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Research Article

On Nonlinear Neutral Fractional Integrodifferential Inclusions with Infinite Delay

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Of concern is a class of nonlinear neutral fractional integrodifferential inclusions with infinite delay in Banach spaces. A theorem about the existence of mild solutions to the fractional integrodifferential inclusions is obtained based on Martelli's fixed point theorem. An example is given to illustrate the existence result.

1. Introduction

As have been seen, the field of the application of fractional calculus is very broad. For instance, we can see it in the study of the memorial materials, earthquake analysis, robots, electric fractal network, fractional sine oscillator, electrolysis chemical, fractional capacitance theory, electrode electrolyte interface description, fractal theory, especially in the dynamic process description of porous structure, fractional controller design, vibration control of viscoelastic system and pliable structure objects, fractional biological neurons, and probability theory. The mathematical modeling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives, naturally leads to differential equations of fractional-order. The main feature of fractional order differential equation is containing the noninteger order derivative. It can effectively describe the memory and transmissibility of many natural phenomena. These differential equations have been studied by many researchers (cf., e.g., [1–11] and references therein).

As an generalization of differential equations, differential inclusions have also been investigated (cf., e.g., [1, 7, 12, 13] and references therein). Moreover, equations with delay are often more useful to describe concrete systems than those without delay. So the study of these equations has been attracted so much attention (cf., e.g., [1, 4, 8, 12, 14–21] and references therein).

In this paper, we pay our attention to the investigation of the existence of mild solutions to the following fractional integrodifferential inclusions of neutral type with infinite delay in a Banach space X:

$$D^q(x(t) - g(t, x_t)) \in A(x(t) - g(t, x_t)) + \int_0^t K(t, s)F(s, x(s), x_s)\, ds, \quad t \in [0, T],$$
$$x_0 = \phi \in \mathcal{P}, \tag{1.1}$$

where $0 < q < 1$, the fractional derivative is understood in the Caputo sense ([2], see Definition 2.3 in Section 2), $\mathcal{P}$ is an admissible phase space, $x_t : (-\infty, 0] \to X$ defined by

$$x_t(\theta) = x(t + \theta), \text{ for } \theta \in (-\infty, 0], \tag{1.2}$$

$T > 0$, $g : [0, T] \times \mathcal{P} \to X$, A generates a compact and uniformly bounded semigroup $S(\cdot)$ on X which implies that there exists $M \geq 1$ such that

$$\|S(t)\| \leq M, \quad \forall t \geq 0, \tag{1.3}$$

$K : [0, T] \times [0, T] \to \mathbf{R}$, ϕ belongs to $\mathcal{P}$ with

$$\phi(0) = 0, \tag{1.4}$$

and F is a multivalued map to be specified later.

2. Preliminaries

Throughout this paper, X is a Banach space with norm $\|\cdot\|$, $L(X)$ is the Banach space of all linear continuous operators on X, $J := [0, T]$, and $C(J, X)$ $(C([0, \infty), X))$ is the space of all X-valued continuous functions on $J([0, \infty))$.

Moreover, we abbreviate $\|u\|_{L^1(J, \mathbf{R}^+)}$ as $\|u\|_{L^1}$, for any $u \in L^1(J, \mathbf{R}^+)$.

We use the notation $\mathfrak{B}(X)$ to denote the family of all nonempty subsets of X. Let $\mathfrak{B}_{\mathrm{bd}}(X), \mathfrak{B}_{\mathrm{cl}}(X), \mathfrak{B}_{\mathrm{cp}}(X), \mathfrak{B}_{\mathrm{cv}}(X)$, and $\mathfrak{B}_{\mathrm{cp,cv}}(X)$ denote, respectively, the family of all nonempty bounded, closed, compact, convex, and compact-convex subsets of X.

See the following definition about admissible phase space according to [8, 14–21].

Definition 2.1. A linear space $\mathcal{P}$ consisting of functions from $\mathbf{R}^-$ into X with norm $\|\cdot\|_{\mathcal{P}}$ is called an admissible phase space if $\mathcal{P}$ has the following properties.

(H1) For any $t_0 \in R$ and $a > 0$, if $x : (-\infty, t_0 + ah] \to X$ is continuous on $[t_0, t_0 + a]$ and $x_{t_0} \in \mathcal{P}$, then $x_t \in \mathcal{P}$, x_t is continuous in $t \in [t_0, t_0 + a]$, and

$$\|x(t)\| \leq \overline{C}\|x_t\|_{\mathcal{P}}, \tag{2.1}$$

for a positive constant $\overline{C}$.

(H2) There exists a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{P}} \leq C_1(t - t_0) \max_{s \in [t_0, t]} \|x(s)\| + C_2(t - t_0)\|x_{t_0}\|_{\mathcal{P}} \tag{2.2}$$

for $t \in [t_0, t_0 + a]$ and x as in (H1).

(H3) The space $(\mathcal{P}, \|\cdot\|_{\mathcal{P}})$ is complete.

Remark 2.2. (H1) is equivalent to that for any $t_0 \in R$ and $a > 0$, if $x : (-\infty, t_0 + a] \to X$ is continuous on $[t_0, t_0 + a]$ and $x_{t_0} \in \mathcal{P}$, then $x_t \in \mathcal{P}$, x_t is continuous in $t \in [t_0, t_0 + a]$, and

$$\|\phi(0)\| \leq \overline{C}\|\phi\|_{\mathcal{P}}, \quad \forall \phi \in \mathcal{P} \tag{2.3}$$

for a positive constant $\overline{C}$.

Now we recall some very basic concepts in the fractional calculus theory. For more details see, for example, [2, 9, 11].

We set for $\beta \geq 0$,

$$g\{\beta\}(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} t^{\beta-1}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{2.4}$$

and $g_0(t) = 0$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.3. Let $f \in L^1(0, \infty; X)$ and $\alpha \geq 0$. Then the expression

$$I^\alpha f(t) := \big(g\{\alpha\} * f\big)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) ds, \quad t > 0,\ \alpha > 0 \tag{2.5}$$

with $I^0 f(t) = f(t)$ is called Riemann-Liouville integral of order α of f.

Definition 2.4. Let $f(t) \in C^{m-1}([0,\infty);X)$, $g\{m-\alpha\} * f \in W^{m,1}(I,X)$ $(m \in \mathbb{N},\ 0 \le m-1 < \alpha < m)$. The Caputo fractional derivative of order α of f is defined by

$$D^{\alpha} f(t) = D^{m} I^{m-\alpha}\left(f(t) - \sum_{i=0}^{m-1} f^{(i)}(0) g_{i+1}(t) \right), \tag{2.6}$$

where $D^m := d^m/dt^m$.

The following concepts are also very basic, which will be used later.

A multivalued map $G : X \to \mathfrak{B}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if

$$G(B) = \bigcup_{x \in B} G(x) \tag{2.7}$$

is bounded in X for all $B \in \mathfrak{B}_{\text{bd}}(X)$, that is,

$$\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty. \tag{2.8}$$

A multivalued map $G : J \to \mathfrak{B}_{\text{cl}}(X)$ is said to be measurable if for each $x \in X$ the function $Y : J \to \mathbf{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\} \tag{2.9}$$

is measurable.

If for each $x \in X$, the set $G(x)$ is a nonempty, closed subset of X, and for each open set B of X containing $G(x)$, there exists an open neighborhood V of x such that $G(V) \subseteq B$, then G is called upper semicontinuous (u.s.c.) on X.

If for every $B \in \mathfrak{B}_{\text{bd}}(X)$, $G(B)$ is relatively compact, then G is said to be completely continuous.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, that is,

$$x_n \longrightarrow x_*, \qquad y_n \longrightarrow y_*, \qquad y_n \in G(x_n) \text{ imply } y_* \in G(x_*). \tag{2.10}$$

We say that G has a fixed point if there is some $x \in X$ such that $x \in G(x)$.

For more details on multivalued maps we refer to the book by Deimling [22].

The following is the multivalued version of the fixed-point theorem due to Martelli [23].

Lemma 2.5. *Let X be a Banach space and let $N : X \to \mathfrak{B}_{\text{cp,cv}}(X)$ be an upper semicontinuous is bounded; then N has a fixed point and completely continuous multivalued map. If the set*

$$\Omega := \{y \in X : \lambda y \in Ny \quad \textit{for some } \lambda > 1\} \tag{2.11}$$

is bounded, then N has a fixed point.

Following Liang and Xiao [14, 15], let $\mathcal{D}^{[0,T]}$ be the set defined by

$$\mathcal{D}^{[0,T]} = \{x : (-\infty, T] \longrightarrow X : x|_J \in C(J, X), \quad x_0 \in \mathcal{D}\}. \tag{2.12}$$

Let $\|\cdot\|_T$ be the norm of $\mathcal{D}^{[0,T]}$ defined by

$$\|y\|_T = \|y_0\|_{\mathcal{D}} + \max\{\|y(s)\| : 0 \le s \le T\}, \qquad y \in \mathcal{D}^{[0,T]}. \tag{2.13}$$

Based on the work in [7, 11], we set

$$\begin{aligned} Q(t) &= \int_0^\infty \xi_q(\sigma) S(t^q \sigma) d\sigma, \\ R(t) &= q \int_0^\infty \sigma t^{q-1} \xi_q(\sigma) S(t^q \sigma) d\sigma, \end{aligned} \tag{2.14}$$

and ξ_q is a probability density function defined on $(0, \infty)$ (see [7]) such that

$$\xi_q(\sigma) = \frac{1}{q} \sigma^{-1-1/q} \varpi_q\left(\sigma^{-1/q}\right) \ge 0, \tag{2.15}$$

where

$$\varpi_q(\sigma) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \sigma^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \qquad \sigma \in (0, \infty). \tag{2.16}$$

Remark 2.6. It is not difficult to verify that for $v \in [0, 1]$,

$$\int_0^\infty \sigma^v \xi_q(\sigma) d\sigma = \int_0^\infty \sigma^{-qv} \varpi_q(\sigma) d\sigma = \frac{\Gamma(1+v)}{\Gamma(1+qv)}. \tag{2.17}$$

Then, we can see

$$\|R(t)\| \le \frac{qM}{\Gamma(1+q)} t^{q-1}, \qquad t > 0. \tag{2.18}$$

We define the mild solution to problem (1.1) as follows.

Definition 2.7. A function $x \in \mathcal{D}^{[0,T]}$ satisfying the equation

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, \ 0], \\ -Q(t) g(0, \phi) + g(t, \ x_t) + \displaystyle\int_0^t \int_0^s R(t-s) K(s, \tau) f(\tau) d\tau \, ds, & t \in J, \end{cases} \tag{2.19}$$

is called a mild solution of problem (1.1), where

$$f \in S_{F,\,x} = \left\{ f \in L^1(J, X) : f(t) \in F(t, x(t), x_t) \text{ for a.e. } t \in J \right\}. \tag{2.20}$$

Remark 2.8. (1) Since we only consider the following case:

$$\phi(0) = 0, \tag{2.21}$$

we define the mild solution to problem (1.1) in the way as mentioned before.

(2) For general $\phi(0)$, one can define the mild solution to problem (1.1) similarly and obtain the same conclusion by the similar arguments given in this paper. So we only pay attention to the essential case:

$$\phi(0) = 0. \tag{2.22}$$

3. Results and Proofs

We will require the following assumptions.

(A1) $F : J \times X \times \mathcal{D} \to \mathfrak{B}_{\mathrm{cp,\,cv}}(X)$; $(t, v, w) \to F(t, v, w)$ is measurable with respect to t for each $(v,\ w) \in X \times \mathcal{D}$; for every $t \in J$, the map $F(t, \cdot,\ \cdot) : X \times \mathcal{D} \to \mathfrak{B}_{\mathrm{cp,\,cv}}(X)$ is u.s.c., and the set

$$S_{F,\,v} = \left\{ f \in L^1(J,\ X) : f(t) \in F(t, v(t), v_t) \text{ for a.e. } t \in J \right\} \tag{3.1}$$

is nonempty.

(A2) There exist two functions $\mu_i \in L^1(J, \mathbf{R}^+)$ $(i = 1, 2)$ such that

$$\begin{aligned} \|F(t,\ v,\ w)\| &:= \sup\{\|f\| : f \in F(t,\ v,\ w)\} \\ &\leq \mu_1(t)\|v\| + \mu_2(t)\|w\|_{\mathcal{D}}, \quad (t,\ v,\ w) \in J \times X \times \mathcal{D}. \end{aligned} \tag{3.2}$$

(A3) There exist positive constants a and b such that

$$\|g(t, \tilde{\varphi})\| \leq a\|\tilde{\varphi}\|_{\mathcal{D}} + b, \quad \text{for } t \in J,\ \ \tilde{\varphi} \in \mathcal{D}. \tag{3.3}$$

(A4) For each $t \in J$, $K(t, \cdot)$ is measurable on $[0, t]$ and

$$K(t) = \operatorname{ess\,sup}\{|K(t, s)|,\ \ 0 \leq s \leq t\} \tag{3.4}$$

is bounded on J. The map $t \to K(t, *)$ is continuous from J to $L^\infty(J, \mathbf{R})$.

The following lemma will be used in the proof of our main result.

Lemma 3.1 (see [24]). *Let I be a compact real interval and let E be a Banach space. Let F be a multivalued map satisfying hypothesis (A1) and let Υ be a linear continuous mapping from $L^1(I,E) \to C(I,E)$. Then,*

$$\Upsilon \circ S_F : C(I,E) \longrightarrow \mathfrak{B}_{\mathrm{cp,cv}}(C(I,E)), \qquad x \longmapsto (\Upsilon \circ S_F)(x) = \Upsilon(S_{F,x}) \tag{3.5}$$

is a closed graph operator in $C(I,E) \times C(I,E)$.

To prove the main result, we consider the multivalued map $\mathcal{N} : \mathcal{D}^{[0,T]} \to \mathfrak{B}(\mathcal{D}^{[0,T]})$ defined by

$$\mathcal{N}(x)(t) = \left\{ \rho \in \mathcal{D}^{[0,T]} : \rho(t) = \begin{cases} \phi(t), & t \in (-\infty,\ 0], \\ -Q(t)g(0,\ \phi) + g(t,\ x_t) & \\ +\displaystyle\int_0^t \int_0^s R(t-s)K(s,\tau)f(\tau)d\tau\, ds, & t \in J, \end{cases} \right\}, \tag{3.6}$$

where $f \in S_{F,\,x}$.

It is clear that the fixed points of $\mathcal{N}$ are mild solutions to problem (1.1).

For $\phi \in \mathcal{D}$, we define the function

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty,\ 0], \\ 0, & t \in J, \end{cases} \tag{3.7}$$

then $y \in \mathcal{D}^{[0,T]}$.

Set $x(t) = u(t) + y(t)$, $t \in (-\infty,\ T]$.

It is obvious that x satisfies (2.19) if and only if u satisfies $u_0 = 0$ and for $t \in J$,

$$u(t) = -Q(t)g(0,\phi) + g(t,\ u_t + y_t) + \int_0^t \int_0^s R(t-s)K(s,\tau)f(\tau)d\tau\, ds, \tag{3.8}$$

where

$$f \in S_{F,u} = \left\{ f \in L^1(J,X) : f(t) \in F(t, u(t) + y(t), u_t + y_t) \text{ for a.e. } t \in J \right\}. \tag{3.9}$$

Let

$$\mathcal{D}_0^{[0,T]} = \left\{ u \in \mathcal{D}^{[0,T]} : u_0 = 0 \right\}. \tag{3.10}$$

For any $u \in \mathcal{D}_0^{[0,T]}$,

$$\|u\|_T = \|u_0\|_{\mathcal{D}} + \max\{\|u(s)\| : 0 \le s \le T\} = \max\{\|u(s)\| : 0 \le s \le T\}. \tag{3.11}$$

Thus $(\mathcal{D}_0^{[0,T]},\ \|\cdot\|_T)$ is a Banach space.

Set

$$B_r = \left\{u \in \mathcal{D}_0^{[0,T]} : \|u\|_T \leq r\right\}, \quad \text{for } r \geq 0. \tag{3.12}$$

For $u \in B_r$, from Definition 2.1, we have

$$\begin{aligned}\|u_t + y_t\|_{\mathcal{D}} &\leq \|u_t\|_{\mathcal{D}} + \|y_t\|_{\mathcal{D}} \\ &\leq C_1(t)\max_{0\leq\tau\leq t}\|u(\tau)\| + C_2(t)\|u_0\|_{\mathcal{D}} + C_1(t)\max_{0\leq\tau\leq t}\|y(\tau)\| + C_2(t)\|y_0\|_{\mathcal{D}} \\ &\leq C_1^* r + C_2^*\|\phi\|_{\mathcal{D}} := r',\end{aligned} \tag{3.13}$$

where

$$C_i^* = \sup_{t\in J} C_i(t) \quad (i = 1,2). \tag{3.14}$$

Define the operator

$$\widetilde{\mathcal{N}} : \mathcal{D}_0^{[0,T]} \longrightarrow \mathfrak{B}\left(\mathcal{D}_0^{[0,T]}\right) \tag{3.15}$$

by

$$\begin{aligned}\widetilde{\mathcal{N}}(u)(t) = \Bigg\{ h \in \mathcal{D}_0^{[0,T]} : h(t) &= -Q(\mathrm{t})g(0,\phi) + g(t,\ u_t + y_t) \\ &+ \int_0^t\int_0^s R(t-s)K(s,\tau)f(\tau)d\tau ds, \quad t \in J\Bigg\},\end{aligned} \tag{3.16}$$

where $f \in S_{F,\,u}$.

We can see that if $\widetilde{\mathcal{N}}$ has a fixed point in $\mathcal{D}_0^{[0,T]}$, then $\mathcal{N}$ has a fixed point in $\mathcal{D}^{[0,T]}$ which is a mild solution of problem (1.1).

Assume the following.

(A5) The function $g : J \times \mathcal{D} \to X$ is completely continuous, and for every bounded set $B \in \mathcal{D}_0^{[0,T]}$, the set $\{t \to g(t,u_t) : u \in B\}$ is equicontinuous in X.

Then we can deduce that $\widetilde{\mathcal{N}}$ has a fixed point under the assumptions (A1)–(A5). For this purpose, we will show that the multivalued operator $\widetilde{\mathcal{N}}$ is completely continuous, u.s.c. with convex values. The proof of this conclusion will be given by proving the following six propositions.

Proposition 3.2. *$\widetilde{\mathcal{N}}u$ is convex for each $u \in \mathcal{D}_0^{[0,T]}$.*

Proof. For $h_1(t),\ h_2(t) \in \widetilde{\mathcal{N}}u$, there exist $f_1,\ f_2 \in S_{F,\,u}$ such that for each $t \in J$ we have

$$h_i(t) = -Q(t)g(0,\phi) + g(t,\ u_t + y_t) + \int_0^t \int_0^s R(t-s)K(s,\tau)f_i(\tau)d\tau\, ds, \qquad i = 1,2. \tag{3.17}$$

Let $\beta \in [0,1]$. Then for each $t \in J$, we get

$$\begin{aligned}
&\beta h_1(t) + (1-\beta)h_2(t) \\
&\quad = -Q(t)g(0,\phi) + g(t,\ u_t + y_t) + \int_0^t \int_0^s R(t-s)K(s,\tau)(\beta f_1(\tau) + (1-\beta)f_2(\tau))d\tau\, ds.
\end{aligned} \tag{3.18}$$

Since F has convex values, $S_{F,\,u}$ is convex, we see that

$$\beta h_1(t) + (1-\beta)h_2(t) \in \widetilde{\mathcal{N}}u. \tag{3.19}$$

□

Proposition 3.3. *$\widetilde{\mathcal{N}}$ maps bounded sets into bounded sets in $\mathcal{D}_0^{[0,T]}$.*

Proof. Let $u \in B_r$. If $\overline{h} \in \widetilde{\mathcal{N}}u$, then there exists $f \in S_{F,\,u}$ such that

$$\overline{h}(t) = -Q(t)g(0,\phi) + g(t,\ u_t + y_t) + \int_0^t \int_0^s R(t-s)K(s,\tau)f(\tau)d\tau ds, \qquad \text{for } t \in J. \tag{3.20}$$

In view of (A3) and (3.13),

$$\|g(t,\ u_t + y_t)\| \le ar' + b. \tag{3.21}$$

Hence from (A2), (A3), and (3.13), it follows that

$$\begin{aligned}
\left\|\overline{h}(t)\right\| &\le \|-Q(t)g(0,\phi)\| + \|g(t,\ u_t + y_t)\| \\
&\quad + \frac{qMK^*}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \int_0^s \left[\mu_1(\tau)\|u(\tau) + y(\tau)\| + \mu_2(\tau)\|u_\tau + y_\tau\|_{\mathcal{D}}\right] d\tau\, ds \\
&\le M\left(a\|\phi\|_{\mathcal{D}} + b\right) + ar' + b + \frac{qMK^*}{\Gamma(1+q)}\left(r\|\mu_1\|_{L^1} + r'\|\mu_2\|_{L^1}\right) \int_0^t (t-s)^{q-1} ds \\
&\le M\left(a\|\phi\|_{\mathcal{D}} + b\right) + ar' + b + \frac{MT^qK^*}{\Gamma(1+q)}\left(r\|\mu_1\|_{L^1} + r'\|\mu_2\|_{L^1}\right) \\
&=: \omega,
\end{aligned} \tag{3.22}$$

where

$$K^* = \sup_{t \in J} K(t). \tag{3.23}$$

Therefore, for each $\overline{h} \in \widetilde{\mathcal{N}}(B_r)$, we have

$$\left\|\overline{h}\right\|_T \leq \omega. \tag{3.24}$$

□

Proposition 3.4. *$\widetilde{\mathcal{N}}$ maps bounded sets into equicontinuous sets in $\mathcal{D}_0^{[0,T]}$.*

Proof. Let $h \in \widetilde{\mathcal{N}}u$ for $u \in B_r$, and let $0 < t_2 < t_1 \leq T$. Then we have

$$\begin{aligned}
&\|h(t_1) - h(t_2)\| \\
&\quad\leq \|Q(t_1) - Q(t_2)\| \cdot \|g(0,\phi)\| + \|g(t_1,\ u_{t_1} + y_{t_1}) - g(t_2,\ u_{t_2} + y_{t_2})\| \\
&\qquad + \left\|\int_0^{t_1}\int_0^s R(t_1 - s)K(s,\tau)f(\tau)d\tau ds - \int_0^{t_2}\int_0^s R(t_2 - s)K(s,\tau)f(\tau)\ d\tau ds\right\| \\
&\quad= I_1 + I_2 + I_3.
\end{aligned} \tag{3.25}$$

It follows from the continuity of $S(t)$ in the uniform operator topology for $t > 0$ that

$$I_1 \text{ tends to } 0, \quad \text{as } t_2 \longrightarrow t_1. \tag{3.26}$$

The equicontinuity of g ensures that

$$I_2 \text{ tends to } 0, \quad \text{as } t_2 \longrightarrow t_1. \tag{3.27}$$

For I_3, we obtain

$$\begin{aligned}
I_3 &\leq K^*\int_0^{t_2}\int_0^s \|R(t_1 - s) - R(t_2 - s)\|\|f(\tau)\|d\tau\, ds + K^*\int_{t_2}^{t_1}\int_0^s \|R(t_1 - s)\|\|f(\tau)\|d\tau\, ds \\
&\leq K^*r^*\left(\int_0^{t_2}\|R(t_1 - s) - R(t_2 - s)\|ds + \frac{qM}{\Gamma(1+q)}\int_{t_2}^{t_1}(t_1 - s)^{q-1}ds\right) \\
&\leq qr^*K^*\int_0^{t_2}\int_0^{\infty}\sigma\left\|\left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right]\xi_q(\sigma)S\left((t_1 - s)^q\sigma\right)\right\|d\sigma\, ds \\
&\quad + qr^*K^*\int_0^{t_2}\int_0^{\infty}\sigma(t_2 - s)^{q-1}\xi_q(\sigma)\left\|S\left((t_1 - s)^q\sigma\right) - S\left((t_2 - s)^q\sigma\right)\right\|d\sigma\, ds \\
&\quad + \frac{qMr^*K^*}{\Gamma(1+q)}\int_{t_2}^{t_1}(t_1 - s)^{q-1}ds
\end{aligned}$$

$$\leq \frac{qMr^*K^*}{\Gamma(1+q)} \int_0^{t_2} \left|(t_1-s)^{q-1} - (t_2-s)^{q-1}\right| ds$$
$$+ qr^*K^* \int_0^{t_2} \int_0^{\infty} \sigma(t_2-s)^{q-1} \xi_q(\sigma) \left\|S\left((t_1-s)^q\sigma\right) - S\left((t_2-s)^q\sigma\right)\right\| d\sigma\, ds$$
$$+ \frac{Mr^*K^*}{\Gamma(1+q)}(t_1-t_2)^q, \tag{3.28}$$

where

$$r^* := r\|\mu_1\|_{L^1} + r'\|\mu_2\|_{L^1}. \tag{3.29}$$

Clearly, the first term and third term on the right-hand side of (3.28) tend to 0 as $t_2 \to t_1$. The second term on the right-hand side of (3.28) tends to 0 as $t_2 \to t_1$ as a consequence of the continuity of $S(t)$ in the uniform operator topology for $t > 0$.

Thus the set $\{\widetilde{\mathcal{N}}u : u \in B_r\}$ is equicontinuous. □

Proposition 3.5. *$(\widetilde{\mathcal{N}}B_r)(t)$ is relatively compact for each $t \in J$, where*

$$\left(\widetilde{\mathcal{N}}B_r\right)(t) = \left\{h(t) : h \in \widetilde{\mathcal{N}}(B_r)\right\}. \tag{3.30}$$

Proof. Fix $t \in (0,\ T]$. For arbitrary $0 < \varepsilon < t$ and arbitrary $\delta > 0$, write

$$\begin{aligned} h_{\varepsilon,\delta}(t) &= -Q(t)g(0,\phi) + g(t,\ u_t + y_t) \\ &\quad + q\int_0^{t-\varepsilon} (t-s)^{q-1} \int_\delta^\infty \sigma\xi_q(\sigma) S\left((t-s)^q\sigma\right) \int_0^s K(s,\tau) f(\tau) d\tau\, d\sigma\, ds \\ &= -Q(t)g(0,\ \phi) + g(t,\ u_t + y_t) \\ &\quad + qS(\varepsilon^q\delta) \int_0^{t-\varepsilon} (t-s)^{q-1} \int_\delta^\infty \sigma\xi_q(\sigma) S\left((t-s)^q\ \sigma - \varepsilon^q\delta\right) \int_0^s K(s,\tau) f(\tau) d\tau\, d\sigma\, ds, \end{aligned} \tag{3.31}$$

where $f \in S_{F,\ u}$. Since $S(t)$ is compact for each $t \in (0,\ T]$ and (A5), the set

$$U_{\varepsilon,\delta} = \left\{h_{\varepsilon,\delta}(t) : h \in \widetilde{\mathcal{N}}(B_r)\right\} \tag{3.32}$$

is relatively compact. Moreover,

$$\begin{aligned}\|h(t)-h_{\varepsilon,\delta}(t)\| &\leq q\int_0^{t-\varepsilon}(t-s)^{q-1}\int_0^{\delta}\sigma\xi_q(\sigma)S\big((t-s)^q\sigma\big)\int_0^s K(s,\tau)f(\tau)d\tau\, d\sigma\, ds\\ &\quad+q\int_{t-\varepsilon}^{t}(t-s)^{q-1}\int_0^{\infty}\sigma\xi_q(\sigma)S\big((t-s)^q\sigma\big)\int_0^s K(s,\tau)f(\tau)d\tau\, d\sigma\, ds\\ &\leq MK^*r^*T^q\int_0^{\delta}\sigma\xi_q(\sigma)d\sigma+\frac{Mr^*K^*\varepsilon^q}{\Gamma(1+q)},\end{aligned} \tag{3.33}$$

which implies that $(\widetilde{\mathcal{N}}B_r)(t)$ is relatively compact. □

Now, it follows from Propositions 3.3–3.5 and the Ascoli-Arzela theorem that

$$\widetilde{\mathcal{N}}:\mathcal{D}_0^{[0,T]}\longrightarrow\mathfrak{B}\left(\mathcal{D}_0^{[0,T]}\right) \tag{3.34}$$

is completely continuous.

Proposition 3.6. *$\widetilde{\mathcal{N}}$ has a closed graph.*

Proof. Suppose that

$$u_n\longrightarrow u_*,\qquad h_n\in\widetilde{\mathcal{N}}u_n\quad\text{with } h_n\longrightarrow h_*. \tag{3.35}$$

We claim that

$$h_*\in\widetilde{\mathcal{N}}u_*. \tag{3.36}$$

In fact, the assumption $h_n\in\widetilde{\mathcal{N}}u_n$ implies that there exists $f_n\in S_{F,\,u_n}$ such that

$$h_n(t)=-Q(t)g\big(0,\phi\big)+g\big(t,\ u_{nt}+y_t\big)+\int_0^t\int_0^s R(t-s)K(s,\tau)f_n(\tau)d\tau\, ds,\qquad t\in J. \tag{3.37}$$

We will show that there exists $f_*\in S_{F,\,u_*}$ such that

$$h_*(t)=-Q(t)g\big(0,\ \phi\big)+g\big(t,\ u_{*t}+y_t\big)+\int_0^t\int_0^s R(t-s)K(s,\tau)f_*(\tau)d\tau\, ds,\qquad t\in J. \tag{3.38}$$

Obviously, as $n\ \to\ \infty$, we have

$$\left\|\big(h_n(t)+Q(t)g\big(0,\ \phi\big)-g\big(t,\ u_{nt}+y_t\big)\big)-\big(h_*(t)+Q(t)g\big(0,\ \phi\big)-g\big(t,\ u_{*t}+y_t\big)\big)\right\|\longrightarrow 0. \tag{3.39}$$

Consider the following linear continuous operator:

$$\begin{aligned} &\Upsilon : L^1(J,\ X) \longrightarrow C(J,\ X), \\ &f \longmapsto \Upsilon(f)(t) = \int_0^t \int_0^s R(t-s)K(s,\ \tau)f(\tau)d\tau\, ds. \end{aligned} \tag{3.40}$$

By virtue of Lemma 3.1, we know that $\Upsilon \circ S_F$ is a closed graph operator. Moreover, we get

$$h_n(t) + Q(t)g(0,\ \phi) - g(t,\ u_{nt} + y_t) \in \Upsilon(S_{F,\ u_n}). \tag{3.41}$$

Since $u_n \to u_*$ and $h_n \to h_*$, it follows from Lemma 3.1 that

$$h_*(t) + Q(t)g(0,\ \phi) - g(t,\ u_{*t} + y_t) = \int_0^t \int_0^s R(t-s)K(s,\ \tau)f_*(\tau)d\tau\, ds, \tag{3.42}$$

for some $f_* \in S_{F,\ u_*}$. □

Now, we can conclude that $\widetilde{\mathcal{N}}$ is a completely continuous multivalued map, u.s.c. with convex values. Next, we give the existence result of problem (1.1).

Theorem 3.7. *Assume that (A1)–(A5) are satisfied; then there exists a mild solution of* (1.1) *on* $(-\infty,\ T]$ *provided that* $aC_1^* < 1$.

Proof. Define

$$\Omega := \left\{ u \in \mathcal{D}_0^{[0,T]} : \lambda u \in \widetilde{\mathcal{N}}u,\ \text{for some } \lambda > 1 \right\}. \tag{3.43}$$

Then, according to the previous propositions and discussions, we see that we only need to prove that the set Ω is bounded.

Take $u \in \Omega$. Then there exists $f \in S_{F,\ u}$ such that

$$u(t) = \lambda^{-1}\left(-Q(t)g(0,\ \phi) + g(t,\ u_t + y_t) + \int_0^t \int_0^s R(t-s)K(s,\ \tau)f(\tau)d\tau\, ds \right). \tag{3.44}$$

It follows from Definition 2.1 and (A2) that

$$\begin{aligned} \|u(t)\| &< M\left(a\|\phi\|_{\mathcal{D}} + b\right) + a\left(C_1^* \max_{0\le\tau\le t}\|u(\tau)\| + C_2^*\|\phi\|_{\mathcal{D}} \right) + b \\ &\quad + \frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \int_0^s |K(s,\ \tau)| \|f(\tau)\| d\tau\, ds \\ &\le M_1 + aC_1^* \max_{0\le\tau\le t}\|u(\tau)\| \\ &\quad + \frac{qMK^*}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \int_0^s \left[\mu_1(\tau)\|u(\tau) + y(\tau)\| + \mu_2(\tau)\|u_\tau + y_\tau\|_{\mathcal{D}} \right] d\tau\, ds \\ &\le M_1 + aC_1^* \max_{0\le\tau\le t}\|u(\tau)\| \end{aligned}$$

$$
\begin{aligned}
&+\frac{qMK^*}{\Gamma(1+q)}\left(\int_0^t (t-s)^{q-1}\int_0^s \mu_1(\tau)\|u(\tau)\|\,d\tau\,ds\right.\\
&\qquad +\int_0^t (t-s)^{q-1}\int_0^s \mu_2(\tau)\|u_\tau\|_{\mathcal{P}}d\tau\,ds\\
&\qquad \left.+C_2^*\int_0^t (t-s)^{q-1}\int_0^s \mu_2(\tau)\|\phi\|_{\mathcal{P}}d\tau\,ds\right)\\
&\le \theta_1 + aC_1^*\max_{0\le\tau\le t}\|u(\tau)\| + \frac{qMK^*\|\mu_1\|_{L^1}}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\max_{0\le\tau\le s}\|u(\tau)\|ds\\
&\quad +\frac{qMK^*\|\mu_2\|_{L^1}C_1^*}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\max_{0\le\tau\le s}\|u(\tau)\|ds\\
&= \theta_1 + aC_1^*\max_{0\le\tau\le t}\|u(\tau)\| + \theta_2\int_0^t (t-s)^{q-1}\max_{0\le\tau\le s}\|u(\tau)\|ds,
\end{aligned}
\tag{3.45}
$$

where

$$
\begin{aligned}
M_1 &:= M\left(a\|\phi\|_{\mathcal{P}}+b\right) + aC_2^*\|\phi\|_{\mathcal{P}} + b,\\
\theta_1 &:= M_1 + \frac{MK^*}{\Gamma(1+q)}\|\mu_2\|_{L^1}C_2^*T^q\|\phi\|_{\mathcal{P}},\\
\theta_2 &:= \frac{qMK^*(\|\mu_1\|_{L^1}+C_1^*\|\mu_2\|_{L^1})}{\Gamma(1+q)}.
\end{aligned}
\tag{3.46}
$$

Denote

$$
\kappa(t) := \max_{0\le s\le t}\|u(s)\|,
\tag{3.47}
$$

and let $\widetilde{t}\in[0,\ t]$ such that $\kappa(t)=\|u(\widetilde{t})\|$. Then, by (3.45), we get

$$
\kappa(t) \le \theta_1 + aC_1^*\kappa(t) + \theta_2\int_0^t (t-s)^{q-1}\,\kappa(s)ds.
\tag{3.48}
$$

Furthermore,

$$
\kappa(t) \le \frac{\theta_1}{1-aC_1^*} + \frac{\theta_2}{1-aC_1^*}\int_0^t (t-s)^{q-1}\,\kappa(s)ds.
\tag{3.49}
$$

It is known from [25, Lemma 7.1.1] that, for any continuous functions $v, w : J \to [0,+\infty)$, if $w(\cdot)$ is nondecreasing and there are constants $\overline{a}>0$ and $0<\overline{\alpha}<1$ such that

$$
v(t) \le w(t) + \overline{a}\int_0^t (t-s)^{-\overline{\alpha}}v(s)ds,
\tag{3.50}
$$

then there exists a constant $k = k(\overline{\alpha})$ such that

$$v(t) \leq w(t) + k\overline{a}\int_0^t (t-s)^{-\overline{\alpha}} w(s)ds, \quad \text{for each } t \in J. \tag{3.51}$$

By virtue of this general fact and (3.49), we see that there exists a constant $\widetilde{k} = \widetilde{k}(q)$ such that

$$\begin{aligned} \kappa(t) &\leq \frac{\theta_1}{1-aC_1^*} + \frac{\widetilde{k}\theta_2}{1-aC_1^*}\int_0^t (t-s)^{q-1}\frac{\theta_1}{1-aC_1^*}ds \\ &\leq \frac{\theta_1}{1-aC_1^*}\left[1 + \frac{\widetilde{k}\theta_2 T^q}{q\left(1-aC_1^*\right)}\right] \\ &=: \zeta. \end{aligned} \tag{3.52}$$

Therefore $\|u\|_T \leq \zeta$. This means that the set Ω is bounded.

Thus, it follows from Lemma 2.5 that $\widetilde{\mathcal{N}}$ has a fixed point in $\mathcal{D}_0^{[0,T]}$. Then $\mathcal{N}$ has a fixed point which gives rise to a mild solution to problem (1.1). □

Example 3.8. Set $X = L^2([0,\ \pi],\ \mathbf{R})$ and define A by

$$\begin{gathered} D(A) = H^2(0,\pi) \cap H_0^1(0,\pi), \\ Au = u''. \end{gathered} \tag{3.53}$$

Then A generates a compact, analytic semigroup $S(\cdot)$ of uniformly bounded linear operators, and $\|S(t)\| \leq 1$ (see [26] for more related information).

Consider the following Cauchy problem for a fractional integrodifferential conclusion:

$$\begin{gathered} \frac{\partial^q}{\partial t^q}\left(v(t,\ \xi) - \textstyle\int_{-\infty}^t \gamma(s-t)v(s,\ \xi)ds\right) \in \frac{\partial^2}{\partial \xi^2}\left(v(t,\ \xi) - \textstyle\int_{-\infty}^t \gamma(s-t)v(s,\ \xi)ds\right) \\ + \textstyle\int_0^t (t-s)\int_{-\infty}^s \eta(s,\tau-s,\ \xi)H(s,\ v(\tau,\ \xi))d\tau\, ds, \quad t \in [0,1], \\ v(t,\ 0) - \textstyle\int_{-\infty}^t \gamma(s-t)v(s,\ 0)ds = 0, \\ v(t,\ \pi) - \textstyle\int_{-\infty}^t \gamma(s-t)v(s,\ \pi)ds = 0, \\ v(\theta,\ \xi) = v_0(\theta,\ \xi), \quad -\infty < \theta \leq 0, \end{gathered} \tag{3.54}$$

where $0 < q < 1$, $\xi \in [0,\ \pi]$, $v_0 : (-\infty,0] \times [0,\pi] \to \mathbf{R}$ is a continuous function and $H : [0,1] \times \mathbf{R} \to \mathfrak{B}(\mathbf{R})$ is a u.s.c. multivalued map with compact convex values.

Let $\varpi < 0$, define the space

$$\mathcal{D} = \left\{\varphi \in C((-\infty,0],X) : \lim_{\theta \to -\infty} e^{\varpi\theta}\varphi(\theta) \text{ exists in } X\right\} \tag{3.55}$$

endowed with the norm

$$\|\varphi\|_{\mathcal{P}} = \sup_{-\infty<\theta\leq 0}\left\{e^{\varpi\theta}\|\varphi(\theta)\|_{L^2}\right\}. \tag{3.56}$$

Clearly, we can see that $\mathcal{P}$ is an admissible phase space which satisfies (H1)–(H3) with

$$C_1(t) = \max\left\{1, e^{-\varpi t}\right\}, \qquad C_2(t) = e^{-\varpi t}. \tag{3.57}$$

For $t \in [0,\ 1]$, $\xi \in [0,\ \pi]$, and $\varphi \in \mathcal{P}$, let

$$\begin{aligned}
&x(t)(\xi) = v(t,\ \xi),\\
&\phi(\theta)(\xi) = v_0(\theta,\ \xi), \qquad \theta \in (-\infty,\ 0],\\
&g(t,\varphi)(\xi) = \int_{-\infty}^{0} \gamma(\theta)\varphi(\theta)(\xi)d\theta,\\
&K(t,\ s) = t - s,\\
&F(t,\ x(t),\ \varphi)(\xi) = \int_{-\infty}^{0} \eta(t,\ \theta,\ \xi)H(t,\ \varphi(\theta)(\xi))d\theta.
\end{aligned} \tag{3.58}$$

Then problem (3.54) can be written in the abstract form (1.1).

Furthermore, we assume the following.

(1) The function $\gamma : (-\infty,\ 0] \to \mathbf{R}$ is continuous and

$$M_2 := \left(-\frac{1}{2\varpi}\int_{-\infty}^{0} \gamma^2(\theta)d\theta\right)^{1/2} < \infty. \tag{3.59}$$

(2) There exists a continuous function $v_1(t)$ such that

$$|H(t,\varphi)| \leq v_1(t)\|\varphi(\theta)\|_{L^2}. \tag{3.60}$$

(3) The function $\eta(t,\theta,\ \xi) \geq 0$ is continuous in $[0,\ 1] \times (-\infty, 0] \times [0,\ \pi]$ and

$$\int_{-\infty}^{0} \eta(t,\theta,\ \xi)\ e^{-\varpi\theta}d\theta = p(t,\ \xi) < \infty. \tag{3.61}$$

Then, we can obtain

$$
\begin{aligned}
\|F(t,\ x(t),\ \varphi)\|_{L^2} &= \left(\int_0^{\pi}\left|\int_{-\infty}^{0}\eta(t,\ \theta,\ \xi)H(t,\ \varphi(\theta)(\xi))d\theta\right|^2 d\xi\right)^{1/2}\\
&\le \left(\int_0^{\pi}\left(\int_{-\infty}^{0}\eta(t,\ \theta,\ \xi)v_2(t)\|\varphi(\theta)\|_{L^2}d\theta\right)^2 d\xi\right)^{1/2}\\
&= \left(\int_0^{\pi}\left(\int_{-\infty}^{0}\eta(t,\ \theta,\ \xi)v_2(t)\ e^{-\varpi\theta}e^{\varpi\theta}\|\varphi(\theta)\|_{L^2}d\theta\right)^2 d\xi\right)^{1/2}\\
&\le \left(\int_0^{\pi}\left(\int_{-\infty}^{0}\eta(t,\ \theta,\ \xi)e^{-\varpi\theta}d\theta\right)^2 d\xi\right)^{1/2}\cdot v_2(t)\cdot\|\varphi\|_{\mathcal{P}}\\
&\le \left(\int_0^{\pi}p^2(t,\ \xi)d\xi\right)^{1/2}\cdot v_2(t)\cdot\|\varphi\|_{\mathcal{P}}\\
&= p(t)v_2(t)\|\varphi\|_{\mathcal{P}},
\end{aligned}
\tag{3.62}
$$

where $p(t) = \|p(t,\ \cdot)\|_{L^2}$.

Moreover,

$$
\begin{aligned}
\|g(t,\varphi)\|_{L^2} &= \left(\int_0^{\pi}\left(\int_{-\infty}^{0}\gamma(\theta)\varphi(\theta)(\xi)d\theta\right)^2 d\xi\right)^{1/2}\\
&\le \left(\int_0^{\pi}\left(\int_{-\infty}^{0}\gamma^2(\theta)d\theta\right)\cdot\left(\int_{-\infty}^{0}\varphi^2(\theta)(\xi)d\theta\right)d\xi\right)^{1/2}\\
&= \left(\int_{-\infty}^{0}\gamma^2(\theta)d\theta\right)^{1/2}\left(\int_0^{\pi}\int_{-\infty}^{0}\varphi^2(\theta)(\xi)d\theta\,d\xi\right)^{1/2}\\
&= \left(\int_{-\infty}^{0}\gamma^2(\theta)d\theta\right)^{1/2}\left(\int_{-\infty}^{0}\|\varphi(\theta)\|_{L^2}^2d\theta\right)^{1/2}\\
&= \left(\int_{-\infty}^{0}\gamma^2(\theta)d\theta\right)^{1/2}\left(\int_{-\infty}^{0}e^{-2\varpi\theta}e^{2\varpi\theta}\|\varphi(\theta)\|_{L^2}^2d\theta\right)^{1/2}\\
&\le \left(\int_{-\infty}^{0}\gamma^2(\theta)d\theta\right)^{1/2}\left(\int_{-\infty}^{0}e^{-2\varpi\theta}\left[\sup_{-\infty<\theta\le 0}e^{\varpi\theta}\|\varphi(\theta)\|_{L^2}\right]^2 d\theta\right)^{1/2}\\
&= \left(\int_{-\infty}^{0}\gamma^2(\theta)d\theta\right)^{1/2}\left(\int_{-\infty}^{0}e^{-2\varpi\theta}d\theta\right)^{1/2}\|\varphi\|_{\mathcal{P}}\\
&= M_2\|\varphi\|_{\mathcal{P}}.
\end{aligned}
\tag{3.63}
$$

Therefore, by virtue of Theorem 3.7, problem (3.54) has a mild solution when $e^{-\varpi}M_2 < 1$.

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