(-A_g^t\right) < e^{\overline{\gamma}} + \lambda. \tag{3.3}$$

That is, condition (1.9) for $\varphi(x, z)$ in Theorem 1.2 is satisfied. Therefore, by the result in Theorem 1.2, the existence of unique solution to (3.1) is established for small $\lambda > 0$.

Set

$$E := \{\lambda > 0;\ (3.1) \text{ has a solution}\}. \tag{3.4}$$

Since $E \neq \emptyset$, we can define

$$\Lambda = \sup_{\lambda \in E} \lambda. \tag{3.5}$$

We claim Λ is finite. Actually,

$$\lambda < P\left(\nabla^2 u + \frac{1-t}{n-2}(\Delta u)g + \frac{2-t}{2}|\nabla u|^2 g - du \otimes du - A_g^t\right). \tag{3.6}$$

If we assume that at x_0, u achieves its maximum, then $\nabla^2 u \leq 0$, and so

$$\lambda < P\left(\nabla^2 u + \frac{1-t}{n-2}(\Delta u)g - A_g^t\right) \leq P\left(-A_g^t\right). \tag{3.7}$$

This means that

$$\Lambda \leq P\left(-A_g^t\right). \tag{3.8}$$

For any sequence $\lambda_i \subset E$ with $\lambda_i \to \Lambda$, let u_{λ_i} be the corresponding solution to (3.1) with $\lambda = \lambda_i$.

First, we claim that

$$\inf_M u_{\lambda_i} \longrightarrow -\infty \quad \text{as } i \longrightarrow \infty. \tag{3.9}$$

Suppose this is not true, that is,

$$\inf_M u_{\lambda_i} \geq -C_0 \tag{3.10}$$

for a positive constant C_0. Then, by (3.1), at any maximum point x_0 of u_{λ_i},

$$\max_M u_{\lambda_i} \leq C \tag{3.11}$$

for some constant C depending only on $P(-A_g^t)$. Then the apriori estimates imply that u_{λ_i} (by taking a subsequence) converges to a smooth function u_0 in C^∞, such that u_0 satisfies (3.1) for $\lambda = \lambda_0$. Since the linearized operator of (3.1) is invertible, by the standard implicit function theorem, we have a solution to (3.1) for

$$\lambda = \lambda_0 + \delta \quad \text{with } \delta > 0 \quad \text{small enough.} \tag{3.12}$$

This is a contradiction. Hence (3.9) holds.

Next, we prove that

$$\max_M u_{\lambda_i} \longrightarrow -\infty \quad \text{as } i \longrightarrow \infty. \tag{3.13}$$

We divided our proof into two steps.

Step 1. Let

$$\Lambda = P\left(-A_g^t\right). \tag{3.14}$$

Then, following the above argument,

$$u_{\lambda_i} \to \phi_0 \quad \text{in } C^\infty, \tag{3.15}$$

and (Λ, u_0) is a solution to (3.1). Assume u_0 attains its maximum at y_0. Then at y_0,

$$\nabla^2 u_0 \leq 0, \qquad \nabla u_0 = 0. \tag{3.16}$$

Therefore,

$$e^{u_0(y_0)} \leq P\left(-A_g^t\right) - \Lambda = 0. \tag{3.17}$$

So

$$u_0(y_0) = -\infty. \tag{3.18}$$

That means that (3.13) holds.

Step 2. Let

$$P\left(-A_g^t\right) - \Lambda = \varpi > 0. \tag{3.19}$$

Then, if (3.13) is not true, that is,

$$\max_M u_{\lambda_i} \geq -C_0 \tag{3.20}$$

for a positive constant C_0, write

$$z_{\lambda_i} := u_{\lambda_i} - \max_M u_{\lambda_i}. \tag{3.21}$$

Then we have

$$\max_M z_{\lambda_i} \longrightarrow 0, \qquad \inf_M z_{\lambda_i} \longrightarrow -\infty, \tag{3.22}$$

as $i \to \infty$.

On the other hand, z_{λ_i} satisfies

$$P\left(\nabla^2 z_{\lambda_i} + \frac{1-t}{n-2}(\Delta z_{\lambda_i})g + \frac{2-t}{2}|\nabla z_{\lambda_i}|^2 - dz_{\lambda_i} \otimes dz_{\lambda_i} - A_g^t\right) = e^{\max_M u_{\lambda_i}} e^{z_{\lambda_i}} + \lambda_i. \tag{3.23}$$

Since at any minimum point z_0 of z_{λ_i},

$$\nabla^2 z_{\lambda_i} \geq 0, \qquad \nabla z_{\lambda_i} = 0. \tag{3.24}$$

Consequently, at z_0, we obtain

$$e^{\max_M u_{\lambda_i}} e^{z_{\lambda_i}} \geq P\left(-A_g^t\right) - \Lambda > 0. \tag{3.25}$$

Thus, it is easy to verify that z_{λ_i} is bounded from below as $i \to \infty$. This is a contradiction. So we see that (3.13) is true.

By a priori estimates results again, we deduce that z_{λ_i} converges to a smooth function z in C^∞ and z satisfies (1.16) with $\lambda = \Lambda$.

Finally, let us prove the uniqueness.

Denote

$$Z := \nabla^2 z + \frac{1-t}{n-2}(\Delta z)g + \frac{2-t}{2}|\nabla z|^2 g - dz \otimes dz - A_g^t, \tag{3.26}$$

and for any smooth functions z_0 and z_1, set

$$\begin{aligned} v &= z_1 - z_0, \\ z_s &= s z_1 + (1-s) z_0, \\ Z_s &= \nabla^2 z_s + \frac{1-t}{n-2}(\Delta z_s)g + \frac{2-t}{2}|\nabla z_s|^2 g - dz_s \otimes dz_s - A_g^t. \end{aligned} \tag{3.27}$$

Then we get

$$P(Z_1) - P(Z_0) = \int_0^1 \frac{d}{ds} P(Z_s) = \int_0^1 \left[P_{ij} + \frac{1-t}{n-2} \sum_l P_{ll} \gamma_{ij} \right] (Z_s) ds v_{ij} + b^l v_l \tag{3.28}$$

for some bounded b^l. Thus, if

$$z_0 = \phi, \qquad z_1 = \phi' \tag{3.29}$$

are two solutions to (1.16) for some λ and λ', respectively, then a^{ij} is positive definite. Therefore,

$$\phi = \phi' + c \tag{3.30}$$

for some constant c by the maximum principle. □

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