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Research Article

Approximation of Common Fixed Points of a Sequence of Nearly Nonexpansive Mappings and Solutions of Variational Inequality Problems

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We introduce an explicit iterative scheme for computing a common fixed point of a sequence of nearly nonexpansive mappings defined on a closed convex subset of a real Hilbert space which is also a solution of a variational inequality problem. We prove a strong convergence theorem for a sequence generated by the considered iterative scheme under suitable conditions. Our strong convergence theorem extends and improves several corresponding results in the context of nearly nonexpansive mappings.

1. Introduction

Let C be a nonempty subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. A mapping $T : C \to H$ is called the following:

(1) *monotone* if

$$\langle Tx - Ty, x - y \rangle \geq 0 \quad \forall x, y \in C, \tag{1.1}$$

(2) η-*strongly monotone* if there exists a positive real number η such that

$$\langle Tx - Ty, x - y \rangle \geq \eta \|x - y\|^2 \quad \forall x, y \in C, \tag{1.2}$$

(3) k-*Lipschitzian* if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k \|x - y\| \quad \forall x, y \in C, \tag{1.3}$$

(4) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C, \tag{1.4}$$

(5) *nearly nonexpansive* [1, 2] with respect to a fixed sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$ if

$$\|T^n x - T^n y\| \leq \|x - y\| + a_n \quad \forall x, y \in C \text{ and } n \in \mathbb{N}. \tag{1.5}$$

In [3], Moudafi proposed viscosity approximation methods of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces (see [4] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_1 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n \quad \forall n \in \mathbb{N}, \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It is proved in [4] that under appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.6) strongly converges to the unique solution $x^* \in C$ of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1.7}$$

where $C = F(T)$, the set of fixed points of T.

In 2006, Marino and Xu [5] introduced the viscosity iterative method for nonexpansive mappings. Starting with an arbitrary initial $x_1 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n \quad \forall n \in \mathbb{N}. \tag{1.8}$$

They proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1.9}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.10}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$), and A is a strongly positive bounded linear operator on H; that is, there is a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2 \quad \forall x \in H. \tag{1.11}$$

The applications of the iterative method (1.8) have been studied by some researchers (see [6, 7]).

Also, Wang [8, 9] and Wang and Hu [10] introduced the iterative method for nonexpansive mappings.

Recently, Tian [11] proposed an implicit and an explicit schemes on combining the iterative methods of Marino and Xu [5] and Yamada [12]. He also proved the strong convergence of these two schemes to a fixed point of a nonexpansive mapping T defined on a real Hilbert space under suitable conditions.

More recently, Ceng et al. [13] introduced an implicit and an explicit schemes using the properties of projection for finding the fixed points of a nonexpansive mapping defined on the closed convex subset of a real Hilbert space. They also proved the strong convergence of the sequences generated by the proposed schemes to a fixed point of a nonexpansive mapping which is also a solution of a variational inequality defined on the set of fixed points.

Aoyama et al. [14] proved strong convergence of an iterative scheme for a sequence of nonexpansive mappings as follows.

Theorem 1.1. *Let X be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and C be a nonempty closed convex subset of X. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let T be a mapping from C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{x_n\}$ be a sequence in C generated by the following iterative process:*

$$\begin{gathered} x_1 = x \in C, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) T_n x_n \quad \forall n \in \mathbb{N}, \end{gathered} \tag{1.12}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) *either* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\alpha_n \in (0,1]$ *and* $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$;

(c) $\sum_{n=1}^{\infty} \sup\{\|T_n z - T_{n+1} z\| : z \in B\} < \infty$ *for any bounded subset B of C.*

Then, the sequence $\{x_n\}$ converges strongly to Qx, where Q is the sunny nonexpansive retraction from X onto $F(T)$.

Let C be a nonempty subset of a real Hilbert space H. Let $\mathcal{T} := \{T_n\}$ be a sequence of mappings from C into itself. We denote by $F(\mathcal{T})$ the set of common fixed points of the sequence $\mathcal{T}$, that is, $F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$. Fix a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$, and let $\{T_n\}$ be a sequence of mappings from C into H. Then, the sequence $\{T_n\}$ is called a *sequence of nearly nonexpansive mappings* [15] with respect to a sequence $\{a_n\}$ if

$$\|T_n x - T_n y\| \leq \|x - y\| + a_n \quad \forall x, y \in C, n \in \mathbb{N}. \tag{1.13}$$

It is obvious that the sequence of nearly nonexpansive mappings is a wider class of sequence of nonexpansive mappings.

In this paper, inspired by Aoyama et al. [14], Ceng et al. [13], and Sahu et al. [15], we introduce an explicit iterative scheme and prove a strong convergence theorem for computing an element of $F(\mathcal{T})$, the set of common fixed points of a sequence $\mathcal{T} = \{T_n\}$ of nearly nonexpansive mappings which is also a solution of a variational inequality over $F(\mathfrak{T})$. Our result generalizes and improves the results of Ceng et al. [13], Tian [11], and many other related works.

2. Preliminaries

Throughout this paper, we denote by I the identity operator of H. Also, we denote by $\rightarrow$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. The symbol $\mathbb{N}$ stands for the set of all natural numbers.

Let C be a nonempty closed convex subset of a real Hilbert space H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \forall y \in C. \tag{2.1}$$

The mapping P_C is called the *metric projection* from H onto C (see [1]).

Let C be a nonempty subset of a real Hilbert space H and $T_1, T_2 : C \rightarrow H$ be two mappings. We denote $\mathcal{B}(C)$, the collection of all bounded subsets of C. The deviation between T_1 and T_2 on $B \in \mathcal{B}(C)$, denoted by $\mathfrak{D}_B(T_1, T_2)$, is defined by

$$\mathfrak{D}_B(T_1, T_2) = \sup\{\|T_1x - T_2x\| : x \in B\}. \tag{2.2}$$

The following lemmas will be needed to prove our main result.

Lemma 2.1 (see [16]). *The metric projection mapping P_C is characterized by the following properties:*

(a) $P_C(x) \in C$ *for all* $x \in H$*;*

(b) $\langle x - P_C(x), P_C(x) - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$*;*

(c) $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$ *for all* $x \in H$ *and* $y \in C$*;*

(d) $\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2$ *for all* $x, y \in H$*.*

Lemma 2.2 (see [13]). *Let $V : C \rightarrow H$ be an L-Lipschitzian mapping and $F : C \rightarrow H$ be a k-Lipschitzian and η-strongly monotone operator. Then, for $0 \leq \gamma L < \mu\eta$,*

$$\langle x - y, (\mu F - \gamma V)x - (\mu F - \gamma V)y\rangle \geq (\mu\eta - \gamma L)\|x - y\|^2 \quad \forall x, y \in C. \tag{2.3}$$

That is, $\mu F - \gamma V$ is strongly monotone with coefficient $\mu\eta - \gamma L$.

Lemma 2.3 (see [12]). *Let C be a nonempty subset of a real Hilbert space H. Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \rightarrow H$ be a k-Lipschitzian and η-strongly monotone operator on C. Define the mapping $G : C \rightarrow H$ by*

$$Gx = x - \lambda\mu Fx \quad \forall x \in C. \tag{2.4}$$

Then G is a contraction that provided $\mu < 2\eta/k^2$. More precisely, for $\mu \in (0, 2\eta/k^2)$,

$$\|Gx - Gy\| \leq (1 - \lambda\tau)\|x - y\| \quad \forall x, y \in C, \tag{2.5}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0, 1]$.

Lemma 2.4 (see [1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero; that is, if $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ converges strongly to 0, then $x \in F(T)$.*

Lemma 2.5 (see [17]). *Assume that $\{t_n\}$ is a sequence of nonnegative real numbers such that*

$$t_{n+1} \leq (1 - \alpha_n)t_n + \alpha_n \beta_n \quad \forall n \in \mathbb{N}, \tag{2.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of nonnegative real numbers which satisfy the following conditions:

(a) $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup_{n\to\infty} \beta_n \leq 0$, *or*

(b′) $\sum_{n=1}^{\infty} \alpha_n \beta_n$ *is convergent.*

Then $\lim_{n\to\infty} t_n = 0$.

Lemma 2.6 (see [18]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $\lambda_i > 0$. $(i = 1, 2, 3, \ldots, N)$ such that $\sum_{i=1}^{N} \lambda_i = 1$. Let $T_1, T_2, T_3, \ldots, T_N : C \to C$ be nonexpansive mappings such that $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $T = \sum_{i=1}^{N} \lambda_i T_i$. Then T is nonexpansive from C into itself and $F(T) = \bigcap_{i=1}^{N} F(T_i)$.*

3. Main Result

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping. Let $\mathfrak{T} = \{T_n\}$ be a sequence of nearly nonexpansive mappings from C into itself with respect to a sequence $\{a_n\}$ such that $F(\mathfrak{T}) \neq \emptyset$ and T be a mapping from C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Suppose that $F(T) = F(\mathfrak{T})$, $0 < \mu < 2\eta/k^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. For an arbitrary $x_1 \in C$, consider the sequence $\{x_n\}$ in C generated by the following iterative process:*

$$\begin{gathered} x_1 \in C, \\ x_{n+1} = P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n\right] \quad \forall n \in \mathbb{N}, \end{gathered} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) *either* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$;

(c) *either* $\sum_{n=1}^{\infty} \mathfrak{D}_B(T_n, T_{n+1}) < \infty$ *or* $\lim_{n\to\infty} \mathfrak{D}_B(T_n, T_{n+1})/\alpha_{n+1} = 0$ *for each* $B \in \mathcal{B}(C)$;

(d) $\lim_{n\to\infty} a_n/\alpha_n = 0$.

Then, the sequence $\{x_n\}$ converges strongly to $\tilde{x} \in F(\mathfrak{T})$, where $\tilde{x}$ is the unique solution of variational inequality

$$\langle (\mu F - \gamma V)\tilde{x}, \tilde{x} - y \rangle \leq 0 \quad \forall y \in F(\mathfrak{T}). \tag{3.2}$$

Proof. Let T be a mapping from C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. It is clear that T is a nonexpansive mapping. So, we have $F(T) \neq \emptyset$. Now, we proceed with the following steps.

Step 1. ($\{x_n\}$ is bounded). Let $z \in F(\mathcal{T})$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \left\| P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n\right] - P_C(z) \right\| \\
&\leq \left\| \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n - z \right\| \\
&\leq \left\| \alpha_n (\gamma V x_n - \mu F z) + (I - \alpha_n \mu F) T_n x_n - (I - \alpha_n \mu F) T_n z \right\| \\
&\leq \alpha_n \gamma L \|x_n - z\| + \alpha_n \left\| (\gamma V - \mu F) z \right\| + (1 - \alpha_n \tau)(\|x_n - z\| + a_n) \\
&\leq (1 - \alpha_n(\tau - \gamma L)) \|x_n - z\| + \alpha_n \left\| (\gamma V - \mu F) z \right\| + (1 - \alpha_n \tau) a_n \\
&\leq (1 - \alpha_n(\tau - \gamma L)) \|x_n - z\| + \alpha_n \left\| (\gamma V - \mu F) z \right\| + a_n.
\end{aligned} \tag{3.3}$$

Note that $\lim_{n\to\infty} a_n/\alpha_n = 0$, so there exists a constant $K > 0$ such that

$$\frac{\alpha_n \left\| (\gamma V - \mu F) z \right\| + a_n}{\alpha_n} \leq K \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Thus, we have

$$\begin{aligned}
\|x_{n+1} - z\| &\leq (1 - \alpha_n(\tau - \gamma L)) \|x_n - z\| + \alpha_n K \\
&\leq \max\left\{ \|x_n - z\|, \frac{K}{\tau - \gamma L} \right\} \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{3.5}$$

Hence, $\{x_n\}$ is bounded. So $\{T_n x_n\}$ and $\{V x_n\}$ are bounded.

Step 2. ($\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$). From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \left\| P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n\right] - P_C\left[\alpha_{n-1} \gamma V x_{n-1} + (I - \alpha_{n-1} \mu F) T_{n-1} x_{n-1}\right] \right\| \\
&\leq \left\| \left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n\right] - \left[\alpha_{n-1} \gamma V x_{n-1} + (I - \alpha_{n-1} \mu F) T_{n-1} x_{n-1}\right] \right\| \\
&\leq \left\| \alpha_n \gamma (V x_n - V x_{n-1}) + \gamma(\alpha_n - \alpha_{n-1}) V x_{n-1} \right. \\
&\quad + (I - \alpha_n \mu F) T_n x_n - (I - \alpha_n \mu F) T_n x_{n-1} \\
&\quad \left. + T_n x_{n-1} - T_{n-1} x_{n-1} + \alpha_{n-1} \mu F T_{n-1} x_{n-1} - \alpha_n \mu F T_n x_{n-1} \right\| \\
&\leq \alpha_n \gamma L \|x_n - x_{n-1}\| + \left\| \gamma(\alpha_n - \alpha_{n-1}) V x_{n-1} \right\| \\
&\quad + (1 - \alpha_n \tau) \|T_n x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\| \\
&\quad + \mu \|\alpha_{n-1} F T_{n-1} x_{n-1} - \alpha_n F T_n x_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\alpha_n(\tau-\gamma L)\right)\|x_n-x_{n-1}\| + \mathfrak{D}_B(T_n,T_{n-1}) + (1-\alpha_n\tau)a_n \\
&\quad + \left\|\gamma(\alpha_n-\alpha_{n-1})Vx_{n-1}\right\| \\
&\quad + \mu\|\alpha_{n-1}(FT_{n-1}x_{n-1}-FT_nx_{n-1}) - (\alpha_n-\alpha_{n-1})(FT_nx_{n-1})\| \\
&\leq \left(1-\alpha_n(\tau-\gamma L)\right)\|x_n-x_{n-1}\| + \mathfrak{D}_B(T_n,T_{n-1})\left(1+\mu\alpha_{n-1}k\right) \\
&\quad + M|\alpha_n-\alpha_{n-1}| + a_n,
\end{aligned}
\tag{3.6}
$$

for some constant $M > 0$. Thus, using Lemma 2.5, we get $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$.

Step 3. We have ($\|x_n - Tx_n\| \to 0$ as $n \to \infty$). Note that

$$
\begin{aligned}
\|x_n-T_nx_n\| &\leq \|x_n-x_{n+1}\| + \|x_{n+1}-T_nx_n\| \\
&= \|x_n-x_{n+1}\| + \left\|P_C\left[\alpha_n\gamma Vx_n + \left(I-\alpha_n\mu F\right)T_nx_n\right] - P_C(T_nx_n)\right\| \\
&\leq \|x_n-x_{n+1}\| + \left\|\alpha_n\gamma Vx_n + \left(I-\alpha_n\mu F\right)T_nx_n - T_nx_n\right\| \\
&= \|x_n-x_{n+1}\| + \alpha_n\left\|\gamma Vx_n - \mu FT_nx_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{3.7}
$$

Since

$$
\begin{aligned}
\|x_n-Tx_n\| &\leq \|x_n-T_nx_n\| + \|T_nx_n-Tx_n\| \\
&\leq \|x_n-T_nx_n\| + \mathfrak{D}_B(T_n,T),
\end{aligned}
\tag{3.8}
$$

it follows that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Step 4. We have ($\limsup_{n\to\infty}\langle x_n-\tilde{x},(\gamma V-\mu F)\tilde{x}\rangle \leq 0$). Let us choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\to\infty}\left\langle x_n-\tilde{x},(\gamma V-\mu F)\tilde{x}\right\rangle = \lim_{k\to\infty}\left\langle x_{n_k}-\tilde{x},(\gamma V-\mu F)\tilde{x}\right\rangle.
\tag{3.9}
$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup z \in C$. By using Lemma 2.4, we get that $z \in F(T)$. Note that $F(T) = F(\mathfrak{T})$, it follows that $z \in F(\mathfrak{T})$. Hence from (3.2), we get the following:

$$
\limsup_{n\to\infty}\left\langle x_n-\tilde{x},(\gamma V-\mu F)\tilde{x}\right\rangle = \left\langle z-\tilde{x},(\gamma V-\mu F)\tilde{x}\right\rangle \leq 0.
\tag{3.10}
$$

Step 5. We have ($x_n \to \tilde{x}$ as $n \to \infty$). Set $y_n = \alpha_n\gamma Vx_n + (I-\alpha_n\mu F)T_nx_n$ and $\gamma_n = \alpha_n(\tau-\gamma L)$. Noticing that $x_{n+1} = P_C(y_n)$. From (3.1), we have

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \left\langle y_n-\tilde{x}, x_{n+1}-\tilde{x}\right\rangle + \left\langle P_C(y_n)-y_n, P_C(y_n)-\tilde{x}\right\rangle \\
&\leq \left\langle y_n-\tilde{x}, x_{n+1}-\tilde{x}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_n \langle \gamma V x_n - \mu F \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\quad + \langle (I - \alpha_n \mu F) T_n x_n - (I - \alpha_n \mu F) T_n \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&= \alpha_n \gamma \langle V x_n - V \tilde{x}, x_{n+1} - \tilde{x} \rangle + \alpha_n \langle \gamma V \tilde{x} - \mu F \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\quad + \langle (I - \alpha_n \mu F) T_n x_n - (I - \alpha_n \mu F) T_n \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\leq \alpha_n \gamma L \| x_n - \tilde{x} \| \| x_{n+1} - \tilde{x} \| + \alpha_n \langle (\gamma V - \mu F) \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\quad + (1 - \alpha_n \tau)(\| x_n - \tilde{x} \| + a_n) \| x_{n+1} - \tilde{x} \| \\
&= (1 - \alpha_n (\tau - \gamma L)) \| x_n - \tilde{x} \| \| x_{n+1} - \tilde{x} \| \\
&\quad + \alpha_n \langle (\gamma V - \mu F) \tilde{x}, x_{n+1} - \tilde{x} \rangle + (1 - \alpha_n \tau) a_n \| x_{n+1} - \tilde{x} \| \\
&\leq (1 - \alpha_n (\tau - \gamma L)) \frac{1}{2} \left(\| x_n - \tilde{x} \|^2 + \| x_{n+1} - \tilde{x} \|^2 \right) \\
&\quad + \alpha_n \langle (\gamma V - \mu F) \tilde{x}, x_{n+1} - \tilde{x} \rangle + a_n \| x_{n+1} - \tilde{x} \|.
\end{aligned}
\tag{3.11}
$$

Hence, we have

$$
\begin{aligned}
\| x_{n+1} - \tilde{x} \|^2 &\leq \frac{1 - \alpha_n (\tau - \gamma L)}{1 + \alpha_n (\tau - \gamma L)} \| x_n - \tilde{x} \|^2 + \frac{2\alpha_n}{1 + \gamma_n} \langle (\gamma V - \mu F) \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\quad + \frac{2a_n}{1 + \gamma_n} \| x_{n+1} - \tilde{x} \| \\
&\leq (1 - \alpha_n (\tau - \gamma L)) \| x_n - \tilde{x} \|^2 + \frac{2\alpha_n}{1 + \gamma_n} \langle (\gamma V - \mu F) \tilde{x}, x_{n+1} - \tilde{x} \rangle \\
&\quad + \frac{2a_n}{1 + \gamma_n} \| x_{n+1} - \tilde{x} \| \\
&= (1 - \gamma_n) \| x_n - \tilde{x} \|^2 + \gamma_n \delta_n + \frac{2a_n}{1 + \gamma_n} \| x_{n+1} - \tilde{x} \|,
\end{aligned}
\tag{3.12}
$$

where

$$
\delta_n = \frac{2}{(1 + \gamma_n)(\tau - \gamma L)} \langle (\gamma V - \mu F) \tilde{x}, x_{n+1} - \tilde{x} \rangle.
\tag{3.13}
$$

Noticing that $\lim_{n\to\infty} a_n/\alpha_n = 0$, it follows from Lemma 2.5 that $\lim_{n\to\infty} x_n = \tilde{x}$. This completes the proof.

Now, we derive the main result of Ceng et al. ([13], Theorem 3.2) as the following corollary. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator and $V : C \to H$ be an L-Lipschitzian mapping. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose that $0 < \mu < 2\eta/k^2$ and*

$0 \leq \gamma L < \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. *For an arbitrary* $x_1 \in C$, *consider the sequence* $\{x_n\}$ *generated by the following iterative process:*

$$x_1 \in C,$$
$$x_{n+1} = P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T x_n\right] \quad \forall n \in \mathbb{N}, \tag{3.14}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying the conditions (a) and (b) of Theorem 3.1.*

Then, the sequence $\{x_n\}$ *converges strongly to* $\tilde{x} \in F(T)$, *where* $\tilde{x}$ *is the unique solution of the following variational inequality:*

$$\langle (\mu F - \gamma V)\tilde{x}, \tilde{x} - y \rangle \leq 0 \quad \forall y \in F(T). \tag{3.15}$$

Again, we derive the result of Tian ([11], Theorem 3.2) as the following corollary.

Corollary 3.3. *Let* H *be a real Hilbert space. Let* f *be an* α*-contraction on* H *and* $F : H \to H$ *be a* k*-Lipschitzian and* η*-strongly monotone operator. Let* $T : H \to H$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Suppose that* $0 < \mu < 2\eta/k^2$ *and* $0 < \gamma\alpha < \tau$, *where* $\tau = \mu(\eta - \mu k^2/2)$. *For an arbitrary* $x_1 \in H$, *consider the sequence* $\{x_n\}$ *generated by the following iterative process:*

$$x_1 \in H,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T x_n \quad \forall n \in \mathbb{N}, \tag{3.16}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying the conditions* (a) *and* (b) *of Theorem 3.1.*

Then, the sequence $\{x_n\}$ *converges strongly to* $\tilde{x} \in F(T)$, *where* $\tilde{x}$ *is the unique solution of the following variational inequality:*

$$\langle (\mu F - \gamma f)\tilde{x}, \tilde{x} - y \rangle \leq 0 \quad \forall y \in F(T). \tag{3.17}$$

The following result obtains immediately from Theorem 3.1.

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $F : C \to H$ *be a* k*-Lipschitzian and* η*-strongly monotone operator and* $V : C \to H$ *be an* L*-Lipschitzian mapping. Let* $\{T_n\}$ *be a sequence of nonexpansive mappings from* C *into itself such that* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *and* T *be a mapping from* C *into itself defined by* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in C$. *Suppose that* $0 < \mu < 2\eta/k^2$ *and* $0 \leq \gamma L < \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. *For an arbitrary* $x_1 \in C$, *consider the sequence* $\{x_n\}$ *in* C *generated by the following iterative process:*

$$x_1 \in C,$$
$$x_{n+1} = P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n\right] \quad \forall n \in \mathbb{N}, \tag{3.18}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *satisfying the conditions (a)–(c) of Theorem 3.1.*

Then, the sequence $\{x_n\}$ *converges strongly to* $\tilde{x} \in \bigcap_{n=1}^{\infty} F(T_n)$, *where* $\tilde{x}$ *is the unique solution of the following variational inequality:*

$$\langle (\mu F - \gamma V)\tilde{x}, \tilde{x} - y \rangle \leq 0 \quad \forall y \in \bigcap_{n=1}^{\infty} F(T_n). \tag{3.19}$$

4. Application

Recall that the so-called problem of image recovery is essentially to find a common element of finitely many nonexpansive retracts $C_1, C_2, \ldots, C_r$ of C with $\bigcap_{i=1}^{r} C_i \neq \emptyset$. It is easy to see that every nonexpansive retraction P_i of C onto C_i is a nonexpansive mapping of C into itself. There is no doubt that the problem of image recovery is equivalent to finding a common fixed point of finitely many nonexpansive mappings $P_1, P_2, \ldots, P_r$ of C into itself. Applying our main result, we obtain the following result which improves a number of results connected to the problem of image recovery.

Theorem 4.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $F : C \to H$ *be a* k*-Lipschitzian and* η*-strongly monotone operator and* $V : C \to H$ *be an* L*-Lipschitzian mapping. Let* $\lambda_i > 0 (i = 1, 2, 3, \ldots, N)$ *such that* $\sum_{i=1}^{N} \lambda_i = 1$ *and* $T_1, T_2, T_3, \ldots, T_N : C \to C$ *be nonexpansive mappings such that* $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Suppose that* $0 < \mu < 2\eta/k^2$ *and* $0 \leq \gamma L < \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. *For an arbitrary* $x_1 \in C$, *consider the sequence* $\{x_n\}$ *in* C *generated by the following iterative process:*

$$\begin{gathered} x_1 \in C, \\ x_{n+1} = P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) \sum_{i=1}^{N} \lambda_i T_i x_n\right] \quad \forall n \in \mathbb{N}, \end{gathered} \tag{4.1}$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *satisfying the conditions* (a) *and* (b) *of Theorem 3.1.*

Then, the sequence $\{x_n\}$ *converges strongly to* $\tilde{x} \in \bigcap_{i=1}^{N} F(T_i)$, *where* $\tilde{x}$ *is the unique solution of the following variational inequality:*

$$\langle (\mu F - \gamma V)\tilde{x}, \tilde{x} - y \rangle \leq 0 \quad \forall y \in \bigcap_{i=1}^{N} F(T_i). \tag{4.2}$$

Proof. Define $T = \sum_{i=1}^{N} \lambda_i T_i$. Then T is nonexpansive mapping from C into itself. Thus, using Lemma 2.6, we get $F(T) = \bigcap_{i=1}^{N} F(T_i)$. Therefore, the proof follows from Corollary 3.2. □

5. Numerical Example

For showing the effectiveness and convergence of the sequence generated by the considered iterative scheme, we discuss the following example.

Example 5.1. Let $H = \mathbb{R}$ and $C = [0, 1]$. Let T be a self-mapping defined by $Tx = 1 - x$ for all $x \in C$. Let $F, V : C \to H$ be two mappings defined by $Fx = 4x$ and $Vx = 2x$ for all $x \in C$,

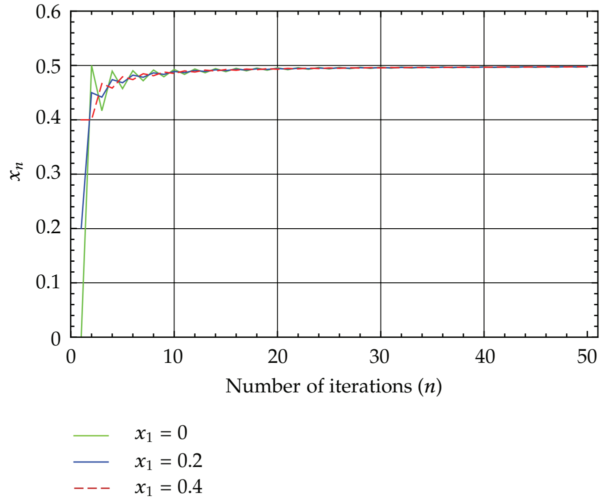

Figure 1

where F is a k-Lipschitzian and η-strongly monotone, and V is an L-Lipschitzian mapping. We take $0 < \mu < 2\eta/k^2$ and $0 \leq \gamma L < \tau$, and we have $\mu = 1/4, \tau = 1$ and $\gamma = 1/4$. Define $\{\alpha_n\}$ in $(0,1)$ by $\alpha_n = 1/n + 1$. Without loss of generality, we may assume that $a_n = 1/n^{3/2}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by

$$T_n x = \begin{cases} 1 - x, & \text{if } x \in [0,1), \\ a_n, & \text{if } x = 1. \end{cases} \tag{5.1}$$

In [15], it is proved that $\mathfrak{T} = \{T_n\}$ is a sequence of nearly nonexpansive mappings from C into itself such that $F(\mathfrak{T}) = \{1/2\}$ and $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, where T is nonexpansive mapping.

It can be observed that all the assumptions of Theorem 3.1 are satisfied and the sequence $\{x_n\}$ generated by (3.1) converges to a unique solution $1/2$ of variational inequality (3.2) over $F(\mathfrak{T})$. The graphical presentation of the convergence of the sequence $\{x_n\}$ generated by the iterative scheme (3.1) is given in Figure 1.

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