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Research Article

General Common Fixed Point Theorems and Applications

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The main result is a common fixed point theorem for a pair of multivalued maps on a complete metric space extending a recent result of Đorić and Lazović (2011) for a multivalued map on a metric space satisfying Ćirić-Suzuki-type-generalized contraction. Further, as a special case, we obtain a generalization of an important common fixed point theorem of Ćirić (1974). Existence of a common solution for a class of functional equations arising in dynamic programming is also discussed.

1. Introduction

Consistent with Nadler [1, page 620], (X, d) will denote a metric space and $\mathrm{CL}(X)$, the collection of all nonempty closed subsets of X. For $A, B \in \mathrm{CL}(X)$ and $\varepsilon > 0$,

$$
\begin{gathered}
N(\varepsilon, A) = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}, \\
E_{A,B} = \{\varepsilon > 0 : A \subseteq N(\varepsilon, B),\ B \subseteq N(\varepsilon, A)\}, \\
H(A, B) = \begin{cases} \inf E_{A,B}, & \text{if } E_{A,B} \neq \emptyset \\ +\infty, & \text{if } E_{A,B} = \emptyset. \end{cases}
\end{gathered} \tag{1.1}
$$

The hyperspace $(\mathrm{CL}(X), H)$ is called the generalized Hausdorff metric space induced by the metric d on X.

For nonempty subsets $A,\ B$ of $X,\ d(A,B)$ denotes the gap between the subsets A and B, while

$$\rho(A,B)=\sup\{d(a,b):a\in A,\ b\in B\},$$
$$BN(X)=\{A:\emptyset\neq A\subseteq X \text{ and the diameter of } A \text{ is finite}\}. \tag{1.2}$$

As usual, we write $d(x,B)$ (resp. $\rho(x,B)$) for $d(A,B)$ (resp. $\rho(A,B)$) when $A=\{x\}$.

Let $S,T:X\to \mathrm{CL}(X)$. Then $u\in X$ is a fixed point of S if and only if $u\in Su$ and a common fixed point of S and T if and only if $u\in Su\cap Tu$.

Let S and T be maps to be defined specifically in a particular context, while x and y are the elements of a metric space (X,d):

$$M(Sx,Ty)=\max\left\{d(x,y),d(x,Sx),d(y,Ty),\frac{d(x,Ty)+d(y,Sx)}{2}\right\}. \tag{1.3}$$

Recently Suzuki [2] and Kikkawa and Suzuki [3] obtained interesting generalizations of the Banach's classical fixed point theorem and other fixed point results by Nadler [4], Jungck [5], and Meir and Keeler [6]. These results have important outcomes (see, e.g., [7–14]). The following result, due to Đorić and Lazović [9], extends and generalizes fixed point theorems from Ćirić [15], Kikkawa and Suzuki [3], Nadler [4], Reich [16], Rus [17], and others.

Theorem 1.1. *Define a nonincreasing function φ from $[0,1)$ onto $(0,1]$ by*

$$\varphi(r)=\begin{cases}1 & \text{if } 0\le r<\dfrac{1}{2}\\ 1-r & \text{if } \dfrac{1}{2}\le r<1.\end{cases} \tag{1.4}$$

Let X be a complete metric space and $T:X\to \mathrm{CL}(X)$. Assume there exists $r\in[0,1)$ such that for every $x,y\in X$,

$$\varphi(r)d(x,Tx)\le d(x,y) \text{ implies } H(Tx,Ty)\le rM(Tx,Ty). \tag{1.5}$$

Then there exists $z\in X$ such that $z\in Tz$.

We remark that, for every $x,y\in X$, the generalized contraction $H(Tx,Ty)\le rM(Tx,Ty)$, $0\le r<1$, was first studied by Ćirić [15]. The following important common fixed point theorem is due to Ćirić [18].

Theorem 1.2. *Let X be a complete metric space and $S,T:X\to X$. Assume there exists $r\in[0,1)$ such that for every $x,y\in X$,*

$$d(Sx,Ty)\le rM(Sx,Ty). \tag{1.6}$$

Then S and T have a unique common fixed point.

For an excellent discussion on several special cases and variants of Theorem 1.2, one may refer to Rus [17]. However, the generality of Theorem 1.2 may be appreciated from the fact that (1.6) in Theorem 1.2 cannot be replaced by

$$d(Sx,Ty) \leq r \max\{d(x,y), d(x,Sx), d(y,Ty), d(x,Ty), d(y,Sx)\}. \tag{1.7}$$

Indeed, Sastry and Naidu [19, Example 5] have shown that maps S and T satisfying (1.7) need not have a common fixed point on a complete metric space. Notice that the condition (1.7) with $S = T$ is the quasicontraction due to Ćirić [20].

The main result of this paper (cf. Theorem 2.2) generalizes Theorems 1.1 and 1.2. Further, a corollary of Theorem 2.2 is used to obtain a unique common fixed point theorem for multivalued maps on a metric space with values in $BN(X)$. As another application, we deduce the existence of a common solution for a general class of functional equations under much weaker conditions than those in [12, 14, 21–24].

2. Main Results

We shall need the following result essentially due to Nadler [4] (see also [15, 25], [26, page 4], [27], [17, page 76]).

Lemma 2.1. *If $A, B \in \mathrm{CL}(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \varepsilon$.*

Theorem 2.2. *Let X be a complete metric space and $S, T : X \to \mathrm{CL}(X)$. Assume there exists $r \in [0,1)$ such that for every $x, y \in X$,*

$$\varphi(r) \min\{d(x,Sx), d(y,Ty)\} \leq d(x,y) \text{ implies } H(Sx,Ty) \leq rM(Sx,Ty). \tag{2.1}$$

Then there exists an element $u \in X$ such that $u \in Su \cap Tu$.

Proof. Obviously $M(Sx,Ty) = 0$ iff $x = y$ is a common fixed point of S and T. So, we may take without any loss of generality that $M(Sx,Ty) > 0$ for distinct $x, y \in X$. Let $\varepsilon > 0$ be such that $\beta = r + \varepsilon < 1$. Let $u_0 \in X$ and $u_1 \in Tu_0$. Then by Lemma 2.1, their exists $u_2 \in Su_1$ such that

$$d(u_2, u_1) \leq H(Su_1, Tu_0) + \varepsilon M(Su_1, Tu_0). \tag{2.2}$$

Similarly, their exists $u_3 \in Tu_2$ such that

$$d(u_3, u_2) \leq H(Tu_2, Su_1) + \varepsilon M(Tu_2, Su_1). \tag{2.3}$$

Continuing in this manner, we find a sequence $\{u_n\}$ in X such that

$$\begin{aligned} &u_{2n+1} \in Tu_{2n}, \quad u_{2n+2} \in Su_{2n+1} \text{ such that} \\ &d(u_{2n+1}, u_{2n}) \leq H(Tu_{2n}, Su_{2n-1}) + \varepsilon M(Tu_{2n}, Su_{2n-1}), \\ &d(u_{2n+2}, u_{2n+1}) \leq H(Su_{2n+1}, Tu_{2n}) + \varepsilon M(Su_{2n+1}, Tu_{2n}). \end{aligned} \tag{2.4}$$

Now, we consider two cases and show that for any $n \in N$,

$$d(u_{2n+1}, u_{2n}) \leq \beta d(u_{2n-1}, u_{2n}). \tag{2.5}$$

Case 1. If $d(u_{2n-1}, Su_{2n-1}) \geq d(u_{2n}, Tu_{2n})$, then

$$\varphi(r)\min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}). \tag{2.6}$$

Therefore by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq rM(Su_{2n-1}, Tu_{2n}). \tag{2.7}$$

Case 2. If $d(u_{2n}, Tu_{2n}) \geq d(u_{2n-1}, Su_{2n-1})$, then

$$\varphi(r)\min\{d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n})\} \leq d(u_{2n-1}, u_{2n}). \tag{2.8}$$

So by the assumption,

$$H(Su_{2n-1}, Tu_{2n}) \leq rM(Su_{2n-1}, Tu_{2n}). \tag{2.9}$$

Hence in either case we obtain by (2.7) and (2.9),

$$\begin{aligned}
&d(u_{2n}, u_{2n+1})\\
&\quad\leq H(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n})\\
&\quad\leq rM(Su_{2n-1}, Tu_{2n}) + \varepsilon M(Su_{2n-1}, Tu_{2n}) = \beta M(Su_{2n-1}, Tu_{2n})\\
&\quad= \beta \max\left\{d(u_{2n-1}, u_{2n}), d(u_{2n-1}, Su_{2n-1}), d(u_{2n}, Tu_{2n}), \frac{d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Su_{2n-1})}{2}\right\}\\
&\quad\leq \beta \max\{d(u_{2n-1}, u_{2n}), d(u_{2n}, u_{2n+1})\}.
\end{aligned} \tag{2.10}$$

This yields (2.5). Analogously, we obtain $d(u_{2n+2}, u_{2n+1}) \leq \beta d(u_{2n+1}, u_{2n})$, and conclude that for any $n \in N$,

$$d(u_{n+1}, un) \leq \beta d(u_n, u_{n-1}). \tag{2.11}$$

Therefore $\{u_n\}$ is a Cauchy sequence and has a limit in X. Call it u.

Now we show that for any $y \in X - \{u\}$,

$$d(u, Ty) \leq r\max\{d(u, y), d(y, Ty)\}, \tag{2.12}$$

$$d(u, Sy) \leq r\max\{d(u, y), d(y, Sy)\}. \tag{2.13}$$

Since $u_n \to u$, there exists $n_0 \in N$ (natural numbers) such that

$$d(u, u_n) \le \frac{1}{3} d(u, y) \quad \text{for } y \ne u \text{ and all } n \ge n_0. \tag{2.14}$$

Then as in [2, page 1862],

$$\begin{aligned}\varphi(r) d(u_{2n-1}, Su_{2n-1}) \le d(u_{2n-1}, Su_{2n-1}) \le d(u_{2n-1}, u_{2n}) &\le d(u_{2n-1}, u) + d(u, u_{2n}) \\ &\le \frac{2}{3} d(y, u) = d(y, u) - \frac{1}{3} d(y, u) \le d(y, u) - d(u_{2n-1}, u) \\ &\le d(u_{2n-1}, y).\end{aligned} \tag{2.15}$$

Therefore

$$\varphi(r) d(u_{2n-1}, Su_{2n-1}) \le d(u_{2n-1}, y). \tag{2.16}$$

Now either $d(u_{2n-1}, Su_{2n-1}) \le d(y, Ty)$ or $d(y, Ty) \le d(u_{2n-1}, Su_{2n-1})$.

So in either case by (2.16),

$$\varphi(r) \min\{d(u_{2n-1}, Su_{2n-1}), d(y, Ty)\} \le d(u_{2n-1}, y). \tag{2.17}$$

Hence by the assumption (2.1),

$$\begin{aligned}d(u_{2n}, Ty) &\le H(Su_{2n-1}, Ty) \le rM(Su_{2n-1}, Ty) \\ &\le r \max\left\{d(u_{2n-1}, y), d(u_{2n-1}, Su_{2n-1}), d(y, Ty), \frac{d(u_{2n-1}, Ty) + d(y, Su_{2n-1})}{2}\right\}.\end{aligned} \tag{2.18}$$

Making $n \to \infty$,

$$\begin{aligned}d(u, Ty) &\le r \max\left\{d(u, y), d(u, u), d(y, Ty), \frac{d(u, Ty) + d(y, u)}{2}\right\} \\ &\le r \max\{d(u, y), d(y, Ty), d(u, Ty)\}.\end{aligned} \tag{2.19}$$

This yields (2.12). Similarly, we can show (2.13).

Now, we show that $u \in Su \cap Tu$.

For $0 \le r < 1/2$, the following cases arise.

Case 1. Suppose $u \notin Su$ and $u \notin Tu$. Then as in [8, page 6], let $a \in Tu$ be such that

$$2rd(a, u) < d(u, Tu), \tag{2.20}$$

and $a \in Su$ be such that $2rd(a, u) < d(u, Su)$.

Since $a \in Tu$ implies $a \neq u$, we have from (2.12) and (2.13),

$$d(u, Ta) \leq r \max\{d(u, a), d(a, Ta)\}, \tag{2.21}$$

$$d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}. \tag{2.22}$$

On the other hand, since $\varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u)$,

$$\varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u). \tag{2.23}$$

Therefore by the assumption (2.1),

$$\begin{aligned} d(Sa, a) \leq H(Sa, Tu) &\leq r \max\left\{d(a, u), d(u, Tu), d(a, Sa), \frac{d(u, Sa) + d(a, Tu)}{2}\right\} \\ &= r \max\left\{d(a, u), d(a, Sa), \frac{1}{2} d(u, Sa)\right\}. \end{aligned} \tag{2.24}$$

This gives $d(a, Sa) \leq H(Sa, Tu) \leq rd(a, u) < d(a, u)$.

So by (2.22), $d(Sa, u) \leq rd(a, u)$. Thus

$$\begin{aligned} d(u, Tu) &\leq d(u, Sa) + H(Sa, Tu) \\ &\leq rd(a, u) + rd(a, u) = 2rd(a, u) < d(u, Tu) \quad (\text{by the assumption of Case 1}). \end{aligned} \tag{2.25}$$

This contradicts $u \notin Tu$. Consequently $u \in Tu$. Similarly $u \in Su$.

Case 2. Let $u \in Su$ and $u \notin Tu$. Then as in the previous case, let $a \in Tu$ be such that

$$2rd(a, u) < d(u, Tu). \tag{2.26}$$

Since $a \neq u$, we have from (2.13),

$$d(u, Sa) \leq r \max\{d(u, a), d(a, Sa)\}. \tag{2.27}$$

On the other hand, Since $\varphi(r)d(u, Tu) \leq d(u, Tu) \leq d(a, u)$,

$$\varphi(r) \min\{d(a, Sa), d(u, Tu)\} \leq d(a, u). \tag{2.28}$$

Therefore by the assumption (2.1),

$$\begin{aligned} d(Sa,a) &\le H(Sa,Tu) \le r\max\left\{d(a,u),d(u,Tu),d(a,Sa),\frac{d(u,Sa)+d(a,Tu)}{2}\right\} \\ &= r\ \max\left\{d(a,u),d(a,Sa),\frac{1}{2}d(u,Sa)\right\}. \end{aligned} \tag{2.29}$$

This gives $d(a,Sa) \le H(Sa,Tu) \le rd(a,u) < d(a,u)$.

So by (2.22), $d(Sa,u) \le rd(a,u)$. Thus

$$\begin{aligned} d(u,Tu) &\le d(u,Sa) + H(Sa,Tu) \\ &\le rd(a,u) + rd(a,u) = 2rd(a,u) < d(u,Tu) \quad \text{(by the assumption of Case 2)}. \end{aligned} \tag{2.30}$$

This contradicts $u \notin Tu$. Consequently $u \in Tu$.

Case 3. $u \in Tu$ and $u \notin Su$. As in the previous case, it follows that $u \in Su$.

Now we consider the case $1/2 \le r < 1$.

First we show that

$$H(Sx,Tu) \le r\max\left\{d(x,u),d(x,Sx),d(u,Tu),\frac{d(x,Tu)+d(u,Sx)}{2}\right\}. \tag{2.31}$$

Assume that $x \neq u$. Then for every $n \in N$, there exists $z_n \in Sx$ such that

$$d(u,z_n) \le d(u,Sx) + \frac{1}{n}d(x,u). \tag{2.32}$$

Therefore

$$\begin{aligned} d(x,Sx) &\le d(x,z_n) \le d(x,u) + d(u,z_n) \\ &\le d(x,u) + d(u,Sx) + \frac{1}{n}d(x,u). \end{aligned} \tag{2.33}$$

Using (2.13) with $y = x$, (2.33) implies

$$d(x,Sx) \le d(x,u) + r\max\{d(x,u),d(x,Sx)\} + \frac{1}{n}d(u,x). \tag{2.34}$$

If $d(x,u) \ge d(x,Sx)$, then (2.34) gives

$$\begin{aligned} d(x,Sx) &\le d(x,u) + rd(x,u) + \frac{1}{n}d(u,x) \\ &= \left(1 + r + \frac{1}{n}\right)d(x,u). \end{aligned} \tag{2.35}$$

Making $n \to \infty$,

$$d(x,Sx) \leq (1+r)d(x,u). \tag{2.36}$$

Thus $\varphi(r)d(x,Sx) = (1- \ r)d(x,Sx) \leq (1/(1+r))d(x,Sx) \leq d(x,u)$.

Then $\varphi(r)\min\{d(x,Sx),d(u,Tu)\} \leq d(x,u)$, and by the assumption (2.1),

$$H(Sx,Tu) \leq r\max\left\{d(x,u),d(x,Sx),d(u,Tu),\frac{d(x,Tu)+d(u,Sx)}{2}\right\}. \tag{2.37}$$

If $d(x,u) < d(x,Sx)$, then (2.34) gives

$$d(x,Sx) \leq d(x,u) + rd(x,Sx) + \frac{1}{n}d(u,x), \tag{2.38}$$

that is, $(1-r)d(x,Sx) \leq (1+(1/n))d(x,u)$. □

Making $n \to \infty$,

$$\varphi(r)d(x,Sx) \leq d(x,u). \tag{2.39}$$

Then $\varphi(r)\min\{d(x,Sx),d(u,Tu)\} \leq d(x,u)$, and by the assumption, we get (2.37).

Taking $x = u_{2n+1}$ in (2.37) and passing to the limit, we obtain

$$d(u,Tu) \leq rd(u,Tu). \tag{2.40}$$

This gives $u \in Tu$. Analogously, $u \in Su$.

The following result generalizes Theorem 1.2.

Corollary 2.3. *Let X be a complete metric space and S,T maps from X into X. Suppose there exists $r \in [0,1)$ such that for every $x,y \in X$,*

$$\varphi(r)\min\{d(x,Sx),d(y,Ty)\} \leq d(x,y) \text{ implies } d(Sx,Ty) \leq rM(Sx,Ty). \tag{2.41}$$

Then S and T have a unique common fixed point.

Proof. For single-valued maps S and T, it comes from Theorem 2.2 that they have a common fixed point. The uniqueness of the common fixed point follows easily. □

Remark 2.4. Theorem 1.1 is obtained as a particular case of Theorem 2.2 when $S = T$.

Now we derive the following result due to Đorić and Lazović [9, Corollary 2.3].

Corollary 2.5. *Let X be a complete metric space and T a map from X into X. Suppose there exists $r \in [0,1)$ such that for every $x, y \in X$,*

$$\varphi(r)d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq rM(Tx,Ty). \tag{2.42}$$

Then T has a unique fixed point.

Proof. It comes from Corollary 2.3 when $S = T$. □

The following example shows the generality of our results.

Example 2.6. Let $X = \{(0,0),(0,4),(4,0),(0,5),(5,0),(4,5),(5,4)\}$ be endowed with the metric d defined by

$$d[(x_1,x_2),(y_1,y_2)] = |x_1 - y_1| + |x_2 - y_2|. \tag{2.43}$$

Let S and T be such that

$$S(x_1,x_2) = \begin{cases} (x_1,0) & \text{if } x_1 \leq x_2 \\ (0,0) & \text{if } x_1 > x_2, \end{cases} \qquad T(x_1,x_2) = \begin{cases} (x_2,0) & \text{if } x_1 \leq x_2 \\ (0,x_2) & \text{if } x_1 > x_2. \end{cases} \tag{2.44}$$

Then S and T do not satisfy the condition (1.6) of Theorem 1.2 at $x = (4,5)$, $y = (5,4)$. However, this is readily verified that all the hypotheses of Corollary 2.3 are satisfied for the maps S and T.

Theorem 2.7. *Let X be a complete metric space and $P, Q : X \to BN(X)$. Assume there exists $r \in [0,1)$ such that for every $x, y \in X$,*

$$\varphi(r)\min\{\rho(x,Px),\rho(y,Qy)\} \leq d(x,y) \tag{2.45}$$

implies

$$\rho(Px,Qy) \leq r \max\left\{d(x,y),\rho(x,Px),\rho(y,Qy),\frac{d(x,Qy)+d(y,Px)}{2}\right\}. \tag{2.46}$$

Then there exsits a unique point $z \in X$ such that $z \in Pz \cap Qz$.

Proof. Choose $\lambda \in (0,1)$. Define single-valued maps $S, T : X \to X$ as follows. For each $x \in X$, let Sx be a point of Px which satisfies

$$d(x,Sx) \geq r^{\lambda}\rho(x,Px). \tag{2.47}$$

Similarly, for each $y \in X$, let Ty be a point of Qy such that

$$d(y,Ty) \geq r^{\lambda}\rho(y,Qy). \tag{2.48}$$

Since $Sx \in Px$ and $Ty \in Qy$,

$$d(x, Sx) \leq \rho(x, Px), \qquad d(y, Ty) \leq \rho(y, Qy). \tag{2.49}$$

So, (2.45) gives

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq \varphi(r) \min\{\rho(x, Px), \rho(y, Qy)\} \leq d(x, y), \tag{2.50}$$

and this implies (2.46). Therefore

$$\begin{aligned} d(Sx, Ty) &\leq \rho(Px, Qy) \\ &\leq r \cdot r^{-\lambda} \max\left\{ r^{\lambda} d(x, y), r^{\lambda} \rho(x, Px), r^{\lambda} \rho(y, Qy), \frac{r^{\lambda} d(x, Qy) + r^{\lambda} d(y, Px)}{2} \right\} \\ &\leq r^{1-\lambda} \max\left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\}. \end{aligned} \tag{2.51}$$

So (2.50), namely, $\varphi(r') \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)$ implies

$$d(Sx, Ty) \leq r' \max\left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\}, \tag{2.52}$$

where $r' = r^{1-\lambda} < 1$.

Hence by Theorem 2.2, S and T have a unique point $z \in X$ such that $Sz = Tz = z$. This implies $z \in Pz \cap Qz$. □

Corollary 2.8. *Let X be a complete metric space and $P : X \to BN(X)$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$\begin{aligned} &\rho(x, Px) \leq (1 + r) d(x, y) \text{ implies} \\ &\rho(Px, Py) \leq r \max\left\{ d(x, y), \rho(x, Px), \rho(y, Py), \frac{d(x, Py) + d(y, Px)}{2} \right\}. \end{aligned} \tag{2.53}$$

Then there exists a unique point $z \in X$ such that $z \in Pz$.

Proof. It comes from Theorem 2.7 when $Q = P$. □

3. Applications

Throughout this section, we assume that Y and Z are Banach spaces, $W \subseteq Y$ and $D \subseteq Z$. Let R denotes the field of reals, $g_1, g_2 : W \times D \to R$ and $G_1, G_2 : W \times D \times R \to R$. Taking W and D

as the state and decision spaces, respectively, the problem of dynamic programming reduces to the problem of solving functional equations:

$$p_i = \sup_{y\in D}\{g_i(x,y) + H_i(x,y,p_i(x,y))\}, \quad x \in W,\ i = 1,2. \tag{3.1}$$

In the multistage process, some functional equations arise in a natural way (cf. [22, 23]; see also [21, 24, 28, 29]). In this section, we study the existence of common solution of the functional equations (3.1) arising in dynamic programming.

Let $B(W)$ denotes the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x\in W}|h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold:

(DP-1) $H_1,\ H_2,\ g_1$, and g_2 are bounded.

(DP-2) There exists $r \in [0,1)$ such that for every $(x,y) \in W \times D, h, k \in B(W)$ and $t \in W$,

$$\varphi(r)\min\{|h(t) - A_1h(t)|, |k(t) - A_2k(t)|\} \le |h(t) - k(t)| \tag{3.2}$$

implies

$$\begin{aligned}&|H_1(x,y,h(t)) - H_2(x,y,k(t))|\\ &\quad\le r\max\left\{|h(t)-k(t)|, |h(t) - A_1h(t)|, |k(t) - A_2k(t)|, \frac{|h(t) - A_2k(t)| + |k(t) - A_1h(t)|}{2}\right\},\end{aligned} \tag{3.3}$$

where $A_1,\ A_2$ are defined as follows:

$$A_ih(x) = \sup_{y\in D} H_i(x,y,h(x,y)), \quad x \in W,\ h \in B(W),\ i = 1,2. \tag{3.4}$$

Theorem 3.1. *Assume the conditions (DP-1) and (DP-2). Then the functional equations* (3.1), $i = 1,2$, *have a unique common solution in* $B(W)$.

Proof. For any $h,k \in B(W)$, let $d(h,k) = \sup\{|h(x) - k(x)| : x \in W\}$. Then $(B(W), d)$ is a complete metric space.

Let λ be any arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$ and choose $y_1, y_2 \in D$ such that

$$A_ih_i < H_i(x, y_i, h_i(x_i)) + \lambda, \tag{3.5}$$

where $x_i = (x, y_i),\ i = 1,2$.

Further,

$$A_1h_1 \ge H_1(x, y_2, h_1(x_2)), \tag{3.6}$$

$$A_2h_2 \ge H_2(x, y_1, h_2(x_1)). \tag{3.7}$$

Therefore, the first inequality in (DP-2) becomes

$$\varphi(r)\min\{|h_1(x)-A_1h_1(x)|,|h_2(x)-A_2h_2(x)|\}\le |h_1(x)-h_2(x)|, \tag{3.8}$$

and this together with (3.5) and (3.7) implies

$$\begin{aligned} A_1h_1-A_2h_2 &< H_1(x,y_1,h_1(x_1))-H_2(x,y,h_2(x_1))+\lambda \\ &\le \left|H_1(x,y_1,h_1(x_1))-H_2(x,y_1,h_2(x_1))\right|+\lambda \\ &\le rM(H_1h_1,H_2h_2)+\lambda. \end{aligned} \tag{3.9}$$

Similarly, (3.5), (3.6), and (3.8) imply

$$A_2h_2(x)-A_1h_1(x)\le rM(A_1h_1,A_2h_2)+\lambda. \tag{3.10}$$

So, from (3.10) and (3.11), we obtain

$$|A_1h_1(x)-A_2h_2(x)|\le r\ M(A_1h_1,A_2h_2)+\lambda. \tag{3.11}$$

Since this inequality is true for any $x\in W$, and $\lambda>0$ is arbitrary, on taking supremum, we find from (3.8) and (3.11) that

$$\varphi(r)\min\{d(h_1,A_1h_1),d(h_2,A_2h_2)\}\le d(h_1,h_2) \tag{3.12}$$

implies

$$d(A_1h_1,A_2h_2)\le rM(A_1h_1,A_2h_2). \tag{3.13}$$

Therefore, Corollary 2.3 applies, wherein A_1 and A_2 correspond, respectively, to the maps S and T. So A_1 and A_2 have a unique common fixed point h^*, that is, $h^*(x)$ is the unique bounded common solution of the functional equations (3.1), $i=1,2$. □

The following result generalizes a recent result of Singh and Mishra [12, Corollary 4.2] which in turn extends certain results from [21, 23, 24].

Corollary 3.2. *Suppose that the following conditions hold.*

(i) *G and g are bounded.*

(ii) *There exists $r\in[0,1)$ such that for every $x,y\in W\times D$, $h,k\in B(W)$ and $t\in W$,*

$$\begin{aligned} &\varphi(r)|h(t)-Kh(t)|\le|h(t)-k(t)| \text{ implies} \\ &\quad \left|G(x,y,h(t))-G(x,y,k(t))\right|\le r\max M(K,h(t),k(t)), \end{aligned} \tag{3.14}$$

where K is defined as

$$Kh(t)=\sup_{y\in D}\{g(t,y)+G(t,y,h(t,y))\},\quad t\in W,\ h\in B(W). \tag{3.15}$$

Then the functional equation (3.1) *with* $H_1 = H_2 = G$ *and* $g_1 = g_2 = g$ *possesses a unique bounded solution in* W.

Proof. It comes from Theorem 3.1 when $g_1 = g_2 = g$ and $H_1 = H_2 = G$. □

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