

Research Article

On Some Properties of New Paranormed Sequence Space of Nonabsolute Type

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Received 25 February 2012; Accepted 24 June 2012

Academic Editor: Qiji J. Zhu

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We introduce some new generalized sequence space related to the space $\ell(p)$. Furthermore we investigate some topological properties as the completeness, the isomorphism, and also we give some inclusion relations between this sequence space and some of the other sequence spaces. In addition, we compute α -, β -, and γ -duals of this space and characterize certain matrix transformations on this sequence space.

1. Introduction

In studying the sequence spaces, especially, to obtain new sequence spaces, in general, the matrix domain μ_A of an infinite matrix A defined by $\mu_A = \{x = (x_k) \in w : Ax \in \mu\}$ is used. In most cases, the new sequence space μ_A generated by a sequence space μ is the expansion or the contraction of the original space μ . In some cases, these spaces could be overlapped. Indeed, one can easily see that the inclusion $\mu_S \subset \mu$ strictly holds for $\mu \in \{\ell_\infty, c, c_0\}$. Similarly one can deduce that the inclusion $\mu \subset \mu_\Delta$ also strictly holds for $\mu \in \{\ell_\infty, c, c_0\}$, where S and Δ are matrix operators.

Recently, in [1], Mursaleen and Noman constructed new sequence spaces by using matrix domain over a normed space. They also studied some topological properties and inclusion relations of these spaces.

It is well known that paranormed spaces have more general properties than the normed spaces. In this work, we generalize the normed sequence spaces defined by Mursaleen and Noman [1] to the paranormed spaces. Furthermore we introduce new sequence space over the paranormed space. Next we investigate behaviors of this sequence

space according to topological properties and inclusion relations. Finally we give certain matrix transformation on this sequence space and its duals.

In the literature, by using the matrix domain over the paranormed spaces, many authors have defined new sequence spaces. Some of them are as follows. For example, Choudhary and Mishra [2] have defined the sequence space $\overline{\ell(p)}$ where the S -transform is in $\ell(p)$, Başar and Altay [3, 4] defined the spaces $\lambda(u, v; p) = \{\lambda(p)\}_G$ for $\lambda \in \{\ell_\infty, c, c_0\}$ and $\ell(u, v; p) = \{\ell(p)\}_G$, respectively, and Altay and Başar [5] have defined the spaces $r_\infty^t(p)$, $r_c^t(p)$, $r_0^t(p)$. In [6], Karakaya and Polat defined and examined the spaces $e_0^r(\Delta; p)$, $e^r(\Delta; p)$, $e_\infty^r(\Delta; p)$, and Karakaya et al. [7] have recently introduced and studied the spaces $\ell_\infty(\lambda, p)$, $c(\lambda, p)$, $c_0(\lambda, p)$, where R^t and E^r denote the Riesz and the Euler means, respectively, Δ denotes the band matrix of the difference operators, and Λ, G are defined in [1, 8], respectively. Also, the information on matrix domain of sequence spaces can be found in [9–13].

By w , we denote the space of all real valued sequences. Any vector subspace of w is called a sequence space. By the spaces ℓ_1, cs , and bs , we denote the spaces of all absolutely convergent series, convergent series, and bounded series, respectively.

A linear topological space X over the real field \mathbb{R} is said to be a paranormed space if there is a subadditivity function $h : X \rightarrow \mathbb{R}$ such that $h(\theta) = 0$, $h(x) = h(-x)$, and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \rightarrow 0$ and $h(x_n - x) \rightarrow 0$ imply $h(\alpha_n x_n - \alpha x) \rightarrow 0$ for all α in \mathbb{R} and x in X , where θ is the zero in the linear space X .

Let μ, ν be any two sequence spaces, and let $A = (a_{nk})$ be any infinite matrix of real number a_{nk} , where $n, k \in \mathbb{N}$ with $\mathbb{N} = \{0, 1, 2, \dots\}$. Then we say that A defines a matrix mapping from μ into ν by writing $A : \mu \rightarrow \nu$, if for every sequence $x = (x_k) \in \mu$, the sequence $Ax = (A_n(x))$, the A -transform of x , is in ν , where

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \quad (1.1)$$

By (μ, ν) , we denote the class of all matrices A such that $A : \mu \rightarrow \nu$. Thus, $A \in (\mu, \nu)$ if and only if the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax \in \nu$ for all $x \in \mu$. A sequence x is said to be A -summable to a if Ax converges to a which is called as the A -limit of x .

Assume here and after that $(p_k), (q_k)$ are bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $M = \max(1, H)$, and also let $p_k = p_k / (p_k - 1)$ for $1 < p_k < \infty$ and for all $k \in \mathbb{N}$. The linear space $\ell(p)$ was defined by Maddox [14] as follows:

$$\ell(p) = \left\{ x = (x_n) \in w : \sum_{n=0}^{\infty} |x_n|^{p_n} < \infty \right\} \quad (1.2)$$

which are the complete space paranormed by

$$h(x) = \left(\sum_{n=0}^{\infty} |x_n|^{p_n} \right)^{1/M}. \quad (1.3)$$

Throughout this work, by F and N_k , respectively, we will denote the collection of all subsets of \mathbb{N} and the set of all $n \in \mathbb{N}$ such that $n \geq k$ and $e = (1, 1, 1, \dots)$.

2. The Sequence Space $\ell(\lambda, p)$

In this section, we define the sequence space $\ell(\lambda, p)$ and prove that this sequence space according to its paranorm is complete paranormed linear space. In [1], Mursaleen and Noman defined the matrix $\Lambda = (\lambda_{nk})_{n,k=0}^\infty$ by

$$\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} & (0 \leq k \leq n) \\ 0 & (k > n), \end{cases} \quad (2.1)$$

where $\lambda = (\lambda_k)_{k=0}^\infty$ is a strictly increasing sequence of positive reals tending to ∞ , that is, $0 < \lambda_0 < \lambda_1 < \dots$ and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. Now, by using (2.1) we define new sequence space as follows:

$$\ell(\lambda, p) = \left\{ x = (x_k) \in w : \sum_{n=0}^\infty \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^{p_n} < \infty \right\}. \quad (2.2)$$

For any $x = (x_n) \in w$, we define the sequence $y = (y_n)$, which will frequently be used, as the Λ -transform of x , that is, $y = \Lambda(x)$, and hence

$$y_n = \sum_{k=0}^n \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) x_k \quad (n \in N). \quad (2.3)$$

We now may begin with the following theorem.

Theorem 2.1. *The sequence space $\ell(\lambda, p)$ is the complete linear metric space with respect to paranorm defined by*

$$h(x) = \left(\sum_{n=0}^\infty \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^{p_n} \right)^{1/M}. \quad (2.4)$$

Proof. The linearity of $\ell(\lambda, p)$ with respect to the coordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x, t \in \ell(\lambda, p)$ (see, [15]):

$$\begin{aligned} \left(\sum_{n=0}^\infty \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) (x_k + t_k) \right|^{p_n} \right)^{1/M} &\leq \left(\sum_{n=0}^\infty \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^{p_n} \right)^{1/M} \\ &+ \left(\sum_{n=0}^\infty \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) t_k \right|^{p_n} \right)^{1/M} \end{aligned} \quad (2.5)$$

and for any $\alpha \in \mathbb{R}$ (see, [16])

$$|\alpha|^{p_k} \leq \max \{ 1, |\alpha|^M \}. \quad (2.6)$$

It is clear that $h(\theta) = 0$, $h(x) = h(-x)$ for all $x \in \ell(\lambda, p)$. Again inequalities (2.5) and (2.6) yield the subadditivity of h and hence $h(\alpha x) \leq \max\{1, |\alpha|^M\}h(x)$. Let $\{x^m\}$ be any sequence of points $x^m \in \ell(\lambda, p)$ such that $h(x^m - x) \rightarrow 0$ and (α_m) also any sequence of scalars such that $\alpha_m \rightarrow \alpha$. Then, since the inequality

$$h(x^m) \leq h(x) + h(x^m - x) \quad (2.7)$$

holds by subadditivity of h , we can write that $\{h(x^m)\}$ is bounded and we thus have

$$\begin{aligned} h(\alpha_m x^m - \alpha x) &= \left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda^n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) (\alpha_m x_k^m - \alpha x_k) \right|^{p_n} \right)^{1/M} \\ &\leq |\alpha_m| h(x^m) + |\alpha| h(x^m - x) \end{aligned} \quad (2.8)$$

which tends to zero as $n \rightarrow \infty$. Therefore, the scalar multiplication is continuous. Hence h is a paranorm on the space $\ell(\lambda, p)$. It remains to prove the completeness of the space $\ell(\lambda, p)$. Let $\{x^j\}$ be any Cauchy sequence in the space $\ell(\lambda, p)$, where $x^j = \{x_0^{(j)}, x_1^{(j)}, x_2^{(j)}, \dots\}$. Then, for a given $\varepsilon > 0$, there exists a positive integer $m_0(\varepsilon)$ such that $h(x^j - x^i) < \varepsilon/2$ for all $i, j > m_0(\varepsilon)$. Using definition of h , we obtain for each fixed $n \in \mathbb{N}$ that

$$\left| \Lambda_n(x^j) - \Lambda_n(x^i) \right| \leq \left(\sum_{n=0}^{\infty} \left| \Lambda_n(x^j) - \Lambda_n(x^i) \right|^{p_n} \right)^{1/M} < \frac{\varepsilon}{2} \quad (2.9)$$

for every $i, j > m_0(\varepsilon)$ which leads us to the fact that $\{\Lambda_n(x^0), \Lambda_n(x^1), \Lambda_n(x^2), \dots\}$ is a Cauchy sequence of real numbers for every fixed $n \in \mathbb{N}$. Since \mathbb{R} is complete, it converges, say $\Lambda_n(x^i) - \Lambda_n(x)$ as $i \rightarrow \infty$. Using these infinitely many limits, we may write the sequence $\{\Lambda_0(x), \Lambda_1(x), \Lambda_2(x), \dots\}$. From (2.9) as $i \rightarrow \infty$, we have

$$\left| \Lambda_n(x^j) - \Lambda_n(x) \right| < \frac{\varepsilon}{2}, \quad (j \geq m_0(\varepsilon)) \quad (2.10)$$

for every fixed $n \in \mathbb{N}$. By using (2.9) and boundedness of Cauchy sequence, we have

$$\left(\sum_{n=0}^{\infty} |\Lambda_n(x)|^{p_n} \right)^{1/M} \leq \left(\sum_{n=0}^{\infty} \left| \Lambda_n(x^j) - \Lambda_n(x^i) \right|^{p_n} \right)^{1/M} + \left(\sum_{n=0}^{\infty} \left| \Lambda_n(x^j) \right|^{p_n} \right)^{1/M} < \infty. \quad (2.11)$$

Hence, we get $x \in \ell(\lambda, p)$. So, the space $\ell(\lambda, p)$ is complete. \square

Theorem 2.2. *The sequence space $\ell(\lambda, p)$ of nonabsolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \leq H < \infty$.*

Proof. To prove the theorem, we would show the existence of linear bijection between the spaces $\ell(\lambda, p)$ and $\ell(p)$. With the notation of (2.3), we define transformation T from $\ell(\lambda, p)$ to $\ell(p)$ by $x \rightarrow y = Tx$. The linearity of T is trivial. Furthermore, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in \ell(p)$ and define the sequence $x = \{x_n\}$

$$x_n(\lambda) = \sum_{k=n-1}^n \left((-1)^{n-k} \frac{\lambda_k}{\lambda_n - \lambda_{n-1}} \right) y_k \quad (n, k \in \mathbb{N}). \quad (2.12)$$

Then, we have

$$h_{\ell(\lambda, p)}(x) = \left(\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^{p_n} \right)^{1/M} = \left(\sum_{n=0}^{\infty} |y_n|^{p_n} \right)^{1/M} = h_{\ell(p)}(y). \quad (2.13)$$

Thus, we have that $x \in \ell(\lambda, p)$ and consequently T is surjective. Hence, T is a linear bijection and this tells us that the spaces $\ell(\lambda, p)$ and $\ell(p)$ are linearly isomorphic. This completes the proof. \square

3. Some Inclusion Relations

In this section, we give some inclusion relations concerning the space $\ell(\lambda, p)$. Before giving the theorems about the section, we give a lemma given in [1].

Lemma 3.1. *For any sequence $x = (x_k) \in w$, the equalities*

$$S_n(x) = x_n - \Lambda_n(x), \quad (3.1)$$

$$S_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} [\Lambda_n(x) - \Lambda_{n-1}(x)] \quad (3.2)$$

hold, where the sequence $S(x) = \{S_n(x)\}$ is defined by

$$S_0(x) = 0, \quad S_n(x) = \frac{1}{\lambda_n} \sum_{k=1}^n \lambda_{k-1} (x_k - x_{k-1}) \quad (n \geq 1). \quad (3.3)$$

Theorem 3.2. *The inclusion $\ell(\lambda, p) \subset c_0(\lambda, p)$ holds.*

Proof. Let $x \in \ell(\lambda, p)$. It can be written $\Lambda x \in \ell(p)$. By the definition of the space $\ell(p)$, $\Lambda_n x \rightarrow \infty$ as $n \rightarrow \infty$, we obtain $\Lambda x \in c_0$. Hence we get $x \in c_0(\lambda, p)$. \square

Theorem 3.3. *The inclusion $\ell(\lambda, p) \subset \ell(p)$ if and only if $S(x) \in \ell(p)$ for every sequence $x \in \ell(\lambda, p)$, where $1 \leq p_k \leq H$.*

Proof. We suppose that $\ell(\lambda, p) \subset \ell(p)$ holds and take any $x \in \ell(\lambda, p)$. Then $x \in \ell(p)$ by hypothesis. Thus we obtain from (3.1) that

$$[h(S(x))]_{\ell(p)} \leq [h(x)]_{\ell(p)} + [h(\Lambda x)]_{\ell(p)} = [h(x)]_{\ell(p)} + [h(x)]_{\ell(\lambda, p)} \quad (3.4)$$

which yields that $S(x) \in \ell(p)$.

Conversely, let $x \in \ell(\lambda, p)$ be given. Then we have by the hypothesis that $S(x) \in \ell(p)$. Again by using (3.1)

$$[h(x)]_{\ell(p)} \leq [h(S(x))]_{\ell(p)} + [h(\Lambda x)]_{\ell(p)} = [h(S(x))]_{\ell(p)} + [h(x)]_{\ell(\lambda, p)} \quad (3.5)$$

which shows that $x \in \ell(p)$. Hence the inclusion $\ell(\lambda, p) \subset \ell(p)$ holds. This completes the proof. \square

Theorem 3.4.

(i) If $p_n > 1$ for all $n \in \mathbb{N}$, then the inclusion $\ell_p^\lambda \subset \ell(\lambda, p)$ holds.

(ii) If $p_n < 1$ for all $n \in \mathbb{N}$, then the inclusion $\ell(\lambda, p) \subset \ell_p^\lambda$ holds.

Proof. (i) If $p = (p_n)$ for all $n \in \mathbb{N}$, then we write ℓ_p^λ in place of $\ell(\lambda, p)$. Let $x \in \ell_p^\lambda$. It is clear that $\Lambda(x) \in \ell_p$. One can find $m \in \mathbb{N}$ such that $|\Lambda_n(x)| < 1$ for all $n \geq m$. Under condition (i), we have $|\Lambda_n(x)|^{p_n} < |\Lambda_n(x)|$ for all $n \geq m$. Hence we get $x \in \ell(\lambda, p)$.

(ii) We suppose that $x \in \ell(\lambda, p)$. Then $\Lambda(x) \in \ell(p)$ and there exists $m \in \mathbb{N}$ such that $|\Lambda_n(x)|^{p_n} < 1$ for all $n \geq m$. To obtain the result, we consider the following inequality:

$$|\Lambda_n(x)| = (|\Lambda_n(x)|^{p_n})^{1/p_n} < |\Lambda_n(x)|^{p_n} \quad (3.6)$$

for all $n \geq m$. So, we get $x \in \ell_p^\lambda$. \square

4. Some Matrix Transformations and Duals of the Space $\ell(\lambda, p)$

In this section, we give the theorems determining the α -, β -, and γ -duals of the space $\ell(\lambda, p)$. In proving the theorem, we apply the technique used in [3]. Also we give some matrix transformations from the space $\ell(\lambda, p)$ into paranormed spaces $\ell(q)$ by using the matrix given in [1].

For the sequence space μ and ν , the set $S(\mu, \nu)$ defined by

$$S(\mu, \nu) = \{a = (a_k) \in w : ax \in \nu \forall x \in \mu\} \quad (4.1)$$

is called the multiplier space of μ and ν . The α -, β -, and γ -duals of a sequence space μ , which are, respectively, denoted by μ^α , μ^β , and μ^γ , are defined by

$$\mu^\alpha = S(\mu, \ell_1), \quad \mu^\beta = S(\mu, cs), \quad \mu^\gamma = S(\mu, bs). \quad (4.2)$$

We may begin with the following theorem which computes the α -dual of the space $\ell(\lambda, p)$.

Theorem 4.1. Let $K_1 = \{k \in \mathbb{N} : p_k \leq 1\}$ and $K_2 = \{k \in \mathbb{N} : p_k > 1\}$. Define the matrix $D^a = (d_{nk}^a)$ by

$$d_{nk}^a = \begin{cases} (-1)^{n-k} \frac{\lambda_k}{\lambda_n - \lambda_{n-1}} a_n, & (n-1 \leq k \leq n) \\ 0, & (0 \leq k \leq n-1) \text{ or } (k > n). \end{cases} \quad (4.3)$$

Then

$$\begin{aligned} \ell_{K_1}^\alpha(\lambda, p) &= \{a = (a_n) \in w : D^a \in (\ell(p); \ell_\infty)\} \\ \ell_{K_2}^\alpha(\lambda, p) &= \{a = (a_n) \in w : D^a \in (\ell(p); \ell_1)\}. \end{aligned} \tag{4.4}$$

Proof. We consider the following equality:

$$a_n x_n = \sum_{k=n-1}^n d_{nk}^a y_k = (D^a y)_n \quad (n \in \mathbb{N}), \tag{4.5}$$

where $D^a = (d_{nk}^a)$ is defined by (4.3).

From (4.5), it can be obtained that $ax = (a_n x_n) \in \ell_1$ or $ax = (a_n x_n) \in \ell_\infty$ whenever $x \in \ell(\lambda, p)$ if and only if $D^a y \in \ell_1$ or $D^a y \in \ell_\infty$ whenever $y \in \ell(p)$. This means $a \in \ell_{K_1}^\alpha(\lambda, p)$ or $a \in \ell_{K_2}^\alpha(\lambda, p)$ if and only if $D^a \in (\ell(p); \ell_1)$ or $D^a \in (\ell(p); \ell_\infty)$. Hence this completes the proof. \square

The result of the Theorem above corresponds the Theorem 5.1(0, 8, 12) given in [17]. As a direct consequence of Theorem 4.1, we have the following.

Corollary 4.2. *Let $K^* = \{k \in \mathbb{N} : n - 1 \leq k \leq n\} \cap K$ for $K \in F$. Then*

- (i) $\ell_{K_1}^\alpha(\lambda, p) = \{a = (a_n) \in w : \sup_N \sup_{k \in \mathbb{N}} |\sum_{n \in K^*} d_{nk}^a|^{p_k} < \infty\}$,
- (ii) $\ell_{K_2}^\alpha(\lambda, p) = \bigcup_{M > 1} \{a = (a_n) \in w : \sup_{K \in F} \sum_k |\sum_{n \in K^*} d_{nk}^a M^{-1}|^{p_k} < \infty\}$

In the following theorem, we characterize the β - and γ -duals of the space $\ell(\lambda, p)$.

Theorem 4.3. *Let $K_1 = \{k \in \mathbb{N} : p_k \leq 1\}$, $K_2 = \{k \in \mathbb{N} : p_k > 1\}$, and let $\Delta x_k = x_k - x_{k+1}$. Define the sequence $s^1 = (s_k^1)$, $s^2 = (s_k^2)$ and the matrix $B^a = (b_{nk}^a)$ by*

$$\begin{aligned} s_k^1 &= \Delta \left(\frac{a_k}{\lambda_k - \lambda_{k-1}} \right) \lambda_k, & s_k^2 &= \frac{a_k \lambda_k}{\lambda_k - \lambda_{k-1}} \\ b_{nk}^a &= \begin{cases} s_k^1, & (0 \leq k \leq n-1) \\ s_k^2, & (k = n) \\ 0, & (k > n) \end{cases} \end{aligned} \tag{4.6}$$

for all $n, k \in \mathbb{N}$. Then

$$\ell_{K_1}^\beta(\lambda, p) = \ell_{K_1}^\gamma(\lambda, p) = \{a = (a_n) \in w : B^a \in (\ell(p); \ell_\infty)\}, \tag{4.7}$$

$$\ell_{K_2}^\beta(\lambda, p) = \ell_{K_2}^\gamma(\lambda, p) = \{a = (a_n) \in w : B^a \in (\ell(p); c)\}. \tag{4.8}$$

Proof. Consider the equality

$$\sum_{k=0}^n a_k x_k = \sum_{k=0}^{n-1} s_k^1 y_k + s_n^2 y_n = (B^a y)_n. \quad (4.9)$$

From (4.9), it can be obtained that $ax = (a_n x_n) \in cs$ or bs whenever $x = (x_n) \in \ell(\lambda, p)$ if and only if $B^a y \in c$ or ℓ_∞ whenever $y = (y_k) \in \ell(p)$. This means that $a = (a_n) \in \{\ell_{K_1}^\beta(\lambda, p)$ or $\ell_{K_2}^\beta(\lambda, p)\}$ or $a = (a_n) \in \{\ell_{K_1}^\gamma(\lambda, p)$ or $\ell_{K_2}^\gamma(\lambda, p)\}$ if and only if $B^a \in (\ell(p); c)$ or $B^a \in (\ell(p); \ell_\infty)$. Hence this completes the proof. \square

We can write the following corollary from Theorem 4.3.

Corollary 4.4. Let $\dot{p}_k = p_k / (p_k - 1)$ for $1 < p_k < \infty$ and for all $k \in \mathbb{N}$. Then

$$(i) \ell_{K_1}^\beta(\lambda, p) = \ell_{K_1}^\gamma(\lambda, p) = \{a = (a_n) \in w : s^1, s^2 \in \ell_\infty(p)\},$$

$$(ii) \ell_{K_2}^\beta(\lambda, p) = \ell_{K_2}^\gamma(\lambda, p) = \bigcup_{M>1} \{a = (a_n) \in w : s^1 M^{-1}, s^2 M^{-1} \in \ell(p') \cap \ell_\infty(p')\}.$$

After this step, we can give our theorems on the characterization of some matrix classes concerning the sequence space $\ell(\lambda, p)$.

Let $x, y \in w$ be connected by the relation $y = \Lambda(x)$. For an infinite matrix $A = (a_{nk})$, we have by using (4.9) of Theorem 4.3 that

$$\sum_{k=0}^m a_{nk} x_k = \sum_{k=0}^{m-1} \tilde{a}_{nk} y_k + \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} a_{nm} y_m \quad (m, n \in \mathbb{N}), \quad (4.10)$$

where

$$\tilde{a}_{nk} = \left(\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} - \frac{a_{n,k+1}}{\lambda_{k+1} - \lambda_k} \right) \lambda_k \quad (n, k \in \mathbb{N}). \quad (4.11)$$

The necessary and sufficient conditions characterizing the matrix mapping of the sequence space $\ell(p)$ of Maddox have been determined by Grosse-Erdmann [17]. Let L and M be the natural numbers and define the sets by $K_1 = \{k \in \mathbb{N} : p_k \leq 1\}$ and $K_2 = \{k \in \mathbb{N} : p_k > 1\}$ and also let us put $\dot{p}_k = p_k / (p_k - 1)$ for $1 < p_k < \infty$ and for all $k \in \mathbb{N}$. Before giving the theorems,

let us suppose that (q_n) is a nondecreasing bounded sequence of positive real numbers and consider the following conditions:

$$\sup_N \sup_{k \in K_1} \left| \sum_{n \in \mathbb{N}} \tilde{a}_{nk} \right|^{q_n} < \infty, \tag{4.12}$$

$$\exists M \sup_N \sum_{k \in K_2} \left| \sum_{n \in \mathbb{N}} \tilde{a}_{nk} M^{-1} \right|^{p_k} < \infty, \tag{4.13}$$

$$\exists M \sup_k \sum_n \left| \tilde{a}_k M^{-1/p_k} \right|^{q_n} < \infty, \tag{4.14}$$

$$\lim_n |\tilde{a}_{nk}|^{q_n} = 0 \quad (\forall k \in \mathbb{N}), \tag{4.15}$$

$$\forall L, \sup_n \sup_{k \in K_1} \left| \tilde{a}_{nk} L^{1/q_n} \right|^{p_k} < \infty, \tag{4.16}$$

$$\forall L, \exists M \sup_n \sum_{k \in K_2} \left| \tilde{a}_k L^{1/q_n} M^{-1} \right|^{p_k} < \infty, \tag{4.17}$$

$$\sup_n \sup_{k \in K_1} |\tilde{a}_{nk}|^{p_k} < \infty, \tag{4.18}$$

$$\exists M \sup_n \sum_{k \in K_2} \left| \tilde{a}_k M^{-1} \right|^{p_k} < \infty, \tag{4.19}$$

$$\forall L, \sup_n \sup_{k \in K_1} \left(|\tilde{a}_{nk} - \tilde{a}_k| L^{1/q_n} \right)^{p_k} < \infty, \tag{4.20}$$

$$\lim_n |\tilde{a}_{nk} - \tilde{a}_k|^{q_n} = 0, \quad \forall k. \tag{4.21}$$

$$\forall L, \exists M \sup_n \sum_{k \in K_2} \left(|\tilde{a}_{nk} - \tilde{a}_k| L^{1/q_n} M^{-1} \right)^{p_k}, \tag{4.22}$$

$$\exists L, \sup_n \sup_{k \in K_1} \left| \tilde{a}_{nk} L^{-1/q_n} \right|^{p_k} < \infty, \tag{4.23}$$

$$\exists L, \sup_n \sum_{k \in K_2} \left| \tilde{a}_{nk} L^{-1/q_n} \right|^{p_k} < \infty, \tag{4.24}$$

$$\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk} \right)_{k=0}^\infty \in c_0(q) \quad (\forall n \in \mathbb{N}) \tag{4.25}$$

$$\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk} \right)_{k=0}^\infty \in c(q) \quad (\forall n \in \mathbb{N}) \tag{4.26}$$

$$\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk} \right)_{k=0}^\infty \in \ell_\infty(q) \quad (\forall n \in \mathbb{N}). \tag{4.27}$$

By using (4.7), (4.10), and Corollary 4.4, we have the following results.

Theorem 4.5. *One has the following:*

- (i) $A \in (\ell(\lambda, p) : \ell(q))$ if and only if (4.12), (4.13), (4.14), and (4.25) hold,
- (ii) $A \in (\ell(\lambda, p) : c_0(q))$ if and only if (4.15), (4.16), (4.17), and (4.25) hold,
- (iii) $A \in (\ell(\lambda, p) : c(q))$ if and only if (4.18), (4.19), (4.20), (4.21), (4.22), and (4.26) hold,
- (iv) $A \in (\ell(\lambda, p) : \ell_\infty(q))$ if and only if (4.23), (4.24), and (4.27) hold.

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