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Research Article

Some Nonlinear Vortex Solutions

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We consider the steady-state two-dimensional motion of an inviscid incompressible fluid which obeys a nonlinear Poisson equation. By seeking solutions of a specific form, we arrive at some interesting new nonlinear vortex solutions.

1. Introduction

The study of wakes behind bodies has generated significant interest due to the problem's important applications. An excellent review of current theories on the evolution of such wakes is given by Chomaz [1]. Numerical models of these flows are typically based on the full three-dimensional Navier-Stokes equations. Still, two-dimensional exact solutions of the Euler equation add significantly to our understanding of such flows [2]. Recently, vortex solutions of the steady two-dimensional inviscid problem have been used as initial conditions for solvers applied to the full three-dimensional time-dependent problem. This approach was adopted by Faddy and Pullin [3] to model the three-dimensional wake behind an aircraft wing. Flows which bear strong resemblance to exact vortex solutions occur in many other applications; a recent experimental study [4] involving cavity flows provides some good examples of this. In the present contribution we examine a class of exact solutions of the two-dimensional inviscid problem which are related to vortex flows. Similar theoretical efforts are well documented; in particular, we note two recent contributions [5, 6].

In plane two-dimensional hydrodynamics, the equation governing the motion of an inviscid incompressible fluid can be written in terms of a stream function $\psi(x, y, t)$ as

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0, \tag{1.1}$$

where $\partial(a, b)/\partial(x, y) = (\partial a/\partial x)\,(\partial b/\partial y) - (\partial a/\partial y)\,(\partial b/\partial x)$ is a Jacobian and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian. The stream function equation (1.1) admits steady-state solutions of the form

$$\nabla^2\psi = \mathcal{F}(\psi) \tag{1.2}$$

for any function $\mathcal{F}$. Equation (1.2) is a nonlinear Poisson or an elliptic Klein-Gordon equation, and a number of solutions are known for two-dimensional hydrodynamics. In particular, the solution corresponding to a row of corotating Stuart vortices [7] is $\psi = \ln[\cosh y - \varepsilon \cos x]$, while the solution corresponding to the counter-rotating Mallier-Maslowe (M&M) vortices [8] is

$$\psi = \ln\left[\frac{\cosh \varepsilon y - \varepsilon \cos x}{\cosh \varepsilon y + \varepsilon \cos x}\right]. \tag{1.3}$$

Stuart vortices satisfy Liouville's equation [9], $\nabla^2\psi = (1-\varepsilon^2)e^{-2\psi}$, and a number of additional solutions of Liouville's equation are known, with some of these given in [10–12]. M&M vortices satisfy the sinh-Poisson, or elliptic sinh-Gordon, equation, $\nabla^2\psi = (-1/2)(1 - \varepsilon^2)\sinh 2\psi$. Again, a number of additional solutions are known [5, 13, 14], some of which involve Jacobian elliptic functions [15] such as cn and sn, which are doubly periodic functions of $z = x + iy$.

The goal of this study is to find solutions to (1.2), that are nonlinear vortex solutions of the form

$$\psi = \ln\Big[A f^2(x) + B g^2(y) + C\Big], \tag{1.4}$$

with A, B, C constants and $g(y) = tu(y \mid a)$ and $f(x) = vw(x \mid b)$. Here we employ a compact notation where t, u, v, w are any four of the letters c, s, d, n, so that tu and vw may represent any of the 12 Jacobian elliptic functions [15, Table 16.2] with parameters a or b. When t and u are the same, or v and w are the same, resulting in cc, for example, the corresponding function is set equal to unity. The original motivation for this form was that the Stuart vortex can be written as $\psi = \ln[\cosh^2 y - \varepsilon\cos^2 x - (1+\varepsilon)/2]$, and (1.4) suggests itself as a likely form for a generalization of the Stuart vortex.

Due to relations between the Jacobian elliptic functions, we need not consider all twelve. By definition $uv(x \mid a) = un(x \mid a)/vn(x \mid a)$. The squares are related according to [15]

$$\begin{aligned} -dn^2(x \mid a) + \tilde{a} &= -acn^2(x \mid a) = asn^2(x \mid a) - a, \\ -\tilde{a}nd^2(x \mid a) + \tilde{a} &= -a\tilde{a}sd^2(x \mid a) = acd^2(x \mid a) - a, \\ \tilde{a}sc^2(x \mid a) + \tilde{a} &= \tilde{a}nc^2(x \mid a) = dc^2(x \mid a) - a, \\ cs^2(x \mid a) + \tilde{a} &= ds^2(x \mid a) = ns^2(x \mid a) - a, \end{aligned} \tag{1.5}$$

where $\tilde{a} = 1 - a$, so that we can write all twelve in terms of cn, cd, dc, cs. We also have expressions for change of argument [15], $cd(x + K_a \mid a) = -sn(x \mid a)$ and $cs(x + K_a \mid a) = -\tilde{a}^{1/2}\mathrm{sc}(x \mid a)$, where K_a is the quarter-period, so that we can write all twelve in terms of just two functions, cn and dc.

The Jacobian elliptic functions reduce to circular or hyperbolic functions, or sometimes constants, when the parameter a is either 0 or 1 [15]. Because of this, we will also look for solutions when either, or both, of the Jacobian functions in (1.4) are replaced by a circular or hyperbolic function, and for these solutions we can make use of the relations $\cosh^2 x - \sinh^2 x = 1$, $\cos^2 x + \sin^2 x = 1$, and $\sin(x + \pi/2) = \cos x$.

2. Analysis

We substituted the assumed form (1.4) into the nonlinear Poison equation (1.2) for the various different $f(x)$ and $g(y)$ under consideration and found the values of the constants A, B, and C for which the assumed form was indeed a solution. The solutions we found are presented in Tables 1, 2, and 4.

2.1. Solutions with No Jacobian Functions

There are a number of solutions of the prescribed form involving only hyperbolic and circular functions, and these are presented in Table 1, with H denoting a hyperbolic function and C denoting a circular function. The solution HC1 is of course the well-known Stuart vortex [7] and is unique amongst the solutions in Table 1 in that it represents a family of solutions, because of the parameter ε, while the other solutions in the table are isolated solutions. The HC solutions satisfy Liouville's equation $\nabla^2\psi = ke^{l\psi}$, while the HH and CC solutions satisfy the equation $\nabla^2\psi = ce^{\psi} + de^{-2\psi}$, where $c = \pm 16/27$ and $d = \pm 2$. It should of course be remembered that if we replace sec by csc in these solutions, we get the same solution displaced a distance $\pi/2$.

The solutions in Table 1 are plotted in Figure 1. HC1 is the familiar Stuart vortex, consisting of a row of cat's eyes in series. HC2 resembles the Stuart vortex, but with each cat's eye rotated by 90°, so that it looks like a row of cat's eyes in parallel, and HC3 looks like HH2 but with the cat's eyes centered on $|y| = \infty$ instead of $y = 0$. HC1 appears to consist of two vortices, one centered on $x = y = 0$ and a second on $|x| = |y| = \infty$, HC2 looks like a single cat's eye centered on $y = 0$ and $|x| = \infty$, and HH3 looks like a vortex centered on $|x| = |y| = \infty$. Finally, CC1 is a double periodic array of vortices.

Table 1: Solutions with only hyperbolic and circular functions.

$\exp(\psi)$	$\nabla^2\psi$	
$\cosh^2 y + \varepsilon\cos^2 x - \dfrac{\epsilon+1}{2}$	$(1-\varepsilon^2)e^{-2\psi}$	HC1
$\sec^2 x - \operatorname{sech}^2 y$	$2e^{\psi}$	HC2
$\sec^2 x + \operatorname{csch}^2 y$	$2e^{\psi}$	HC3
$\operatorname{sech}^2 y + \operatorname{sech}^2 x - \dfrac{2}{3}$	$-\dfrac{16}{27}e^{-2\psi} - 2e^{\psi}$	HH1
$\operatorname{sech}^2 y - \operatorname{csch}^2 x - \dfrac{2}{3}$	$-\dfrac{16}{27}e^{-2\psi} - 2e^{\psi}$	HH2
$\operatorname{csch}^2 y + \operatorname{csch}^2 x + \dfrac{2}{3}$	$-\dfrac{16}{27}e^{-2\psi} + 2e^{\psi}$	HH3
$\sec^2 y + \sec^2 x - \dfrac{2}{3}$	$\dfrac{16}{27}e^{-2\psi} + 2e^{\psi}$	CC1

Table 2: Solutions with one Jacobian function and one hyperbolic or circular function; $\alpha = (1-a\tilde{a})^{1/4}$, $c_1 = (4/27)(2a - 2\alpha^2 - 1)(2a + \alpha^2 - 1)^2$ and $c_2 = (4/27)(2a + 2\alpha^2 - 1)(2a - \alpha^2 - 1)^2$.

$\exp(\psi)$	$\nabla^2\psi$	
$acn^2(y \mid a) + \alpha^2\operatorname{sech}^2\alpha x + \dfrac{1-2a-\alpha^2}{3}$	$c_1e^{-2x} - 2e^{x}$	HE1
$dc^2(y \mid a) - \alpha^2\operatorname{sech}^2\alpha x + \dfrac{\alpha^2-1-a}{3}$	$c_1e^{-2x} + 2e^{x}$	HE2
$acn^2(y \mid a) - \alpha^2\operatorname{csch}^2\alpha x + \dfrac{1-2a-\alpha^2}{3}$	$c_1e^{-2x} - 2e^{x}$	HE3
$dc^2(y \mid a) + \alpha^2\operatorname{csch}^2\alpha x + \dfrac{\alpha^2-1-a}{3}$	$c_1e^{-2x} + 2e^{x}$	HE4
$acn^2(y \mid a) - \alpha^2\sec^2\alpha x + \dfrac{1-2a+\alpha^2}{3}$	$c_2e^{-2x} - 2e^{x}$	CE1
$\mathrm{dc}^2(y \mid a) + \alpha^2\sec^2\alpha x - \dfrac{\alpha^2+1+a}{3}$	$c_2e^{-2x} + 2e^{x}$	CE2

Whereas the Stuart vortex HC1 is a smooth solution, the other solutions have singularities. HC2 is singular on the lines $x = \pm\pi/2, \pm 3\pi/2, \ldots$ and $y = 0$. HC3 is singular on the lines $x = \pm\pi/2, \pm 3\pi/2, \ldots$ and $y = 0$. HH1 is singular on the curve $\operatorname{sech}^2 y + \operatorname{sech}^2 x = 2/3$. HH2 is singular on the line $x = 0$ and on the curve $\operatorname{sech}^2 y = \operatorname{csch}^2 x + 2/3$. HH3 is singular on the lines $x = 0$ and $y = 0$. CC1 is singular on the lines $x = \pm\pi/2, \pm 3\pi/2, \ldots$ and $y = \pm\pi/2, \pm 3\pi/2, \ldots$.

Table 3: Limits of e^{ψ} from Table 2; †means x and y are interchanged.

	$a = 0$	$a = 1$	$a = 0$ $y + K_a$	$a = 1$ $y + K_a$
HE1	$\operatorname{sech}^2 x$	HH1	$\operatorname{sech}^2 x$	$\operatorname{sech}^2 x - \frac{2}{3}$
HE2	HC2†	$\frac{2}{3} - \operatorname{sech}^2 x$	HC2†	HH2†
HE3	$-\operatorname{csch}^2 x$	HH2	$\operatorname{csch}^2 x$	$\frac{2}{3} + \operatorname{csch}^2 x$
HE4	HC3†	$\operatorname{csch}^2 x + \frac{2}{3}$	HC3†	HH3
CE1	$\frac{2}{3} - \sec^2 x$	HC2	$\sec^2 x - \frac{2}{3}$	$\sec^2 x$
CE2	CC1	$\sec^2 x$	CC1	HC3

Table 4: Solutions with two Jacobian functions; $\gamma = \alpha\beta$ and $\chi = (4/27)[\gamma^6(b-2)(2b-1)(b+1)+(a-2)(2a-1)(a+1)]$.

$\exp(\psi)$	$\nabla^2\psi$	
$acn^2(y \mid a) + b\,\gamma^2 cn^2(\gamma x \mid b) + \dfrac{1-2a+\gamma^2(1-2b)}{3}$	$\chi e^{-2\psi} - 2e^{\psi}$	EE1
$acn^2(y \mid a) - \gamma^2 dc^2(\gamma x \mid b) + \dfrac{1-2a+\gamma^2(1+b)}{3}$	$\chi e^{-2\psi} - 2e^{\psi}$	EE2
$dc^2(y \mid a) + \gamma^2 dc^2(\gamma x \mid b) - \dfrac{1+a+\gamma^2(1+b)}{3}$	$\chi e^{-2\psi} + 2e^{\psi}$	EE3

2.2. Solutions with One Jacobian Function

There are also a number of solutions of the prescribed form involving one Jacobian elliptic function and one hyperbolic or circular function. For the reasons given earlier in this section, we need only consider cn and dc, and the resulting solutions are presented in Table 2 and plotted in Figure 2. HE1 looks like a row of cat's eyes centered on $x = 0$ and a second staggered row centered on $|x| = \infty$. HE2 looks like HC2 rotated through 90°, HE3 looks like a row of cat's eyes centered on $|x| = \infty$, HE4 looks like HC3 rotated through 90°, CE1 looks like a stack of rows of cat's eyes, or equivalently a row of cat's eyes in a bounded channel, and CE2 looks like CC1. When we set the parameter a to either 0 or 1, the Jacobian functions become either circular or hyperbolic functions, or constant, so in these limits, the solutions in Table 2 should reduce to either one of the solutions of Section 2.1 or to a function of either x alone or y alone, and Table 3 confirms that this does indeed occur. Regrettably, none of the solutions in Table 2 reduces to the Stuart vortex in either limit. The two right-hand columns of Table 3 labelled $y + K_a$ merit further explanation. The expressions for change of argument, $cd(x + K_a \mid a) = -sn(x \mid a)$ and $cs(x + K_a \mid a) = -\tilde{a}^{1/2} sc(x \mid a)$, mean that the solutions involving cd look the same as those involving cn, but the limits when $a \to 0$ and $a \to 1$ differ for the two functions, and this is what is shown in those two right-hand columns.

Each of the solutions in Table 2 obeys the equation $\nabla^2\psi = ce^{\psi} + de^{-2\psi}$, where $c = c_1$ or c_2 and $d = \pm 2$. This equation reduces to Liouville's equation when $a = 0$ for the HE solutions and when $a = 1$ for the CE solutions.

The solutions in Table 2 each have singularities. HE1 is singular on the curve $acn^2(y \mid a) + \alpha^2 \operatorname{sech}^2 \alpha x + (1 - 2a - \alpha^2)/3 = 0$. HE2 is singular on the lines $y = \pm K_a, \pm 3K_a, \ldots$, where

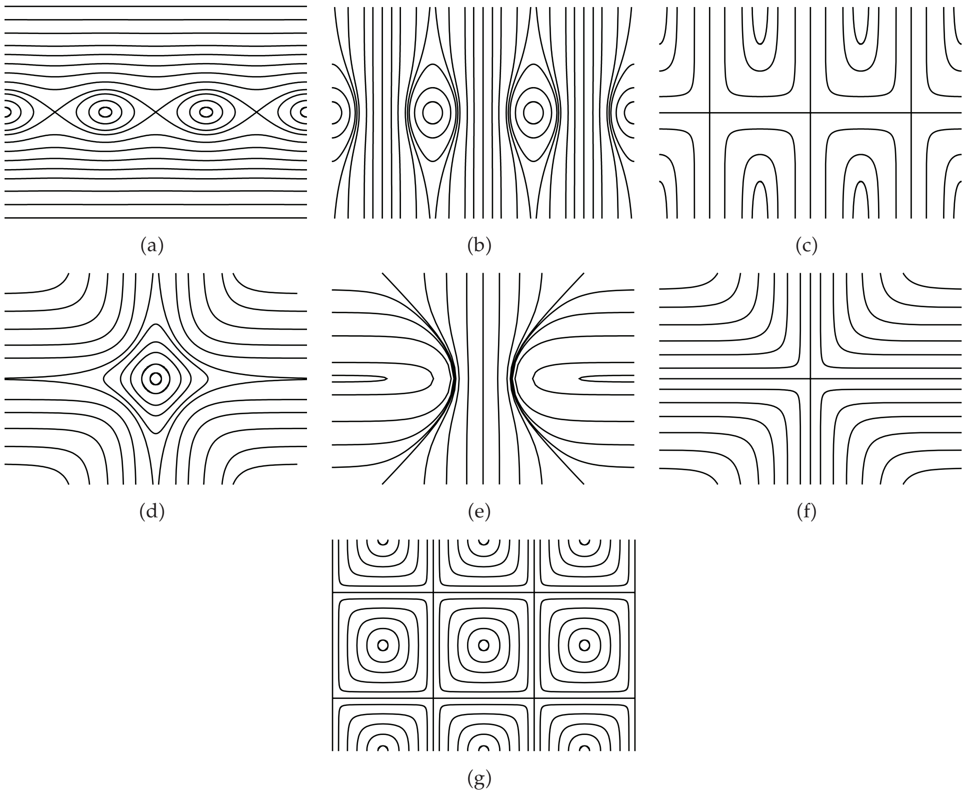


Figure 1: Solutions with only hyperbolic and circular functions: (a) HC1; (b) HC2; (c) HC3; (d) HH1; (e) HH2; (f) HH3; (g) CC1; definitions in Table 1.

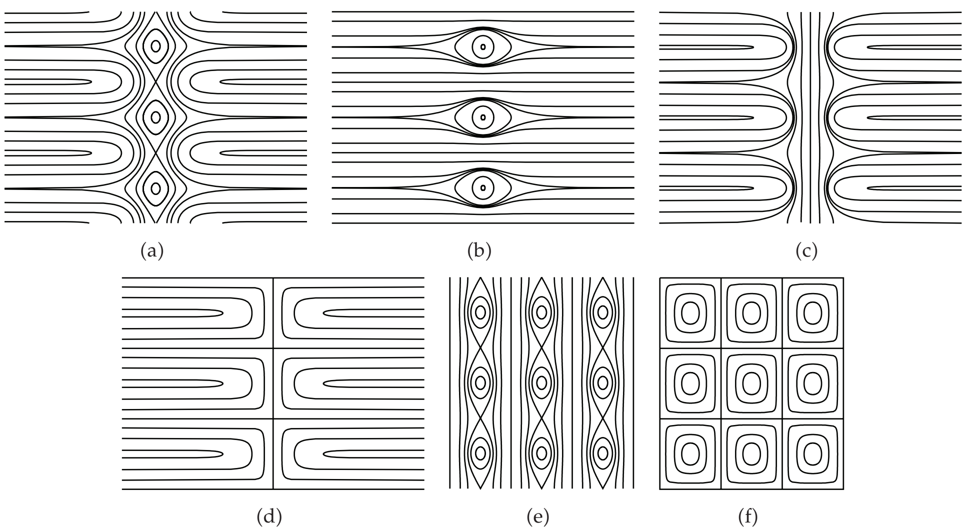
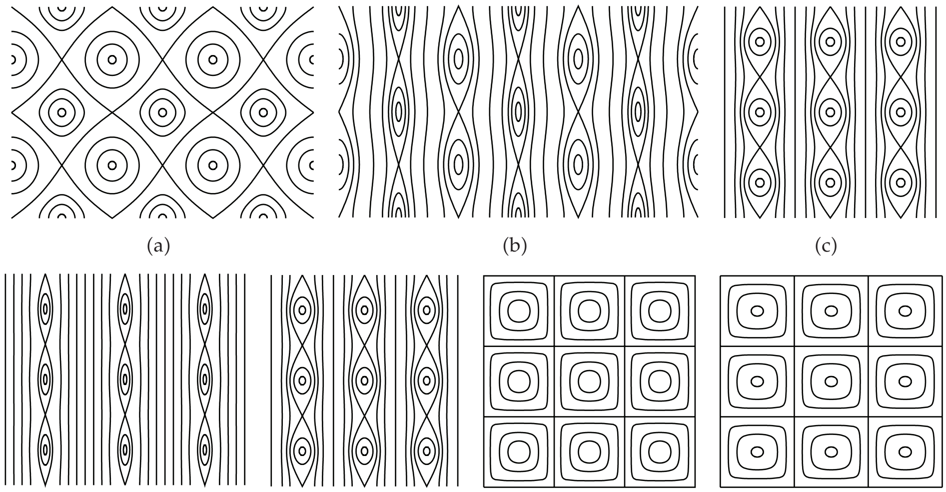


Figure 2: Solutions with one Jacobian function and one hyperbolic or circular function, $a = 1/2$: (a) HE1; (b) HE2; (c) HE3; (d) HE4; (e) CE1; (f) CE2; definitions in Table 2.



(a) (b) (c) (d) (e) (f) (g)

Figure 3: Solutions with two Jacobian functions: (a) EE1, $a = b = 1/2$; (b) EE1, $a = 1/16$, $b = 9/16$; (c) EE2, $a = b = 1/2$; (d) EE2, $a = 1/16$, $b = 9/16$; (e) EE2, $a = 9/16$, $b = 1/16$; (f) EE3, $a = b = 1/2$; (g) EE3, $a = 1/16$, $b = 9/16$; definitions in Table 4.

K_a is the quarter-period of $\mathrm{dc}(y \mid a)$. HE3 is singular at $x = 0$ and on the curve $dc^2(y \mid a) + \alpha^2 \mathrm{csch}^2 \alpha x + (\alpha^2 - 1 - a)/3$. HE4 is singular on the lines $x = 0$ and $y = \pm K_a, \pm 3K_a, \ldots$. CE1 is singular on the lines $x = \pm\pi/2, \pm 3\pi/2, \ldots$. CE2 is singular on the lines $x = \pm\pi/2, \pm 3\pi/2, \ldots$ and $y = \pm K_a, \pm 3K_a, \ldots$.

2.3. Solutions with Two Jacobian Functions

Finally, we come to solutions of the prescribed form involving two Jacobian elliptic functions. For the reasons given earlier in this section, we need only consider cn and dc, and the resulting solutions are presented in Table 4 and plotted in Figure 3. The streamlines of EE2 resemble those of CE1 in Figure 2, with the distance between the rows of cat's eyes varying as we vary the parameters a and b. The streamlines of EE3 resembles those of CE2. EE1 is perhaps the most interesting of the solutions. When the parameters are equal, $a = b$, as in Figure 3(a), we get an interesting array of vortices with wavy walls between the cells. As the difference between the parameters increases, as in Figure 3(b), the flow looks a little like CE1, but with alternating rows of vortices staggered instead of symmetric.

When we set either of the parameters a or b to either 0 or 1, the Jacobian functions become either circular or hyperbolic functions, or constant, so in these limits, the solutions in Table 4 should reduce to either one of the solutions of Sections 2.1 and 2.2 or to a function of either x alone or y alone, and Table 5 confirms that this does indeed occur. As with Table 3, the columns in Table 5 labelled $y + K_a$ mean that we have $cn^2(y + K_a \mid a)$ and $dc^2(y + K_a \mid a)$ instead of $cn^2(y \mid a)$ and $dc^2(y \mid a)$, and similarly for the columns labelled $x + K_b$.

Each of the solutions in Table 2 obeys the equation $\nabla^2\psi = \chi e^{\psi} + de^{-2\psi}$, where $d = \pm 2$. This equation reduces to Liouville's equation when $b^2 = \tilde{a}^2$.

If we set the parameter $b = \tilde{a}$, the solutions in Table 4 have some special cases, which are given in Table 6; in some of these solutions, we have removed a common factor or replaced

Table 5: Limits of e^{ψ} from Table 4; †means x and y are interchanged.

	$a = 0$	$a = 1$
EE1	$b\beta^2 cn^2(\beta x \mid b) + \dfrac{1+\beta^2(1-2b)}{3}$	HE1†
EE2	$-\beta^2 dc^2(\beta x \mid b) + \dfrac{\beta^2(1+b)+1}{3}$	HE2†
EE3	CE2†	$dc^2(y \mid a) + \dfrac{1-\beta^2(1+b)}{3}$
	$a = 0; y + K_a$	$a = 1; y + K_a$
EE1	$b\beta^2 cn^2(\beta x \mid b) + \dfrac{1+\beta^2(1-2b)}{3}$	$b\beta^2 cn^2(\beta x \mid b) + \dfrac{\beta^2(1-2b)-1}{3}$
EE2	$-\beta^2 dc^2(\beta x \mid b) + \dfrac{\beta^2(1+b)+1}{3}$	$-\beta^2 dc^2(\beta x \mid b) + \dfrac{\beta^2(1+b)-1}{3}$
EE3	CE2†	HE4†
	$b = 0$	$b = 1$
EE1	$acn^2(y \mid a) + \dfrac{1-2a+\alpha^2}{3}$	HE1
EE2	CE1	$acn^2(y \mid a) + \dfrac{1-2a-\alpha^2}{3}$
EE3	CE2	$dc^2(y \mid a) + \dfrac{\alpha^2-1-a}{3}$
	$b = 0; x + K_b$	$b = 1; x + K_b$
EE1	$acn^2(y \mid a) + \dfrac{1-2a+\alpha^2}{3}$	$acn^2(y \mid a) + \dfrac{1-2a-\alpha^2}{3}$
EE2	CE1	HE3
EE3	CE2	HE4
	$b = \tilde{a}$	
EE1	$acn^2(y \mid a) + \tilde{a}\,\mathrm{cn}^2(x \mid \tilde{a})$	$\nabla^2\psi = -2e^{\psi}$
EE2	$adc^2(y \mid a) - \mathrm{dc}^2(x \mid \tilde{a}) + \tilde{a}$	$\nabla^2\psi = -2e^{\psi}$
EE3	$dc^2(y \mid a) + \mathrm{dc}^2(x \mid \tilde{a}) - 1$	$\nabla^2\psi = 2e^{\psi}$

Table 6: Special cases of the solutions from Table 4 with $b = \tilde{a}$: EE1A and EE1B are special cases of EE1; EE2A and EE2B are special cases of EE2.

$\exp(\psi)$	$\nabla^2\psi$	
$nd^2(y \mid a) - sn^2(x \mid \tilde{a})$	$2(a-1)e^{\psi}$	EE1A
$sd^2(y \mid a) - sd^2(x \mid \tilde{a})$	$2a(a-1)e^{\psi}$	EE1B
$nc^2(y \mid a) - cn^2(x \mid \tilde{a})$	$2(1-a)e^{\psi}$	EE2A
$nc^2(y \mid a) - asd^2(x \mid \tilde{a})$	$2(1-a)e^{\psi}$	EE2B

y by $y + K_a$ or x by $x + K_b$. These special cases are interesting because of the limits of these solutions when we set the parameter a to either 0 or 1, which are given in Table 7; some of these limits are $\ln[\cosh^2 y - \cos^2 x]$, or $\ln[\cosh^2 y - \sin^2 x]$, which are point vortex limits of the Stuart vortex.

Table 7: Limits of e^{ψ} from Table 6.

	$a=0$	$a=1$
EE1A	$\operatorname{sech}^2 x$	$\cosh^2 y - \sin^2 x$
EE1B	$\cosh^2 y - \cos^2 x$	$\cosh^2 y - \cos^2 x$
EE2A	$\sec^2 y - \operatorname{sech}^2 x$	$\cosh^2 y - \cos^2 x$
EE2B	$\sec^2 y$	$\cosh^2 y - \sin^2 x$

2.4. *The General Case*

So far in this section, we have sought solutions of the form (1.4), with f and g Jacobian functions, or their limits circular and hyperbolic functions. For each of the functions considered, we had

$$\begin{aligned} f'(x) &= \pm\sqrt{f_0 + f_2 f^2(x) + f_4 f^4(x)}, \\ g'(y) &= \sqrt{g_0 + g_2 g^2(y) + g_4 g^4(y)}. \end{aligned} \tag{2.1}$$

In order for this to be a solution, if $g_4 \neq 0$, we require $g_0 = (1/3g_4)(3f_0 f_4 + g_2^2 - f_2^2)$, $B = -f_4 A/g_4$ and $C = A(f_2 + g_2)/3g_4$, and then we have $\nabla^2\psi = D_1 e^{\psi} + D_2 e^{-2\psi}$, with $D_1 = 2g_4/A$ and $D_2 = (4A^2/27g_4^2)(f_2 + g_2)(g_2^2 - f_2 g_2 - 2f_2^2 + 9f_0 f_4)$. If $g_4 = 0$, we require $f_4 = 0$, $g_2 = -f_2$ and $C = (Bf_0 - Ag_0)/2f_2$ and then we have $\nabla^2\psi = D_3 e^{-2\psi}$, with $D_3 = (B^2 f_0^2 - A^2 g_0^2)/f_2$, which again is Liouville's equation, and the only real-valued solution in this case is the Stuart vortex.

3. Discussion

In the previous section, we sought solutions to the nonlinear Poisson equation $\nabla^2\psi = \digamma(\psi)$ of the form $\psi = \ln[Ag^2(y) + Bf^2(x) + C]$, where f and g where either Jacobian functions, or their limits circular or hyperbolic functions. These solutions are presented in Tables 1, 2 and 4, and plotted in Figures 1–3. Amongst these solutions, the well-known Stuart vortex [7] was unique in that it represented a family of solutions, because of the amplitude parameter ε, while the remaining solutions were isolated solutions. The HC solutions from Table 1 satisfied Liouville's equation, with the remaining solutions satisfying the nonlinear Poisson equation $\nabla^2\psi = ce^{\psi} + de^{-2\psi}$, which reduced to Liouville's equation when the parameter $a = 0$ for the HE solutions and when $a = 1$ for the HC1 solutions from Table 2 and when $b = \tilde{a}$ for the solutions in from Table 4. The form of this nonlinear Poisson equation was slightly unexpected: at the outset, we suspected that any solutions we found would likely obey Liouville's equation, in part because so many other solutions obey Liouville's equation, but in retrospect perhaps the reason so many solutions to Liouville's equation have been found in the past can in part be attributed to the fact that a number of studies have actively sought such solutions while excluding solutions to other nonlinear Poisson equations from their search.

As can be seen from Figures 1–3, some of the solutions have very interesting flow patterns. For example, CE1 looks like a row of cat's eyes in a bounded channel and is regular except on the walls of that channel. CC1, CE2 and EE3 are all cellular flow fields. HC2 and

HE2 look like a stack, as opposed to a row, of cat's eyes. Finally, for some parameter values, EE1 gives us an array of vortices with wavy walls.

We remark in closing that there no doubt remain many exact nonlinear vortex solutions yet to be discovered. In our study, we sought solutions of a very specific form, and we would suggest that further searches could well prove profitable and unearth more vortex solutions.

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