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Research Article

An Inverse Problem for a Class of Linear Stochastic Evolution Equations

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An inverse problem for a linear stochastic evolution equation is researched. The stochastic evolution equation contains a parameter with values in a Hilbert space. The solution of the evolution equation depends continuously on the parameter and is Fréchet differentiable with respect to the parameter. An optimization method is provided to estimate the parameter. A sufficient condition to ensure the existence of an optimal parameter is presented, and a necessary condition that the optimal parameter, if it exists, should satisfy is also presented. Finally, two examples are given to show the applications of the above results.

1. Introduction

The purpose of this paper is to study an inverse problem for the following linear stochastic evolution equation:

$$\begin{gathered} dy + [A(t;p) + B(t;p)]y\,dt = f(t;p)dt + \sigma(t;p)dw(t), \quad t \in (t_0, t_f) \equiv T, \\ y(t_0) = \varphi + \xi, \end{gathered} \tag{1.1}$$

where $t_f < \infty$, $p \in P_{ad} \subset P$ is a parameter to be determined, and P_{ad} is a convex domain in P. The solution of (1.1) corresponding to p can be denoted as $y = y(p) = y(t;p)$ to explicitly show the dependence of y on p.

The problem of this paper is to determine the unknown parameter p based on the measurement $g(t)$, which is defined by the following:

$$g(t) = \Lambda(t;p)y(t), \quad t \in T, \tag{1.2}$$

where V, H, K, W, and P are Hilbert spaces.

There are many papers dealing with parameter estimation problems for stochastic partial differential equations, for instance, see [1–7], but only a few papers to estimate directly parameters involved in stochastic evolution equations in infinite dimensional spaces, for example, [8, 9]. In particular, Lototsky and Rosovskii [9] consider a problem estimating a constant parameter and obtain an estimate that is consistent and asymptotically normal.

Denote by $\mathcal{L}(X;Y)$ the linear continuous operator space on X to Y, by $(\cdot,\cdot)_X$ the inner product of X, and by $\langle\cdot,\cdot\rangle_{X',X}$ the dual product of X' and X, where X' is the dual of X.

V, V', and H make up an evolution triple, namely, they should satisfy

$$V \subset H \subset V', \tag{1.3}$$

where each space is dense in the following space and has a continuous injection, $H' = H$, and

$$\langle y,x\rangle_{V',V} = (y,x)_H, \quad \forall x \in V,\ y \in H. \tag{1.4}$$

For any $t \in T$ and $p \in P_{ad}$, $A(t;p) \in \mathcal{L}(V;V')$, $B(t;p) \in \mathcal{L}(V;H)$, $\sigma(t,p) \in \mathcal{L}(W;H)$, $\Lambda(t;p) \in \mathcal{L}(V;K,)$, $f(t;p) \in H$, and

$$\langle A(t;p)z,z\rangle_{V',V} \geq \alpha\|z\|_V^2, \quad z \in V, \tag{1.5}$$

where the constant α is independent of $p \in P_{ad}$ and $t \in T$. Let $(\Omega,\mathcal{F},\mu)$ be a complete probability space and $\mathcal{F}^t$ an increasing family of sub σ-algebras of $\mathcal{F}(\mathcal{F} = \mathcal{F}^\infty)$.

$M(\Omega,\mu,\Phi)$ denotes the space of random variables with values in a Hilbert space Φ.

$$L^r(\Omega;\Phi) \equiv \left\{ x \in M(\Omega,\mu,\Phi);\ \|x\| \equiv \left[E\|x\|_\Phi^r\right]^{1/r} = \left[\int_\Omega \|x(\omega)\|_\Phi^r d\mu(\omega)\right]^{1/r} < \infty \right\},$$
$$L^r(\Omega\times T;\Phi) \equiv \left\{ x(t) \in M(\Omega,\mu,\Phi), t \in T;\ \|x\| \equiv \left[\int_\Omega\int_T \|x(t,\omega)\|_\Phi^r dt d\mu(\omega)\right]^{1/r} < \infty \right\}. \tag{1.6}$$

$w(t) \in M(\Omega,\mu,W)$ is a Wiener process with values in a separable Hilbert space W, which is adapted to $\mathcal{F}^t$, that is, for all $e \in W$, $(w(t),e)_W$ is a real Wiener process and an $\mathcal{F}^t$-martingale, with the correlation function

$$E\{(w(t_1),e_1)_W(w(t_2),e_2)_W\} = \int_{t_0}^{\min(t_1,t_2)} (Q(\tau)e_1,e_2)d\tau, \tag{1.7}$$

where $t_0 \leq t_1$, $t_2 \leq t_f$ and $Q(t)$ is a positive self-adjoint nuclear operator almost everywhere on W. $Q(t)$ is called the covariance operator. Moreover, assume that $Q(t)$ satisfies

$$Q(\cdot) \in L^\infty(T;\mathcal{L}(W)), \tag{1.8}$$

where $\varphi_0 \in H$, $\xi \in L^2(\Omega,H)$ is independent of $w(t)$.

Now, determine the parameter p in the system (1.1) and (1.2). It is transformed into an optimization problem as most researchers expect. That is, seek an optimal parameter $\hat{p} \in P_{ad}$ such that the cost functional

$$J(p) = E_g \int_T \|g(t) - \Lambda(t;p)y(t;p)\|_K^2 dt \tag{1.9}$$

reaches its minimum over the admissible parameter set P_{ad} at $\hat{p}$, that is,

$$J(\hat{p}) = \min_{p \in P_{ad}} J(p), \tag{1.10}$$

where E_g is a conditional expectation, that is,

$$E_g f = E\{f \mid G^t\}, \tag{1.11}$$

and G^t is the sub-σ-algebra induced by the stochastic process $g(s)$, $0 \le s \le t$, which is adapted to $\mathcal{F}^t$.

If there exists a neighbourhood $U \subset P_{ad}$ of p_0 such that

$$J(p_0) = \inf_{p \in U} J(p), \tag{1.12}$$

then P_o is called a relative optimal parameter.

In Section 2, the base of this paper is given, under certain conditions the function $p \to y(p)$ is continuous and Fréchet continuously differentiable.

In Section 3, the main results of this paper are proposed. The problem estimating the parameter is transformed into the optimization problem. The above optimization problem, such as existence of the optimal parameter and necessary conditions, is studied.

In Section 4, the results in Sections 2 and 3 are applied to parabolic stochastic partial differential equations to identify certain parameters involved in those equations.

2. Continuity and Differentiability with Respect to a Parameter

In this section the continuity and the differentiability of the solution of the system (1.1) with respect to the parameter p are studied.

Before studying the properties of the solution to (1.1), it must be shown that the system (1.1) is well-behaved in some sense on certain conditions. There are many papers dealing with solvability of Stochastic evolution equation (1.1), for example, see [10–12]. From Bensoussan [10] the following lemma is useful.

Lemma 2.1. *Besides the assumptions for A, B, f, φ, $w(t)$, and σ in Section 1, one assumes that, for any $p \in P_{ad}$,*

$$\begin{aligned}
&A(\cdot;p) \in L^2(T;\mathcal{L}(V;V')) \cap L^\infty(T;\mathcal{L}(V;V')),\\
&B(\cdot;p) \in L^\infty(\Omega \times T;\mathcal{L}(V;H)),\\
&f(\cdot,p) \in L^2(\Omega \times T;H),\\
&\sigma(\cdot,p) \in L^2(T,\mathcal{L}(W;H)).
\end{aligned} \tag{2.1}$$

Then there exists a unique generalized solution, y, in the Ladyzenskaja sense of (1.1) *almost every $t \in T$ such that*

$$y \in L^2(\Omega;C(T;H)) \cap L^2(\Omega \times T;V) \cap C\left(\overline{T};L^2(\Omega;H)\right) \equiv \mathfrak{S}, \tag{2.2}$$

$y(t)$ is adapted to $\mathcal{F}^t$ (as a process with values in H), and $y(t)$ is $\mathcal{F}^t$ measurable with values in V.

In the above lemma the space $C^k(X,Y)$ with $k = 0, 1$ consists of all continuous functions $u : X \to Y$ that have continuous Fréchet derivatives up to order k on X, with the norm

$$\|u\| = \sum_{i=0}^{k} \max_{t\in X} \left\| u^{(i)}(t) \right\|. \tag{2.3}$$

Remark 2.2. The generalized solution in the Ladyzenskaja sense is the solution of the following variational equation:

$$\begin{aligned}
(y(t),\eta(t))_H = (\varphi+\xi,\eta(t_0))_H &+ \int_{t_0}^{t} \langle y(t), d_t\eta(t)\rangle_{V,V'}\\
&+ \int_{t_0}^{t} \langle [A(t;p)+B(t;p)]y(t)dt, \eta(t)\rangle_{V',V}\\
&+ \int_{t_0}^{t} (\sigma(t;p)dw(t),\eta(t))_H, \quad \text{a.s.}, \ \forall \eta \in W^1(T;V,H),
\end{aligned} \tag{2.4}$$

where the space $W^1(T;V,H)$ is a Hilbert space that is defined by

$$W^1(T;V,H) = \left\{ \eta;\ \eta \in L^2(T;V),\ \frac{d\eta}{dt} \in L^2(T;V') \right\}, \tag{2.5}$$

and it is well known that $W^1(T;V,H) \subset C(\overline{T};H)$.

Now, it is time to give the main results in this section.

Theorem 2.3. *If the assumptions of Lemma 2.1 are satisfied and*

$$\begin{aligned} &A : p \longrightarrow L^2(T; \mathcal{L}(V; V')), \qquad B : p \longrightarrow L^\infty(\Omega \times T; \mathcal{L}(V; H)), \\ &f : p \longrightarrow L^2(\Omega \times T; H), \qquad \sigma : p \longrightarrow L^2(T, \mathcal{L}(W; H)), \end{aligned} \tag{2.6}$$

are continuous, the solution of (1.1)

$$y : P_{ad} \longrightarrow \mathfrak{S} \tag{2.7}$$

is continuous, that is, $y \in C(P_{ad}; \mathfrak{S})$ *or the following equalities are true:*

$$\lim_{p \to p_0} \| y(p) - y(p_0) \|_{C(\overline{T}; L^2(\Omega, H))} = 0, \tag{2.8}$$

$$\lim_{p \to p_0} \| y(p) - y(p_0) \|_{L^2(\Omega; C(\overline{T}; H))} = 0, \tag{2.9}$$

$$\lim_{p \to p_0} \| y(p) - y(p_0) \|_{L^2(\Omega \times T; V)} = 0. \tag{2.10}$$

Before proving Theorem 2.3, from Bensoussan [10] the following $It\hat{o}$ formula in a Hilbert space is quoted.

Lemma 2.4. *Let* $\Phi(z, t)$ *be a functional on* $H \times \overline{T}$*, which is twice continuously Fréchet differentiable in* $z \in H$ *and continuously differentiable in* $t \in T$*. Assume* $z(t)$ *has the stochastic differential:*

$$dz(t) = a(t)dt + b(t)dw(t), \quad z(t_0) = z_0, \tag{2.11}$$

where $a(t)$ *is a stochastic process with values in* H*, which is adapted and satisfies the condition*

$$a.s \int_{t_0}^{t} \| a(t) \|_H dt < +\infty, \quad \forall t \in T, \tag{2.12}$$

where $b(t)$ *is an adapted process with values in* $\mathcal{L}(W, H)$ *such that* $t, \omega \longrightarrow b$ *is measurable and*

$$E \int_{t_0}^{t_1} \| b(t) \|^2_{\mathcal{L}(W,H)} dt < \infty, \quad \forall t_1 < t_f, \tag{2.13}$$

then one has the following $It\hat{o}$ *formula in the Hilbert space:*

$$\begin{aligned} \Phi(z(t), t) = \Phi(z(t_0), t_0)) &+ \int_{t_0}^{t} \left(\frac{\partial \Phi}{\partial z}, a(s) \right)_H ds + \int_{t_0}^{t} \left(\frac{\partial \Phi}{\partial z}, b dw(s) \right)_H \\ &+ \frac{1}{2} \int_{t_0}^{t} \operatorname{tr} \left[b^* \frac{\partial^2 \Phi}{\partial z^2} b Q \right] ds + \int_{t_0}^{t} \frac{\partial \Phi}{\partial t} ds, \end{aligned} \tag{2.14}$$

where b^* *is the adjoint of* b *and the symbol "tr" is the trace operator, of which definition for a nuclear operator* $U \in \mathcal{L}(X)$ *is as follows:*

$$\text{tr } U = \sum_{n=1}^{\infty} (U f_n, f_n)_X, \tag{2.15}$$

where $\{f_n\}$ *is an orthonormal basis of* X*.*

The following two lemmas are obvious, so their proofs are omitted.

Lemma 2.5. *Supposes that* $U \in \mathcal{L}(X)$ *is a nuclear operator and* $B \in \mathcal{L}(X, Y)$*, then*

$$\text{tr } (B^* B U) \le \|B\|^2 \text{ tr } U. \tag{2.16}$$

Lemma 2.6. *If* $\psi \in L^2(\Omega; L^2(T))$*,*

$$E \int_T \psi(t) dw(t) = 0. \tag{2.17}$$

Lemma 2.7. *If* $f \in L^2(\Omega; L^2(T; W)) \equiv L^2(\Omega \times T, W)$*,*

$$E \left| \int_T (f(t), dw(t))_W \right|^2 = E \int_T (R(t) f(t), f(t))_W dt. \tag{2.18}$$

Proof. First, suppose that f is a step function, that is,

$$f(t) = f_r, \qquad t_r \le t < t_{r+1}, \ r = 0, \ldots, N-1, \tag{2.19}$$

where $f_r \in L^2(\Omega, W)$ and $t_0 < t_1 < \cdots < t_N = t_f$.

Let $\{e_n\}$ be an orthomomal basis of W. Obviously,

$$w(t) = \sum_{i=1}^{\infty} (e_n, w(t))_W e_n, \tag{2.20}$$

and $w_n(t) = (e_n, w(t))$ is a Wiener process. Furthermore,

$$\begin{aligned} E \left| \int_T (f(t), dw(t)) \right|^2 &= E \sum_{i,j} \sum_{r,s} \int_{t_r}^{t_{r+1}} (f_r, e_i dw_i) \int_{t_s}^{t_{s+1}} (f_s, e_j dw_j) \\ &= E \sum_{i,j} \sum_{r,s} (f_r, e_i) [w_i(t_{r+1}) - w_i(t_r)] (f_s, e_j) [w_j(t_{s+1}) - w_j(t_j)] \end{aligned}$$

$$= E\int_T \left(R(t) \sum_i (f, e_i)e_i, \sum_j (f, e_j)e_j \right) dt$$

$$= E\int_T (R(t)f(t), f(t))_W dt, \tag{2.21}$$

So (2.18) is proved.

For the general case, (2.18) can be obtained as most lectures on stochastic integral have done, which is omitted. □

New, the proof of Theorem 2.3 can be given as follows.

Proof of Theorem 2.3. Denote $y(t) = y(t,p)$ and $y_0(t) = y(t;p_0)$, and then

$$dy + [A(t;p) + B(t;p)]y dt = f(t;p)dt + \sigma(t;p)dw(t), \quad y(t_0) = \varphi + \zeta, \tag{2.22}$$

$$dy_0 + [A(t;p_0) + B(t;p_0)]y_0 dt = f(t;p_0)dt + \sigma(t;p_0)dw(t), \quad y_0(t_0) = \varphi + \zeta. \tag{2.23}$$

Let $z(t) = y(t) - y_0(t)$, and then from the above equalities, it follows

$$\begin{aligned} dz(t) + [A(t;p) + B(t;p)]z dt = \{&f(t;p) - f(t;p_0) - [A(t;p) - A(t;p_0)]y_0 \\ &-[B(t;p) - B(t;p_0)]y_0\}dt + [\sigma(t;p) - \sigma(t;p_0)]dw(t), \\ &z(t_0) = 0. \end{aligned} \tag{2.24}$$

Setting $\Phi(z) = (1/2)\|z\|^2,\ z \in W^!(T;V,H)$ according to Lemma 2.4, it gets

$$\begin{aligned} &\frac{1}{2}\|z(t)\|_H^2 + \int_{t_0}^t \langle z(s), A(s;p)z(s)\rangle_{V,V'} ds \\ &\quad = -\int_{t_0}^t (z(s), B(s;p)z(s))_H ds \\ &\quad\quad - \int_{t_0}^t \langle z(s), [A(s;p) - A(s;p_0)]y_0(s)\rangle_{V,V'} ds \\ &\quad\quad + \int_{t_0}^t (z(s), \{f(s;p) - f(s;p_0) - [B(s;p) - B(s;p_0)]y_0(s)\})_H ds \\ &\quad\quad + \int_{t_0}^t (z(s), [\sigma(s;p) - \sigma(s;p_0)]dw(s))_H \\ &\quad\quad + \frac{1}{2}\int_{t_0}^t \operatorname{tr}\{[\sigma(s,p) - \sigma(s;p_0)]^*[\sigma(s,p) - \sigma(s;p_0)]Q(s)\}ds, \end{aligned} \tag{2.25}$$

which is

$$\begin{aligned}
&\|z(t)\|_H^2 + 2\int_{t_0}^{t} \langle z(s), A(s;p)z(s)\rangle_{V.V'} ds \\
&\quad = -2\int_{t_0}^{t} (z(s), B(s;p)z(s))_H ds + 2\int_{t_0}^{t} (z(s), f(s;p) - f(s;p_0))_H ds \\
&\quad\quad - 2\int_{t_0}^{t} \langle z(s), [A(s;p) - A(s;p_0)] y_0(s)\rangle_{V,V'} ds \\
&\quad\quad - 2\int_{t_0}^{t} (z(s); [B(s;p) - B(s;p_0)] y_0(s))_H ds \\
&\quad\quad + 2\int_{t_0}^{t} (z(s), [\sigma(s;p) - \sigma(s;p_0)] dw(s))_H \\
&\quad\quad + \int_{t_0}^{t} \operatorname{tr}\{[\sigma(s,p) - \sigma(s;p_0)]^* [\sigma(s,p) - \sigma(s;p_0)] Q(s)\} ds.
\end{aligned} \tag{2.26}$$

Taking the expectation from the above and considering Lemma 2.6, it has

$$\begin{aligned}
&E\|z(t)\|_H^2 + 2E\int_{t_0}^{t} \langle z(s), A(s;p)z(s)\rangle_{V,V'} ds \\
&\quad = -2E\int_{t_0}^{t} (z(s), B(s;p)z(s))_H ds + 2E\int_{t_0}^{t} (z(s), f(s;p) - f(s;p_0))_H ds \\
&\quad\quad - 2E\int_{t_0}^{t} \langle z(s), [A(s;p) - A(s;p_0)] y_0(s)\rangle_{V,V'} ds \\
&\quad\quad - 2E\int_{t_0}^{t} (z(s); [B(s;p) - B(s;p_0)] y_0(s))_H ds \\
&\quad\quad + \int_{t_0}^{t} \operatorname{tr}\{[\sigma(s,p) - \sigma(s;p_0)]^* [\sigma(s,p) - \sigma(s;p_0)] Q(s)\} ds.
\end{aligned} \tag{2.27}$$

From the assumptions of the spaces H and V, there exists a constant γ such that

$$\|x\|_H \le \gamma \|x\|_V, \quad \forall x \in V. \tag{2.28}$$

Furthermore, according to the assumptions of the operator $A(p)$ and using (2.28) and $2ab \leq \varepsilon a^2 + b^2/\varepsilon$, we can obtain

$$\begin{aligned}
&E\|z(t)\|_H^2 + 2\alpha E\int_{t_0}^t \|z(s)\|_V^2 ds \\
&\leq \alpha E\int_{t_0}^t \|z(s)\|_V^2 ds + \frac{4\gamma}{\alpha}\ \|B(\cdot;p)\|^2_{L^\infty(T;\mathcal{L}(V;H))} E\int_{t_0}^t \|z(s)\|_H^2 ds \\
&\quad + \frac{4\gamma}{\alpha}\ E\int_{t_0}^t \Big\{ \|f(s;p) - f(s;p_0)\|_H^2 + \|[A(s;p) - A(s;p_0)]y_0(s)\|_{V'}^2 \\
&\qquad + \|[B(s;p) - B(s;p_0)]y_0(s)\|_H^2 \Big\} ds. \\
&\quad + \|\operatorname{tr} Q\| \int_{t_0}^t \|\sigma(s,p) - \sigma(s;p_0)\|^2 ds.
\end{aligned} \tag{2.29}$$

By the assumptions of Theorem 2.3 there is a constant c such that

$$\begin{aligned}
&E\|z(t)\|_H^2 + \alpha E\int_{t_0}^t \|z(s)\|_V^2 ds \\
&\leq c\int_{t_0}^t E\|z(s)\|_H^2 ds + c\int_{t_0}^t \|\sigma(s,p) - \sigma(s;p_0)\|^2_{\mathcal{L}(W;H)} ds \\
&\quad + cE\int_{t_0}^t \Big\{ \|f(s;p) - f(s;p_0)\|_H^2 + \|[A(s;p) - A(s;p_0)]y_0(s)\|_{V'}^2 \\
&\qquad + \|[B(s;p) - B(s;p_0)]y_0(s)\|_H^2 \Big\} ds.
\end{aligned} \tag{2.30}$$

Obviously,

$$\begin{aligned}
E\|z(t)\|_H^2 &\leq c\int_{t_0}^t E\|z(s)\|_H^2 ds \\
&\quad + cE\int_{t_0}^t \Big\{ \|f(s;p) - f(s;p_0)\|_H^2 \\
&\qquad + \|[A(s;p) - A(s;p_0)]y_0(s)\|_{V'}^2 \\
&\qquad + \|[B(s;p) - B(s;p_0)]y_0(s)\|_H^2 \Big\} ds \\
&\quad + c\int_{t_0}^t \|\sigma(s,p) - \sigma(s;p_0)\|^2_{\mathcal{L}(W;H)} ds.
\end{aligned} \tag{2.31}$$

Using the Gronwall inequality (see [13] for the definition), and the assumptions of Theorem 2.3, letting $p \to p_0$, it has

$$\max_{t\in\overline{T}} E\|z(s)\|_H^2 = o(1), \tag{2.32}$$

which is (2.8). Furthermore, from (2.31), it follows

$$E\int_T \|z(s)\|_V^2 = o(1), \tag{2.33}$$

which is just (2.10).

From (2.26), it has the following estimate:

$$\begin{aligned}
&\|z(t)\|_H^2 + 2\alpha\int_{t_0}^t \|z(s)\|_V^2 ds \\
&\le \frac{\alpha}{\gamma}\int_{t_0}^t \|z(s)\|_H^2 ds \\
&\quad + \frac{4\gamma}{\alpha}\int_{t_0}^t \Big\{\|f(s;p) - f(s;p_0)\|_H^2 \\
&\qquad + \|[A(s;p) - A(s;p_0)]y_0(s)\|_{V'}^2 \\
&\qquad + \|[B(s;p) - B(s;p_0)]y_0(s)\|_H^2\Big\} ds \\
&\quad + \|B(\cdot;p)\|_{L^\infty(T;\mathcal{L}(V;H))}^2 \int_{t_0}^t \|z(s)\|_H^2 ds \\
&\quad + 2\int_{t_0}^t (z(s), [\sigma(s;p) - \sigma(s;p_0)]dw(s))_H \\
&\quad + \|\operatorname{tr} Q\|_{L^\infty(T)} \int_{t_0}^t \|\sigma(s,p) - \sigma(s;p_0)\|_{\mathcal{L}(W;H)}^2 ds.
\end{aligned} \tag{2.34}$$

Similarly, it also obtains

$$\begin{aligned}
\|z(t)\|_H^2 \le c\int_{t_0}^t \Big\{&\|f(s;p) - f(s;p_0)\|_H^2 + \|[A(p) - A(p_0)]y_0\|_{V'}^2 \\
&+ \|[B(p) - B(p_0)]y_0\|_H^2 + \|\sigma(p) - \sigma(p_0)\|_{\mathcal{L}(W;H)}^2\Big\} ds \\
&+ c\int_{t_0}^t (z(s), [\sigma(s,p) - \sigma(s;p_0)]dw(s))_H, \quad t \in \overline{T}.
\end{aligned} \tag{2.35}$$

On the other hand, the indefinite stochastic integral

$$X(t) = \int_{t_0}^{t} \left([\sigma(s;p) - \sigma(s;p)]^* z(s), dw(s) \right)_H \tag{2.36}$$

is a continuous martingale, so

$$\begin{aligned} E \max_{t\in\overline{T}} X(t) &\le E \max_{t} |X(t)| \le \left[E\left(\max_{t} |X(t)| \right)^2 \right]^{1/2} \le 2\left\{ E\left| X(t_f) \right|^2 \right\}^{1/2} \\ &= 2\left\{ E \int_T (R(t)[\sigma(s;p) - \sigma(s;p_0)]^* z(s), [\sigma(s;p) - \sigma(s;p_0)]^* z(s)) dt \right\}^{1/2} \\ &\le c\left\{ \int_T \|\sigma(t;p) - \sigma(t;p_0)\|^2 dt \right\}^{1/2}, \end{aligned} \tag{2.37}$$

where the following martingale inequality is used:

$$E\left\{ \max_{\overline{T}} |X(t)|^{\alpha} \right\} \le \left(\frac{\alpha}{\alpha - 1} \right)^2 E\left| X(t_f) \right|^{\alpha}, \tag{2.38}$$

with $\alpha = 2$.

Combining (2.35) with (2.37), it has

$$\begin{aligned} E \max_{t\in\overline{T}} \|z(t)\|_H^2 \le cE \int_{t_0}^{t_f} &\left\{ \|f(s;p) - f(s;p_0)\|_H^2 + \|[A(s;p) - A(s;p_0)] y_0(s)\|_{V'}^2 \right. \\ &\left. + \|[B(s;p) - B(s;p_0)] y_0(s)\|_H^2 \right\} ds \\ &+ c\int_T \|\sigma(s,p) - \sigma(s;p_0)\|_{\mathcal{L}(W;H)}^2 ds \\ &+ c\left\{ \int_T \|\sigma(s,p) - \sigma(s;p_0)\|_{\mathcal{L}(W;H)}^2 ds \right\}^{1/2}. \end{aligned} \tag{2.39}$$

Letting $p \to p_0$ in P, by the assumptions of Theorem 2.3, it immediately obtains

$$E \max_{t\in\overline{T}} \|z(t)\|_H^2 = o(1), \tag{2.40}$$

which is (2.9). □

Theorem 2.8. *Besides the assumptions of Theorem 2.3, suppose that the mappings*

$$\begin{aligned} &A : T \times P_{ad} \longrightarrow \mathcal{L}(V;V'), \qquad B : T \times P_{ad} \longrightarrow \mathcal{L}(V;H), \\ &f : T \times P_{ad} \longrightarrow H, \qquad \sigma : T \times P_{ad} \longrightarrow \mathcal{L}(W;H) \end{aligned} \tag{2.41}$$

are continuously Fréchet differentiable, then the solution of (1.1)

$$y : p \longrightarrow y(p) \in \mathfrak{S} \tag{2.42}$$

is continuously Fréchet differentiable and its Fréchet derivative operators at $p_0 \in P_{ad}$*,* $y'(p_0) \in \mathcal{L}(P, \mathfrak{S})$*, are determined by the following system:*

$$d\dot{y} + [A(t;p_0) + B(t;y_0)]\dot{y}dt = \Big[f'_p(t;p_0)h - A'_p(t;p_0)hy_0 - B'_p(t;p_0)hy_0\Big]dt + \sigma'_p(t;p_0)hdw(t),$$
$$\dot{y}(t_0) = 0, \tag{2.43}$$

where $\dot{y} = y'(p_0)h$*,* $h \in P$*,* $y_0 = y(t;p_0)$ *is determined by* (1.1)*, and* $f'_p(t;p_0) \in \mathcal{L}(P;H)$*,* $A'_p(p_0) \in \mathcal{L}(P;\mathcal{L}(V;V'))$*,* $B'_p(p_0) \in \mathcal{L}(P;\mathcal{L}(V;H))$*, and* $\sigma'_p(p_0) \in \mathcal{L}(P;\mathcal{L}(W;H))$ *are the Fréchet derivative operators of* $p \to f(t;p)$*,* $p \to A(p)$*,* $p \to B(p)$*, and* $p \to \sigma(p)$*, at* $p = p_0$*, respectively.*

Proof. By Lemma 2.1, there exists a unique solution to (2.43), $\dot{y} \in \mathfrak{S}$. Taking $p_0 \in P_{ad}$, for any $p \in P_{ad}$, setting $h = p - p_0$ and $z = y - y_0 - \dot{y}$, where $y = y(p)$ and $y_0 = y(p_0)$ are defined by (2.22) and (2.23), respectively, it has

$$\begin{aligned} dz + (A_0 + B_0)zdt = \{&(f - f_0 - f'_0h) - (A - A_0 - A'_0h)y \\ &-(B - B_0 - B'_0h)yA'_0h(y - y_0) - B'_0h(y - y_0)\}dt + (\sigma - \sigma_0 - \sigma'_0h)dw(t), \\ &z(t_0) = 0, \end{aligned} \tag{2.44}$$

where $A = A(p)$, $A_0 = A(p_0)$, $A'_0 = A'_p(p_0), \ldots, \sigma = \sigma(p)$, $\sigma_0 = \sigma(p_0)$, and $\sigma'_0 = \sigma'_p(p_0)$.

Letting $p \to p_0$ or $\|h\| \to 0$, according to the definition of the Fréchet differentiability, it has

$$\begin{aligned} &\|f - f_0 - f'_0h\|_H = o(\|h\|), \\ &\|A - A_0 - A'_0h\|_{\mathcal{L}(V;V')} = o(\|h\|), \\ &\|B - B_0 - B'_0h\|_{\mathcal{L}(V;H)} = o(\|h\|), \\ &\|\sigma - \sigma_0 - \sigma'_0h\|_{\mathcal{L}(w;H)} = o(\|h\|). \end{aligned} \tag{2.45}$$

Moreover, by Theorem 2.3 it gets $\|y - y_0\|_{\mathfrak{S}} = o(1)$ and $\|y\|_{\mathfrak{S}} = \|y - y_0\| + \|y_0\| \le c\|y_0\| + c$.

Using the deduction similar to Theorem 2.3 it obtains

$$\|z\|_{\mathfrak{S}} = o(\|h\|), \tag{2.46}$$

which is just $\|y - y_0 - y'_0\| = o(\|h\|)$.

So, $y(p)$ is Fréchet differentiable at p_0 and its Fréchet derivative operator y'_0 is determined by (2.43).

The continuity of $y'(p)$ can be proved in a way similar to proof of Theorem 2.3, which is omitted here. □

3. Existence and Necessary Conditions for Optimality

In this section the optimization problem (1.10) is researched. First, we prove that the cost functional $p \to J(p)$ is continuous and continuously Fréchet differentiable. Next, we prove that under certain sufficient conditions there exists an optimal parameter $\widehat{p}$, at which the cost functional reaches its minimum over the admissible parameter set P_{ad}, and derive necessary conditions for optimality, which means that the optimal parameter should satisfy some inequalities.

Theorem 3.1. *Let the assumptions of Theorem 2.3 be satisfied, $g \in L^2(\Omega; L^2(T; K)) = L^2(\Omega \times T; K)$, and let*

$$\Lambda : p \to L^\infty(T; \mathcal{L}(V; K)) \tag{3.1}$$

be continuous, then the mapping

$$J : p \to R^+ \tag{3.2}$$

is continuous.

Proof. Take $p_0, p \in P_{ad}$ and set $h = p - p_0$, $y = y(p)$, $y_0 = y(p_0)$, $\Lambda = \Lambda(p)$, and $\Lambda_0 = \Lambda(p_0)$, then it has

$$\begin{aligned} &\left|J(p) - J(p_0)\right| \\ &\quad = \left|E_g \int_T \left\{ \|\Lambda y - g\|_K^2 - \|\Lambda_0 y_0 - g\|_K^2 \right\} dt \right| \\ &\quad = \left|E_g \int_T (\Lambda y + \Lambda_0 y_0 - 2g, \Lambda(y - y_0) + (\Lambda - \Lambda_0) y_0)_K dt \right| \\ &\quad \le 2\left(E_g \int_T \|\Lambda y + \Lambda_0 y_0 - 2g\|_K^2 dt\right)^{1/2} \left(E_g \int_T \left\{ \|\Lambda(y - y_0)\|_K^2 + \|(\Lambda - \Lambda_0) y_0\|_K^2 \right\} dt\right)^{1/2}. \end{aligned} \tag{3.3}$$

Letting $p \to p_0$, that is, $\|h\| \to 0$, using Theorem 2.3 and the assumptions of Theorem 3.1, it obtains at once

$$\left|J(p) - J(p_0)\right| = o(1). \tag{3.4}$$

Hence, $J(p)$ is continuous. □

From Zeidler [14] the following lemma is quoted.

Lemma 3.2. *The minimum problem*

$$\min_{u\in M} F(u) = \alpha \tag{3.5}$$

has a solution, where $F : M \subseteq X \to (-\infty, +\infty]$, $M \neq \emptyset$, *if one of the following conditions is fulfilled:*

(a) X *is a topological space and* F *is lower semicompact;*

(b) X *is a topological space,* M *is compact, and* F *is continuous.*

Lemma 3.3. *The extreme problem*

$$\inf_{p\in P_{ad}} I(p) \equiv E\int_T \|U(t)p - g(t)\|_K^2 dt = d^2, \quad U \in L^2(\Omega\times T; \mathcal{L}(P;K)) \tag{3.6}$$

has a solution, $p^* \in P_{ad} \subset P$, *if one of the following conditions is fulfilled:*

(a) P_{ad} *is a closed subspace of* P*;*

(b) P_{ad} *is a closed and convex in* P*.*

Proof. Obviously, the condition (a) is a special case of the condition (b). So, it needs only proof that the conclusion holds in the case (b).

Assume $K_{ad} = UP_{ad}$. Obviously, K_{ad} is closed and convex.

Suppose $\{p_n\} \subset P_{ad}$ is a minimizing sequence, that is, $I(p_n) \to d^2$.

In order to prove that $\{Up_n\}$ is a Cauchy sequence, we prove the following inequality:

$$\begin{aligned} &E\int_T \|UP' - UP''\|^2 dt \\ &\quad\le \left\{\sqrt{E\int_T \|UP' - g\|^2 dt - d^2} + \sqrt{E\int_T \|UP'' - g\|^2 dt - d^2}\right\}^2, \quad \forall p', p'' \in p. \end{aligned} \tag{3.7}$$

Obviously, we have the following inequality:

$$\left\{E\int_T (UP' - g, UP'' - g)_K dt - d^2\right\}^2 \le \left\{E\int_T \|UP' - g\|_K^2 dt - d^2\right\}\left\{E\int_T \|UP'' - g\|^2 dt - d^2\right\}. \tag{3.8}$$

Therefore,

$$\begin{aligned} &E\int_T \|UP' - UP''\|^2 dt \\ &\quad= \left\{E\int_T \|UP' - g\|^2 dt - d^2\right\} - 2\left\{E\int_T (UP' - g, UP'' - g) dt - d^2\right\} \\ &\qquad+ \left\{E\int_T \|UP'' - g\|^2 dt - d^2\right\} \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ E\int_T \|UP' - g\|^2 dt - d^2 \right\} \\
&\quad + 2\sqrt{E\int_T \|UP' - g\|^2 dt - d^2}\sqrt{E\int_T \|UP'' - g\|^2 dt - d^2} \\
&\quad + \left\{ E\int_T \|UP'' - g\|^2 dt - d^2 \right\} \\
&= \left\{ \sqrt{E\int_T \|UP' - g\|^2 dt - d^2} + \sqrt{E\int_T \|UP'' - g\|^2 dt - d^2} \right\}^2,
\end{aligned} \tag{3.9}$$

which is exactly (3.7).

For any $m, n \in \mathbb{N}$, from (3.7) it has

$$E\int_T \|Up_{n+m} - Up_n\|^2 dt \leq \left\{ \sqrt{E\int_T \|Up_{n+m} - g\|^2 dt - d^2} + \sqrt{E\int_T \|Up_n - g\|^2 dt - d^2} \right\}^2. \tag{3.10}$$

Letting $n \to +\infty$, it obtains

$$Up_{n+m} - Up_n = o(1). \tag{3.11}$$

Hence, $\{Up_n\} \subset K_{ad}$ is a Cauchy sequence. Since K_{ad} is closed, there exists $y^* \in K_{ad}$ such that

$$Up_n \longrightarrow y^*, \quad \text{in } L^2\left(\Omega; L^2(T;K)\right). \tag{3.12}$$

By the definition of K_{ad}, there exists $p^* \in P_{ad}$ such that $y^* = Up^*$. Obviously, it has

$$I(p^*) = \lim_{n\to\infty} I(p_n) = d^2. \tag{3.13}$$

□

Theorem 3.4. *Let the assumptions of Theorem 3.1 be true and let one of the following be fulfilled:*

(a) *P is a finite-dimensional Euclidean space and P_{ad} is convex, bounded, and closed;*

(b) *the function $p \to y(p)$ is linear, Λ is independent of p, and P_{ad} is closed and convex;*

(c) *P_{ad} is compact in P,*

then the optimization problem (1.9) *has a solution, $p^* \in P_{ad}$, namely,*

$$J(p^*) = \min_{p\in P_{ad}} J(p). \tag{3.14}$$

Proof. Obviously, the result can be obtained by Lemma 3.2 and Theorem 3.1 when (a) or (c) is fulfilled.

If the condition (b) holds, there is an operator $Y(t) \in \mathcal{L}(P_{ad}; \mathfrak{S})$, for all $t \in T$, such that for all $p \in P_{ad}$, $Y(t)p = y(t;p)$ is the solution of (1.1). Therefore, $U = \Lambda Y \in L^2(\Omega \times T; \mathcal{L}(P;K))$ satisfies the assumptions of Lemma 3.3. So, by Lemma 3.3 the result immediately can be obtained. □

Furthermore, we also can obtain the smoothness of the mapping $J(p)$.

Theorem 3.5. *Let the assumptions of Theorem 2.8 be satisfied,* $g \in L^2(\Omega \times T; K)$, *and let*

$$\Lambda : p \longrightarrow L^\infty(T; \mathcal{L}(V;K)) \tag{3.15}$$

be Fréchet differentiable, then

$$J : p \longrightarrow \mathbb{R}^+ \tag{3.16}$$

is continuously Fréchet differentiable and the Fréchet differential of J at p along the direction $h \in P$ *is determined by the following:*

$$J'(p)h = 2E_g \int_T (\Lambda'(p)hy(t;p) + \Lambda(p)y'(t;p)h, \Lambda(p)y(t;p) - g(t))_K dt, \tag{3.17}$$

where $\Lambda'(p) \in \mathcal{L}(P; \mathcal{L}(\mathfrak{S};K))$ *and* $y'(t;p)h = \dot{y}(t)$ *is determined by* (2.43).

Proof. Take $p, \overline{p} \in P_{ad}$, set $h = \overline{p} - p$, then we have

$$\begin{aligned}
&J(\overline{p}) - J(p) - 2E_g \int_T (\Lambda'(p)hy(t;p) + \Lambda(p)y'(t;p)h, \Lambda(p)y(t;p) - g(t))_K dt \\
&\quad = E_g \int_T \Big\{ \Big(\Big(\overline{\Lambda} - \Lambda - \Lambda'_p h \Big)\overline{y} + \Lambda'_p h(\overline{y} - y), \overline{\Lambda}\overline{y} - g \Big)_K \\
&\qquad + \Big(\Lambda \Big(\overline{y} - y - y'_p h \Big), \overline{\Lambda}\overline{y} - g \Big)_K \\
&\qquad + \Big(\Lambda y - g, \Big(\overline{\Lambda} - \Lambda - \Lambda'_p h \Big)\overline{y} + \Lambda'_p h(\overline{y} - y) \Big)_k \\
&\qquad + \Big(\Lambda y - g, \Lambda \Big(\overline{y} - y - y'_p h \Big) \Big)_k \\
&\qquad + \Big(\Lambda'_p hy + \Lambda y'_p h, \overline{\Lambda}(\overline{y} - y) + \Big(\overline{\Lambda} - \Lambda \Big) y \Big)_k \Big\} dt,
\end{aligned} \tag{3.18}$$

where $\overline{y} = y(\overline{p})$ and $\overline{\Lambda} = \Lambda(\overline{p})$. Letting $\overline{p} \to p$, that is, $\|h\| \to 0$, it has

$$\left| J(\overline{p}) - J(p) - 2E_g \int_T \Big(\Lambda'_p hy + \Lambda y' h, \Lambda y - g \Big)_K dt \right| = o(\|h\|). \tag{3.19}$$

So, the functional $J(p)$ is Fréchet differentiable and $J'(p)h$ is determined by (3.17), obviously, $p \to J'(p)$ is continuous. □

It is now in the position to state necessary conditions for the optimization problem (1.9).

Theorem 3.6. *Let the assumptions of Theorem 3.5 be satisfied. If a point $p_0 \in P_{ad}$ is a relative optimal parameter for* (1.9)*, then p_0 is characterized by*

$$J'(p_0)(p-p_0) \equiv 2E_g \int_T (\Lambda'(p_0)(p-p_0)y(t;p_0) + \Lambda(p_0)y'(t;p_0)(p-p_0), \Lambda(p_0)y(t;p_0) - g(t))_K dt \geq 0, \quad p \in U \subset P_{ad}, \tag{3.20}$$

where $U \in P_{ad}$ is a neighbourhood of p_0.

Proof. Firstly, let p_0 be the relative optimal parameter, then, it has

$$J(p_0) \leq J(p) = J(p_0) + J'(p_0)(p-p_0) + o(\|p-p_0\|), \quad \forall p \in U, \tag{3.21}$$

from which it immediately obtains (3.20).

Alternatively suppose (3.20) is true. Using the Taylor formula with the Peano remainder

$$J(p) = J(p_0) + J'(p_0)(p-p_0) + o(\|p-p_0\|), \tag{3.22}$$

it at once obtains

$$J(p) \geq J(p_0), \quad p \in U. \tag{3.23}$$

□

Theorem 3.7. *Let the assumptions of Theorem 3.5 be satisfied and let the functional $J(p)$ be convex. If $p_0 \in P_{ad}$ is an extreme point, then p_0 is an optimal parameter and p_0 is characterized by the following inequality*

$$J'(p_0)(p-p_0) \geq 0, \quad \forall p \in P_{ad}. \tag{3.24}$$

In particular, if $p \to y$ is linear, (3.24) *is true.*
(The results of this theorem are obvious, whom proof is omitted here.)

Theorem 3.8. *Let the assumptions of Theorem 3.5 be satisfied and let the observation operator Λ be independent of p. Then the optimal parameter p_0 that minimizes $J(p)$ over P_{ad} is characterized by the following optimization system:*

$$dy_0 + \left[A(p_0) + B(p_0)\right] y_0 dt = f(p_0)dt + \sigma(p_0)dw_t, \quad y_0 = \varphi + \xi, \tag{3.25}$$

$$-dz + \left[A(p_0) + B(p_0)\right]^* z dt = \Lambda^*(\Lambda y - g)dt, \quad z(t_f) = 0, \tag{3.26}$$

$$\frac{1}{2}J'(p_0)(p - p_0) = E_g \int_T \left\{ \left(f_p' h - B_p' h y_0, z\right)_H - \left\langle A_p' h y_0, z\right\rangle_{v',v} \right\} dt \geq 0, \tag{3.27}$$

$p \in P_{ad}$, where $h = p - p_0$.

Proof. Firstly, using the flow of time reversed (changing t to $t_f - t$) according to Lemma 2.1, it is easy to show the problem (3.26) is well posed. By Theorems 3.5 and 3.6

$$\begin{aligned} 0 \leq \frac{1}{2}J'(p_0)(p - p_0) &= E_g \int_T \left(\Lambda y'(t; p_0)h, \Lambda y(t; p_0) - g(t)\right)_K dt \\ &= E_g \int_T \langle y'(t; p_0)h, \Lambda^*[\Lambda y - g]\rangle_{V,V'} dt, \quad p \in P_{ad}. \end{aligned} \tag{3.28}$$

Setting $\dot{y} = y'(t, p_0)h$ and using (3.26) and (2.43), it has

$$\begin{aligned} 0 \leq J'(p_0)h &= E_g \int_T \langle \dot{y}, -dz + (A + B)^* z dt\rangle_{V,V'} \\ &= E_g \int_T \langle d\dot{y} + (A + B)\dot{y} dt, z\rangle_{V',V} \\ &= E_g \int_T \left\langle \left[f_p' h - A_p' h y_0 - B_p' h y_0\right] dt + \sigma_p' h dw, z \right\rangle_{V'V} \\ &= E_g \int_T \left\{ \left(f_p' h - B_p' h y_0, z\right)_H - \left\langle A_p' h y_0, z\right\rangle_{V',V} \right\} dt. \end{aligned} \tag{3.29}$$

□

4. Applications

In this section the above results are applied to systems governed by stochastic partial differential equations. The following symbols are used:

$D \subset \mathbb{R}^n$: a bounded open set;

∂D: the boundary of D, which is smooth;

d_t: the partial differential with respect to t;

$\partial_i = (\partial/\partial x_i)(i = 1, \ldots, n)$;

$W^{m,2}(s)$: Sobolev space, its definition can be found in [15].

4.1. The System Governed by a Stochastic Parabolic Partial Differential Equations

Consider the following stochastic parabolic partial differential equation:

$$d_t u(x,t) = \sum_{i,j=1}^{n} \partial_i\left(a_{ij}(x)\partial_j u\right)dt - c(x,t)u\,dt + f(x,t)dt + \sigma(x,t)dw_t, \quad (x,t) \in D \times T,$$
$$u|_{\partial D} = 0, \tag{4.1}$$
$$u\,|_{t=t_0} = \varphi(x) + \xi(x), \quad x \in D,$$

where $a_{ij} \in W^{m,2}(D)$ with $m > n/2$, $c \in W^{m,2}(D\times T)$, $f \in L^2(\Omega\times D\times T)$, $\sigma(\cdot,t) \in \mathcal{L}(W; L^2(D))$, for all $t \in T$, with $\int_T \|\sigma(\cdot,t)\|^2_{L^2(D)}dt < \infty$, $\varphi \in L^2(D)$, $\xi \in L^2(\Omega \times D)$, $w_t = w(t)$ is a Wiener process in W.

The problem we shall deal with is to determine the unknown coefficients a_{ij} $(i,j = 1,\ldots,n)$ and c based on the measurement

$$z(x) = u\left(x,t_f\right), \quad x \in D. \tag{4.2}$$

Suppose $H = L^2(D)$ and $V = H_0^1(D)$, which is the subspace of $W^{1,2}(D)$ consisting of all elements that vanish on the boundary, then $V' = H^{-1}(D)$. Note that $H = H'$. V, H, and V' make up the evolutional tripe.

Take the unknown parameter $p = (a_{11},\ldots,a_{nn},c)$ and define the parameter space by

$$P = \prod_{i,j=1}^{n} W^{m,2}(D) \times W^{m,2}(D \times T), \tag{4.3}$$

with the norm

$$\|p\|_P^2 = \sum_{i,j=1}^{n} \left\|a_{i,j}\right\|^2_{W^{m,2}(D)} + \|c\|^2_{W^{m,2}(D\times T)}, \tag{4.4}$$

then P is a Hilbert space.

In order to make sense of the problem (4.1), it assumes the admissible parameter set P_{ad} as follows:

$$P_{ad} \equiv \left\{ p \in P; \gamma\left|y\right|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x)y_i y_j \le \delta\left|y\right|^2, \forall y = \left(y_1,\ldots,y_n\right) \in \mathbb{R}^n, \text{ a.e. } x \in D \right\}, \tag{4.5}$$

where γ and $\delta (0 < \gamma \le \delta)$ are given constants.

Obviously, P_{ad} is a convex and closed set in P. By the Sobolev imbedding theorem (see Chapter [15], 6), the imbedding

$$W^{m,2}(D) \longrightarrow C^\lambda\left(\overline{D}\right) \subset C\left(\overline{D}\right) \tag{4.6}$$

is compact, where $C^\lambda(\overline{D})$ is the Hölder space and $\lambda \in (0, m - n/2)$.

Next, define the operator

$$A \equiv -\sum_{i,j=1}^{n} \partial_i \left(a_{i,j}\partial_j \cdot\right), \qquad B(t) \equiv c(t), \tag{4.7}$$

where $c(t) \equiv c(\cdot, t)$, and then, obviously, $A(p) \in \mathcal{L}(V, V')$. Using the generalized Green formula, for all $y, z \in V$, it has

$$\begin{aligned} \left|\langle Ay, z\rangle_{V',V}\right| &= \left|\left\langle -\sum_{i,j=1}^{n} \partial_i \left(a_{i,j}\partial_j y\right), z\right\rangle_{V',V}\right| \\ &= \left|\left(\sum_{i,j=1}^{n} a_{i,j}\partial_j y, \partial_i z\right)_H\right| \\ &\le \max_{i,j} \max_{x\in\overline{D}} \left|a_{i,j}(x)\right| \left\|\nabla y\right\|_{L^2} \|\nabla z\|_{L^2} \\ &\le c_1 \left\|y\right\|_V \|z\|_V. \end{aligned} \tag{4.8}$$

Due to $p \in P_{ad}$, it has

$$\langle Ay, y\rangle_{V'V} = \left(\sum_{i,j=1}^{n} a_{i,j}\partial_j y, \partial_i y\right)_H \ge \gamma \left\|\nabla y\right\|_{L^2}^2. \tag{4.9}$$

By the Poincaré inequality it has

$$\langle Ay, y\rangle_{V',V} \ge c_2 \left\|y\right\|_V^2, \tag{4.10}$$

which, along with (4.8), shows that A satisfies (1.5).

Next, for all $z \in V$

$$\begin{aligned} \|B(t)z\|_H^2 &= \int_D |c(x,t)z(x)|^2 dx \le \|c\|_{L^\infty(D\times T)}^2 \int_D |z(x)|^2 dx \\ &\le c_0 \|z\|_H^2 \le c_1 \|z\|_V^2. \end{aligned} \tag{4.11}$$

Therefore, $B(t) \in \mathcal{L}(V, H)$, for all $t \in T$.

Finally, set $u(t) = u(\cdot, t)$, $f(t) = f(\cdot, t)$, and $\sigma(t) = \sigma(\cdot, t)$. Thus, (4.1) can be written as (3.25).

Summing up the above reasoning, it has the following theorem.

Theorem 4.1. *If $f \in L^2(\Omega \times D \times T)$, then* (4.1) *has a unique solution $u(p) = u(x, t; p)$:*
$u \in L^2(\Omega; C(T; H) \cap L^2(\Omega \times T; V) \cap C(\overline{T}; L^2(\Omega; H)) \equiv \mathfrak{S}$,
$u(\cdot, t)$ is adapted to $\mathcal{F}^t$ (as a process with values in H),
$u(\cdot, t)$ is measurable with values in V.

Theorem 4.2. *If $f \in L^2(\Omega \times D \times T)$, then for any $p \in P_{ad}$, the mapping $u(p)$ is continuously Fréchet differentiable and its Fréchet differential $u'(p)h = \dot{u}$ $(h \in P)$ is determined by the following system:*

$$
\begin{gathered}
d_t \dot{u} - \sum_{i,j} \partial_i (a_{ij} \partial_j \dot{u}) dt + c\dot{u}dt = \sum_{i,j} \partial_i (\Delta a_{i,j} \partial_j u) dt - \Delta c u dt, \\
\dot{u} \mid_{\partial\Omega} = 0, \\
\dot{u} \mid_{t=t_0} = 0,
\end{gathered} \tag{4.12}
$$

where $h = (\Delta a_{11}, \ldots, \Delta a_{nn}, \Delta c) \in P$ and u is determined by (4.1)*. Furthermore, the mapping $u(p)$ is infinitely Fréchet differentiable.*

Because $u \in C(\overline{T}; L^2(\Omega; H))$, $u(\cdot, t_f) \in L^2(\Omega; H)$. So, we can use the following cost functional:

$$
J(p) = E \int_D \left| u(x, t_f; p) - z(x) \right|^2 dx, \quad p \in P_{ad}, \tag{4.13}
$$

in order to determine p. The operator Λ

$$
\Lambda u = u(\cdot, t_f), \quad x \in D, \qquad \Lambda : C\left(\overline{T}; L^2(\Omega; H)\right) \longrightarrow L^2(\Omega; H) \tag{4.14}
$$

obviously satisfies $\Lambda \in \mathcal{L}(\mathfrak{S}; L^2(\Omega; H))$.

Theorem 4.3. *The mapping $J(p)$ defined by* (4.13) *is continuously Fréchet differentiable and its Fréchet differential is determined by*

$$
\begin{aligned}
J'(p)h &= -2E \int_D \int_T \left\{ \sum_{i,j=1}^{n} \Delta a_{i,j} \partial_i v \partial_j u + \Delta c u v \right\} dt\, dx \\
&= \langle \operatorname{grad} J, h \rangle_{P', P},
\end{aligned} \tag{4.15}
$$

where $h = (\Delta a_{11}, \ldots, \Delta a_{nn}, \Delta c) \in P$, the gradient operator is

$$
\operatorname{grad} J = -2E \int_T (\partial_1 v \partial_1 u, \partial_1 v \partial_2 u, \ldots, \partial_n v \partial_n u) dt \in P', \tag{4.16}
$$

$u = u(p)$ is the solution of (4.1)*, and $v = v(p)$ is defined by the following system:*

$$
\begin{gathered}
-d_t v = \sum_{i,j}^{n} \partial_j (a_{ij} \partial_i v) dt - c v dt, \quad (x, t) \in D \times T, \\
v \mid_{\partial D} = 0, \\
v \mid_{t=t_f} = u|_{t=t_f} - z.
\end{gathered} \tag{4.17}
$$

4.2. Example 2

Consider the following system:

$$d_t u - \Delta u dt = \sum_{i=1}^{m} a_i f_i(x,t)dt + \sigma(x,t)dw_t, \quad (x,t) \in D \times T,$$
$$u|_{\partial D} = 0, \quad t \in T, \tag{4.18}$$
$$u\,|_{t=t_0} = \varphi(x) + \zeta(x), \quad x \in D,$$

where $D \subset \mathbb{R}^n$ with $n \leq 3$.

The problem addressed is to identify the unknown parameter $p = (a_1, \ldots, a_m) \in \mathbb{R}^m$, which varies in an admissible parameter set $P_{ad} \equiv \{p \in \mathbb{R}^m; 0 \leq \gamma \leq \|p\|_{R^m} \leq \delta\}$ based on the approximate point measurement:

$$z_i(x,t) = u(x,t), \quad \text{a.e. } x \in D_i, \ \forall t \in T, \ i = 1, \ldots, L, \tag{4.19}$$

where D_i is the ε-neighbourhood of $x_i \in D$ and $D_i \cap D_j = \emptyset$ $(i \neq j)$.

In order to make sense of (4.19), it assumes

$$f_i \in L^2\left(T; W^{1,2}(D)\right), \quad i = 1, \ldots, m,$$
$$\partial_j \sigma(t) \in \mathcal{L}(W,H), \quad \int_T \left\|\partial_j \sigma(t)\right\|^2_{\mathcal{L}(W,H)} dt < \infty, \quad j = 1, \ldots, n. \tag{4.20}$$

It is easy to prove that $u(t) \in W^{2,2}(D) \cap H_0^1(D)$. Using the Sobolev embedding theorem [15], it has $W^{2,2}(D) \subset C(\overline{D})$. So (4.19) is sensible.

Introduce the characteristic function of D_i:

$$\chi_{D_i}(x) = \begin{cases} 1, & x \in D_i, \\ 0, & \text{otherwise}, \end{cases} \tag{4.21}$$

and the cost functional

$$\min_{p \in P_{ad}} J(p) \equiv \sum_{i=1}^{L} E \int_D \int_T \left[\chi_{D_i}(x)\left[u(x,t;p) - z_i(x,t)\right]\right]^2 dt dx. \tag{4.22}$$

In the manner similar to reasoning in Section 4.1 it can be proved that the function $u(p)$ is continuously Fréchet differentiable and its Fréchet differential, $\dot{u} = u'(p)h$, $h = (\Delta a_1, \ldots, \Delta a_m) \in \mathbb{R}^m$, is determined by the following:

$$\begin{gathered} d_t\dot{u} - \Delta\dot{u}dt = \sum_{i=1}^{m} \Delta a_i f_i dt, \quad (x,t) \in D \times T, \\ \dot{u}\mid_{\partial D} = 0, \\ \dot{u}\mid_{t=t_0} = 0. \end{gathered} \tag{4.23}$$

Furthermore, it can also be proved that the functional $J(p)$ is continuously Fréchet differentiable and its Fréchet differential is

$$J'(p)h = 2\sum_{i=1}^{L} E\int_T\int_D \chi_{D_i}(x)\left[u(x,t;p) - z_i(x,t)\right]\dot{u}(x,t;p)dx\,dt. \tag{4.24}$$

Now, the main results are given as followings.

Theorem 4.4. *The functional $J(p)$ is continuously Fréchet differentiable and its Fréchet differentiable is*

$$J'(p)h = (\text{grad}\,J, h)_{\mathbb{R}^m}, \tag{4.25}$$

where grad J *is the gradient of $J(p)$ determined by the following:*

$$\text{grad}\,J = \left(E\int_T\int_D v f_1 dx\,dt, \ldots, E\int_T\int_D v f_m dx\,dt\right) \in \mathbb{R}^m, \tag{4.26}$$

where v is the solution to the following initial-boundary problem:

$$\begin{gathered} -d_t v - \Delta v dt = 2\sum_{i=1}^{L} \chi_{D_i}(x)\left[u(x,t;p) - z(x,t)\right]dt, \quad (x,t) \in D \times T, \\ v\mid_{\partial\Omega} = 0, \qquad v|_{t=t_f} = 0 \end{gathered} \tag{4.27}$$

and $u(x,t;p)$ is the solution to the problem (4.18).

Proof. By the formulae (4.24), (4.23), and (4.27), and using the integration by parts, it has

$$\begin{aligned} J'(p)h &= E\int_T\int_D (-d_t v - \Delta v dt)\dot{u} dx \\ &= E\int_T\int_D v(d_t\dot{u} - \Delta\dot{u} dt)dx = E\int_T\int_D \sum_{i=1}^{m} v\Delta a_i f_i dx\, dt \\ &= \sum_{i=1}^{m}\Delta a_i E\int_T\int_D v f_i dx\, dt = (\text{grad } J, h)_{\mathbb{R}^m}. \end{aligned} \tag{4.28}$$

□

5. Conclusion

In this paper we consider solving of the parameter contained in a linear stochastic evolution equation (LSEE) by means of smooth optimization methods. We prove that the solution to the LSEE continuously depends on the parameter and is continuously Fréchet differentiable with respect to parameter. We also prove that the cost functional is continuously Fréchet differentiable with respect to parameter. Based on the above results the necessary conditions, which the optimal parameter should satisfy, are presented. Moreover, the sufficient conditions, under which there exists an optimal parameter, also are presented. Finally we apply the results to stochastic partial differential equations with a final measurement and an approximate point measurement, respectively.

It should be pointed out that we only consider the linear stochastic evolution equation (LSEE) without measurement errors. The case with measurement errors, such as filtering of diffusion processes [16], is worth investigating in the future.

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