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Research Article

Iterative Method for Solving the Second Boundary Value Problem for Biharmonic-Type Equation

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Solving boundary value problems (BVPs) for the fourth-order differential equations by the reduction of them to BVPs for the second-order equations with the aim to use the achievements for the latter ones attracts attention from many researchers. In this paper, using the technique developed by ourselves in recent works, we construct iterative method for the second BVP for biharmonic-type equation, which describes the deflection of a plate resting on a biparametric elastic foundation. The convergence rate of the method is established. The optimal value of the iterative parameter is found. Several numerical examples confirm the efficiency of the proposed method.

1. Introduction

Solving BVPs for the fourth-order differential equations by the reduction of them to BVPs for the second-order equations with the aim to use a lot of efficient algorithms for the latter ones attracts attention from many researchers. Namely, for the biharmonic equation with the Dirichlet boundary condition, there is intensively developed the iterative method, which leads the problem to two problems for the Poisson equation at each iteration (see e.g., [1–3]). Recently, Abramov and Ul'yanova [4] proposed an iterative method for the Dirichlet problem for the biharmonic-type equation, but the convergence of the method is not proved. In our previous works [5, 6] with the help of boundary or mixed boundary-domain operators appropriately introduced, we constructed iterative methods for biharmonic and biharmonic-type equations associated with the Dirichlet boundary condition. For the biharmonic-type equation with Neumann boundary conditions in [7] an iterative method also was proposed. It is proved that the iterative methods are convergent with the rate of geometric progression.

In this paper we develop our technique in the above-mentioned papers for the second BVP for the biharmonic-type equation. Namely, we consider the following problem

$$\Delta^2 u - a\Delta u + bu = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g_1 \quad \text{on } \Gamma, \tag{1.2}$$

$$\Delta u = g_2, \quad \text{on } \Gamma, \tag{1.3}$$

where Δ is the Laplace operator, Ω is a bounded domain in R^n $(n \geq 2)$, Γ is the sufficiently smooth boundary of Ω, and a, b are nonnegative constants. This problem has not yet considered in [4].

It should be noticed that when $a = 0,\ b = 0$ (1.1) is the equation for a thin plate, and problem (1.1)–(1.3) is decomposed into two consecutive problems for the Poisson equations.

In this paper we suppose that $a \geq 0, b \geq 0,\ a + b > 0$. Then (1.1) describes the deflection of a plate resting on biparametric elastic foundation. For solving this equation several methods such as the boundary element, the finite element methods [8–10], domain/boundary element technique [11], and boundary differential integral equation (BDIE) method [12] were used. It should be noticed that at present the boundary element method is intensively developed and is used for solving more complex problems of plates and shells (see e.g., [13–15]).

In this paper we use completely different approach to (1.1). Two cases will be treated in dependence on the sign of $a^2 - 4b$. In the case if $a^2 - 4b \geq 0$ we can immediately decompose the problem into two problems for second-order equations. In the opposite case we propose an iterative method for reducing problem (1.1)–(1.3) to a sequence of second-order problems. The convergence of the method is established and verified on examples.

2. Case $a^2 - 4b \geq 0$

In this case we always can lead the original problem (1.1)–(1.3) to two problems for second-order equations. To do this, we put

$$\mu = \frac{1}{2}\left(a + \sqrt{a^2 - 4b}\right). \tag{2.1}$$

Then problem (1.1)–(1.3) is reduced to the following problems:

$$\begin{aligned} \mu\Delta v - bv &= f, \quad \text{in } \Omega, \\ v &= \frac{1}{\mu}g_2 - g_1, \quad \text{on } \Gamma, \\ \frac{1}{\mu}\Delta u - u &= v, \quad \text{in } \Omega, \\ u &= g_1, \quad \text{on } \Gamma. \end{aligned} \tag{2.2}$$

These Dirichlet problems can be solved by known methods such as finite element method, boundary element method, or finite difference method. Some fast Poisson solvers in [16, 17] can be applied for the above problems.

3. Case $a^2 - 4b < 0$

This case is very important in mechanics because (1.1) describes the bending plate on elastic foundation (see [18]).

3.1. Construction of Iterative Method on Continuous Level

As in [6], we set

$$\Delta u = v, \tag{3.1}$$

$$\varphi = -bu. \tag{3.2}$$

Then problem (1.1)–(1.3) leads to the following second-order problems

$$\begin{aligned} \Delta v - av &= f + \varphi, \quad \text{in } \Omega, \\ v &= g_2, \quad \text{on } \Gamma, \\ \Delta u &= v, \quad \text{in } \Omega, \\ u &= g_1, \quad \text{on } \Gamma, \end{aligned} \tag{3.3}$$

where all the functions u, v, and φ are unknown but they are related with each other by (3.2).

Now consider the following iterative process for finding φ and simultaneously for finding v, u.

(1) Given $\varphi^{(0)} \in L_2(\Omega)$, for example,

$$\varphi^{(0)} = 0 \quad \text{in } \Omega. \tag{3.4}$$

(2) Knowing $\varphi^{(k)}(x)$ on Ω $(k = 0, 1, \ldots)$ solve consecutively two problems

$$\begin{aligned} \Delta v^{(k)} - av^{(k)} &= f + \varphi^{(k)} \quad \text{in } \Omega, \\ v^{(k)} &= g_2 \quad \text{on } \Gamma, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \Delta u^{(k)} &= v^{(k)} \quad \text{in } \Omega, \\ u^{(k)} &= g_1 \quad \text{on } \Gamma. \end{aligned} \tag{3.6}$$

(3) Compute the new approximation

$$\varphi^{(k+1)} = (1 - \tau_{k+1})\varphi^{(k)} - b\tau_{k+1}u^{(k)} \quad \text{in } \Omega, \tag{3.7}$$

where τ_{k+1} is an iterative parameter to be chosen later.

3.2. Investigation of Convergence

In order to investigate the convergence of the iterative process (3.4)–(3.7), firstly we rewrite (3.7) in the canonical form of two-layer iterative scheme [19]

$$\frac{\varphi^{(k+1)} - \varphi^{(k)}}{\tau_{k+1}} + \left(\varphi^{(k)} + bu^{(k)}\right) = 0. \tag{3.8}$$

Now, we introduce the operator A defined by the formula

$$A\varphi = u, \tag{3.9}$$

where u is the function determined from the problems

$$\begin{aligned} \Delta v - av &= \varphi \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \Delta u &= v \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{3.11}$$

The properties of the operator A will be investigated in the sequel.

Now, let us return to the problem (3.3). We represent their solutions in the form

$$u = u_1 + u_2, \qquad v = v_1 + v_2, \tag{3.12}$$

where u_1, u_2, v_1, v_2 are the solutions of the problems

$$\begin{aligned} \Delta v_1 - av_1 &= \varphi \quad \text{in } \Omega, \\ v_1 &= 0 \quad \text{on } \Gamma, \\ \Delta u_1 &= v_1 \quad \text{in } \Omega, \\ u_1 &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \Delta v_2 - av_2 &= f \quad \text{in } \Omega, \\ v_2 &= g_2 \quad \text{on } \Gamma, \\ \Delta u_2 &= v_2 \quad \text{in } \Omega, \\ u_2 &= g_1 \quad \text{on } \Gamma. \end{aligned} \tag{3.14}$$

According to the definition of A we have

$$A\varphi = u_1. \tag{3.15}$$

Since φ should satisfy the relation (3.2), taking into account the representation (3.12) we obtain the equation

$$(I + bA)\varphi = F, \tag{3.16}$$

where I is the identity operator, and

$$F = -bu_2. \tag{3.17}$$

Thus, we have reduced the original problem (1.1)–(1.3) to the operator equation (3.16), whose right-hand side is completely defined by the data functions f, g, and h, and coefficients a, b.

Proposition 3.1. *The iterative process* (3.4)–(3.7) *is a realization of the two-layer iterative scheme*

$$\frac{\varphi^{(k+1)} - \varphi^{(k)}}{\tau_{k+1}} + (I + bA)\varphi^{(k)} = F, \quad k = 0, 1, 2, \dots \tag{3.18}$$

for the operator equation (3.16).

Proof. Indeed, if in (3.5), (3.6) we put

$$u^{(k)} = u_1^{(k)} + u_2, \qquad v^{(k)} = v_1^{(k)} + v_2, \tag{3.19}$$

where v_2, u_2 are the solutions of problem (3.14), then we get

$$\begin{gathered}
\Delta v_1^{(k)} - a v_1^{(k)} = \varphi^{(k)} \quad \text{in } \Omega, \\
v_1^{(k)} = 0 \quad \text{on } \Gamma, \\
\Delta u_1^{(k)} = v_1^{(k)} \quad \text{in } \Omega, \\
u_1^{(k)} = 0 \quad \text{on } \Gamma.
\end{gathered} \tag{3.20}$$

From here it is easy to see that

$$A\varphi^{(k)} = u_1^{(k)}. \tag{3.21}$$

Therefore, taking into account the first relation of (3.19), the above equality, and (3.17), from (3.8) we obtain (3.18). Thus, the proposition is proved. □

Proposition 3.1 enables us to lead the investigation of convergence of processs (3.4)–(3.7) to the study of scheme (3.18). For this reason we need some properties of the operator A.

Proposition 3.2. *The operator A defined by* (3.9)–(3.11) *is linear, symmetric, positive, and compact operator in the space $L_2(\Omega)$.*

Proof. The linearity of A is obvious. To establish the other properties of A let us consider the inner product $(A\varphi,\overline{\varphi})$ for two arbitrary functions φ and $\overline{\varphi}$ in $L_2(\Omega)$. Recall that the operator A is defined by (3.9)–(3.11). We denote by $\overline{u}$ and $\overline{v}$ the solutions of (3.10) and (3.11), where instead of φ there stands $\overline{\varphi}$. We have

$$\begin{aligned}(A\varphi,\overline{\varphi}) &= \int_\Omega u\overline{\varphi}dx = \int_\Omega u(\Delta\overline{v} - a\overline{v})dx \\ &= \int_\Omega u\Delta\overline{v}dx - a\int_\Omega u\overline{v}dx.\end{aligned} \tag{3.22}$$

Applying the Green formula for the functions u and $\overline{v}$, vanishing on the boundary Γ, we obtain

$$\begin{aligned}&\int_\Omega u\Delta\overline{v}dx = \int_\Omega \overline{v}\Delta u dx = \int_\Omega \overline{v}v dx, \\ &\int_\Omega u\overline{v} = \int_\Omega u\Delta\overline{u} = -\int_\Omega \nabla u\cdot\nabla\overline{u}dx.\end{aligned} \tag{3.23}$$

Hence,

$$(A\varphi,\overline{\varphi}) = \int_\Omega v\overline{v}dx + a\int_\Omega \nabla u\cdot\nabla\overline{u}dx. \tag{3.24}$$

Clearly,

$$\begin{gathered}(A\varphi,\overline{\varphi}) = (A\overline{\varphi},\varphi), \\ (A\varphi,\varphi) = \int_\Omega v^2dx + a\int_\Omega |\nabla u|^2dx \geq 0\end{gathered} \tag{3.25}$$

are due to $a \geq 0$. Moreover, it is easy seen that $(A\varphi,\varphi) = 0$ if and only if $\varphi = 0$. So, we have shown that the operator A is symmetric and positive in $L_2(\Omega)$.

It remains to show the compactness of A. As is well known that if $\varphi \in L_2(\Omega)$ then problem (3.10) has a unique solution $v \in H^2(\Omega)$, therefore, problem (3.11) has a unique solution $v \in H^4(\Omega)$. Consequently, the operator A maps $L_2(\Omega)$ into $H^4(\Omega)$. In view of the compact imbedding $H^4(\Omega)$ into $L_2(\Omega)$ it follows that A is compact operator in $L_2(\Omega)$.

Thus, the proof of Proposition 3.2 is complete. □

Due to the above proposition the operator

$$B = I + bA \tag{3.26}$$

is linear, symmetric, positive definite, and bounded operator in the space $L_2(\Omega)$. More precisely, we have

$$\gamma_1 I < B \leq \gamma_2 I, \tag{3.27}$$

where

$$\gamma_1 = 1, \qquad \gamma_2 = 1 + b\|A\|. \tag{3.28}$$

Notice that since the operator A is defined by (3.9)–(3.11) its norm $\|A\|$ depends on the value of a but not on b in (1.1).

From the theory of elliptic problems [20] we have the following estimates for the functions v, u given by (3.10), (3.11):

$$\|v\|_{H^2(\Omega)} \le C_1\|\varphi\|_{L_2(\Omega)}, \qquad \|u\|_{H^4(\Omega)} \le C_2\|v\|_{H^2(\Omega)}, \tag{3.29}$$

where C_1, C_2 are constants. Therefore,

$$\|u\|_{H^4(\Omega)} \le C_1C_2\|\varphi\|_{L_2(\Omega)}. \tag{3.30}$$

Before stating the result of convergence of the iterative process (3.5)–(3.7) we assume the following regularity of the data of the original problem (1.1)–(1.3):

$$f \in L_2(\Omega), \qquad g_1 \in H^{7/2}(\Gamma), \qquad g_2 \in H^{5/2}(\Gamma). \tag{3.31}$$

Then the problem (1.1)–(1.3) has a unique solution $u \in H^4(\Omega)$. For the function u_2 determined by (3.14) we have also $u_2 \in H^4(\Omega)$.

Theorem 3.3. *Let u be the solution of problem* (1.1)–(1.3) *and φ be the solution of* (3.16). *Then, if $\{\tau_{k+1}\}$ is the Chebyshev collection of parameters, constructed by the bounds γ_1 and γ_2 of the operator B, namely*

$$\begin{gathered}\tau_0 = \frac{2}{\gamma_1+\gamma_2}, \qquad \tau_k = \frac{\tau_0}{\rho_0 t_k + 1}, \qquad t_k = \cos\frac{2k-1}{2M}\pi, \quad k = 1,\ldots,M, \\ \rho_0 = \frac{1-\xi}{1+\xi}, \qquad \xi = \frac{\gamma_1}{\gamma_2},\end{gathered} \tag{3.32}$$

we have

$$\left\|u^{(M)} - u\right\|_{H^4(\Omega)} \le Cq_M, \tag{3.33}$$

where

$$C = C_1C_2\left\|\varphi^{(0)} - \varphi\right\|_{L_2(\Omega)}, \tag{3.34}$$

with C_1, C_2 being the constant in (3.30),

$$q_M = \frac{2\rho_1^M}{1+\rho_1^{2M}}, \qquad \rho_1 = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}. \tag{3.35}$$

In the case of the stationary iterative process, $\tau_k = \tau_0 \quad (k = 1, 2, \ldots)$ *we have*

$$\left\| u^{(k)} - u \right\|_{H^4(\Omega)} \le C \rho_0^k, \quad k = 1, 2, \ldots. \tag{3.36}$$

Proof. According to the general theory of the two-layer iterative schemes (see [21]) for the operator equation (3.16) we have

$$\left\| \varphi^{(M)} - \varphi \right\|_{L_2(\Omega)} \le q_M \left\| \varphi^{(0)} - \varphi \right\|_{L_2(\Omega)}, \tag{3.37}$$

if the parameter $\{\tau_{k+1}\}$ is chosen by the formulae (3.32) and

$$\left\| \varphi^{(k)} - \varphi^* \right\|_{H^4(\Omega)} \le \rho_0^k \left\| \varphi^{(0)} - \varphi^* \right\|_{L_2(\Omega)}, \quad k = 1, 2, \ldots \tag{3.38}$$

if $\tau_k = \tau_0 \quad (k = 1, 2, \ldots)$. In view of these estimates the corresponding estimates (3.33) and (3.36) follow from (3.30) applied to the problems

$$\begin{gathered} \Delta\left(v^{(k)} - v\right) - a\left(v^{(k)} - v\right) = \varphi^{(k)} - \varphi, \quad \text{in } \Omega, \\ v^{(k)} - v = 0 \quad \text{on } \Gamma, \\ \Delta\left(u^{(k)} - u\right) = v^{(k)} - v, \quad \text{in } \Omega, \\ u^{(k)} - u = 0 \quad \text{on } \Gamma, \end{gathered} \tag{3.39}$$

which are obtained in the result of the subtraction of (3.3) from (3.5) and (3.6), respectively. The theorem is proved. □

Remark 3.4. From the above theorem it is easy to see that for each fixed value of a the numbers ρ_0 and q_M characterizing the rate of convergence of the iterative method decrease with the decrease of b. So, the smaller b is, the faster the iterative process converges. In the special case when $b = 0$ the mentioned above numbers also are zero, hence the iterative process converges at once and the original problem (1.1)–(1.3) is decomposed into two second-order problems.

3.3. Computation of the Norm $\|A\|$

As we see in Theorem 3.3 for determining the iterative parameter τ we need the bounds γ_1 and γ_2 of the operator B, and in its turn the latter bound requires to compute $\|A\|$. Therefore, below we consider the problem of computation $\|A\|$.

Suppose the domain $\Omega = [0, l_1] \times [0, l_2]$ in the plane xOy. In this case by Fourier method we found that the system of functions

$$e_{mn}(x, y) = \frac{2}{\sqrt{l_1 l_2}} \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2} \quad (m, n = 1, 2, \ldots) \tag{3.40}$$

is the eigenfunctions of the spectral problem

$$\begin{aligned} \Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma \end{aligned} \tag{3.41}$$

corresponding to the eigenvalues

$$\lambda_{mn} = -\pi^2 \left(\frac{m^2}{l_1^2} + \frac{n^2}{l_2^2} \right). \tag{3.42}$$

Moreover, this system is orthogonal and complete in $L_2(\Omega)$.

Now let a function $\varphi \in L_2(\Omega)$ have the expansion

$$\varphi(x,y) = \sum_{m,n=1}^{\infty} \varphi_{mn} e_{mn}(x,y). \tag{3.43}$$

Then we have

$$\|\varphi\|^2 = (\varphi,\varphi) = \sum_{m,n=1}^{\infty} |\varphi_{mn}|^2. \tag{3.44}$$

Representing the solution v of (3.10) in the form of the double series

$$v(x,y) = \sum_{m,n=1}^{\infty} v_{mn} e_{mn}(x,y) \tag{3.45}$$

we found

$$v_{mn} = \frac{\varphi_{mn}}{\lambda_{mn} - a}. \tag{3.46}$$

Next, we seek the solution of (3.11) in the form

$$u(x,y) = \sum_{m,n=1}^{\infty} u_{mn} e_{mn}(x,y). \tag{3.47}$$

Then from (3.45) we find

$$u_{mn} = \frac{v_{mn}}{\lambda_{mn}}. \tag{3.48}$$

From the definition of the operator $\|A\|$ by (3.9)–(3.11) we have

$$\begin{aligned}(A\varphi,\varphi) = (u,\varphi) &= \left(\sum_{m,n=1}^{\infty} u_{mn}e_{mn}, \sum_{m,n=1}^{\infty}\varphi_{mn}e_{mn}\right)\\ &= \sum_{m,n=1}^{\infty} u_{mn}\varphi_{mn} = \sum_{m,n=1}^{\infty}\frac{|\varphi_{mn}|^2}{\lambda_{mn}(\lambda_{mn}-a)}\\ &= \sum_{m,n=1}^{\infty}\frac{|\varphi_{mn}|^2}{\left[\pi^2\left(m^2l_1^{-2}+n^2l_2^{-2}\right)\right]\left[\pi^2\left(m^2l_1^{-2}+n^2l_2^{-2}\right)+a\right]}\\ &\le \sum_{m,n=1}^{\infty}\frac{|\varphi_{mn}|^2}{\pi^2\left(l_1^{-2}+l_2^{-2}\right)\left[\pi^2\left(l_1^{-2}+l_2^{-2}\right)+a\right]}\end{aligned} \tag{3.49}$$

due to the orthogonality of the system $\{e_{mn}\}$ and (3.46), (3.48), and (3.42). Thus, there holds the estimate

$$(A\varphi,\varphi) \le \frac{1}{\pi^2\left(l_1^{-2}+l_2^{-2}\right)\left[\pi^2\left(l_1^{-2}+l_2^{-2}\right)+a\right]}(\varphi,\varphi). \tag{3.50}$$

The sign of equality occurs for $\varphi = e_{11}(x,y)$. Since A is shown to be symmetric and positive in $L_2(\Omega)$ it follows:

$$\|A\| = \sup_{\varphi\neq 0}\frac{(A\varphi,\varphi)}{(\varphi,\varphi)} = \frac{1}{\pi^2\left(l_1^{-2}+l_2^{-2}\right)\left[\pi^2\left(l_1^{-2}+l_2^{-2}\right)+a\right]}. \tag{3.51}$$

4. Numerical Realization of the Iterative Method

In the previous section we proposed and investigated an iterative method for problem (1.1)–(1.3) in the case if $a^2 - 4b < 0$. Now we study numerical realization of the method.

For simplicity we consider the case, where the domain Ω is a rectangle, $\Omega = [0,l_1] \times [0,l_2]$ in the plane xOy. In this domain we construct an uniform grid

$$\overline{\Omega}_h = \left\{(x,y),\ x = ih_1,\ y = jh_2,\ 0 \le i \le N_1,\ 0 \le j \le N_2\right\}, \tag{4.1}$$

where $h_1 = l_1/N_1,\ h_2 = l_2/N_2$.

Denote by Ω_h the set of inner nodes, by Γ_h the set of boundary nodes of the grid, and by $v_h, u_h, \ldots$ the grid functions defined on Ω_h.

Now consider a discrete version of the iterative method (3.4)–(3.7) when $\tau_k \equiv \tau$.

(1) Given a starting $\varphi_h^{(0)}$, for example,

$$\varphi_h^{(0)} = 0 \quad \text{in } \Omega_h. \tag{4.2}$$

(2) Knowing $\varphi_h^{(k)}$ on Ω_h $(k = 0, 1, \ldots)$ solve consecutively two difference problems

$$\begin{aligned} \Lambda v_h^{(k)} - a v_h^{(k)} &= f_h + \varphi_h^{(k)} \quad \text{in } \Omega_h, \\ v_h^{(k)} &= g_{2h} \quad \text{on } \Gamma_h, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \Lambda u_h^{(k)} &= v_h^{(k)} \quad \text{in } \Omega_h, \\ u_h^{(k)} &= g_{1h} \quad \text{on } \Gamma_h, \end{aligned} \tag{4.4}$$

where Λ is the discrete Laplace operator,

$$(\Lambda v)_{ij} = \frac{v_{i-1,j} - 2v_{ij} + v_{i+1,j}}{h_1^2} + \frac{v_{i,j-1} - 2v_{ij} + v_{i,j+1}}{h_2^2}. \tag{4.5}$$

(3) Compute the new approximation

$$\varphi_h^{(k+1)} = (1 - \tau_h)\varphi_h^{(k)} - b\tau_h u_h^{(k)} \quad \text{in } \Omega_h. \tag{4.6}$$

It is easy to see that the convergence of the above iterative method is related to the discrete version A_h of the operator A, defined by the formula

$$A_h \varphi_h = u_h, \tag{4.7}$$

where u_h is determined from the difference problems

$$\begin{aligned} \Lambda v_h - a v_h &= \varphi_h \quad \text{in } \Omega_h, \\ v_h &= 0 \quad \text{on } \Gamma_h, \\ \Lambda u_h &= v_h \quad \text{in } \Omega_h, \\ u_h &= 0 \quad \text{on } \Gamma_h. \end{aligned} \tag{4.8}$$

Using the results of the spectral problem for the discrete Laplace operator Λ (see [19]) we find the bounds of A_h:

$$\frac{1}{\beta_2(\beta_2 + a)} I \le A_h \le \frac{1}{\beta_1(\beta_1 + a)} I, \tag{4.9}$$

where

$$\begin{aligned} \beta_1 &= \frac{4}{h_1^2}\sin^2\frac{\pi h_1}{2l_1} + \frac{4}{h_2^2}\sin^2\frac{\pi h_2}{2l_2}, \\ \beta_2 &= \frac{4}{h_1^2}\cos^2\frac{\pi h_1}{2l_1} + \frac{4}{h_2^2}\cos^2\frac{\pi h_2}{2l_2}. \end{aligned} \tag{4.10}$$

Table 1: Convergence of the method in Example 4.3 for $a = 0$.

b	Grid 65×65		Grid 129×129		Grid 257×257		$E1/E2$	$E2/E3$
	k	$E1$	k	$E2$	k	$E3$		
0.3	5	$3.7378e-4$	5	$9.3527e-5$	5	$2.3477e-5$	3.9965	3.9838
0.7	6	$3.4125e-4$	6	$8.4859e-5$	6	$2.0771e-5$	4.0214	4.0855
1.0	7	$3.2179e-4$	7	$8.0793e-5$	7	$2.0550e-5$	3.9829	3.9315
1.5	8	$2.9118e-4$	8	$7.2101e-5$	8	$1.7337e-5$	4.0385	4.1588
2.0	9	$2.6904e-4$	9	$6.8215e-5$	9	$1.8013e-5$	3.9440	3.7870
2.5	10	$2.4561e-4$	10	$6.0247e-5$	10	$1.3909e-5$	4.0767	4.3315

Therefore, for the operator B_h, the discrete version of B, we obtain the estimate

$$\gamma_1^h I \le B_h \le \gamma_2^h I, \tag{4.11}$$

where

$$\gamma_1^h = 1 + \frac{b}{\beta_2(\beta_2 + a)}, \qquad \gamma_2^h = 1 + \frac{b}{\beta_1(\beta_1 + a)}. \tag{4.12}$$

Hence, we choose

$$\tau_h = \frac{2}{\gamma_1^h + \gamma_2^h}, \tag{4.13}$$

which is the optimal parameter of the iterative process (4.2)–(4.6).

Now we study the deviation of $u_h^{(k)}$ from $u^{(k)}$ obtained by the iterative process (3.4)–(3.7). In the future for short we will write $\|\cdot\|$ instead of $\|\cdot\|_\infty$.

Proposition 4.1. *For any $k = 0, 1, \ldots$ there holds the estimate*

$$\left\|\varphi_h^{(k)} - \varphi^{(k)}\right\| = O\left(h^2\right), \qquad \left\|u_h^{(k)} - u^{(k)}\right\| = O\left(h^2\right), \tag{4.14}$$

where $h^2 = h_1^2 + h_2^2$, $u^{(k)}$, $\varphi^{(k)}$ are computed by the process (3.4)–(3.7) *and $u_h^{(k)}$, $\varphi_h^{(k)}$ are computed by* (4.2)–(4.6).

Proof. We shall prove this proposition by induction in k.

For $k = 0$ we have $\|\varphi_h^{(0)} - \varphi^{(0)}\| = 0$ and the second estimate in (4.14) is valid due to the the second-order accuracy of the iterative schemes (4.3) and (4.4) for the problems (3.5) and (3.6) (see [21]).

Now suppose (4.14) is valid for $k - 1 \ge 0$. We shall show that it also is valid for k. For this purpose we recall that $\varphi^{(k)}$ is computed by the formula

$$\varphi^{(k)} = (1 - \tau)\varphi^{(k-1)} - b\tau u^{(k-1)} \quad \text{on } \Omega, \tag{4.15}$$

Table 2: Convergence of the method in Example 4.3 for $a = 0.5$.

b	Grid 65×65		Grid 129×129		Grid 257×257		$E1/E2$	$E2/E3$
	k	$E1$	k	$E2$	k	$E3$		
0.3	5	$3.7378e-4$	5	$8.5291e-5$	5	$2.1354e-5$	3.9990	3.9941
0.7	6	$3.4125e-4$	6	$7.9097e-5$	6	$1.9648e-5$	4.0069	4.0257
1.0	7	$3.2179e-4$	7	$7.5415e-5$	7	$1.8938e-5$	3.9959	3.9822
1.5	8	$2.9118e-4$	8	$6.9316e-5$	8	$1.7184e-5$	4.0086	4.0338
2.0	9	$2.6904e-4$	9	$6.4782e-5$	9	$1.6374e-5$	3.9892	3.9564
2.5	9	$2.4561e-4$	9	$6.1521e-5$	9	$1.6340e-5$	3.9378	3.7651

Table 3: Convergence of the method in Example 4.3 for $a = 1$.

b	Grid 65×65		Grid 129×129		Grid 257×257		$E1/E2$	$E2/E3$
	k	$E1$	k	$E2$	k	$E3$		
0.3	5	$3.1879e-4$	5	$7.9702e-5$	5	$1.9938e-5$	3.9998	3.9975
0.7	6	$2.9968e-4$	6	$7.4868e-5$	6	$1.8672e-5$	4.0028	4.0096
1.0	6	$2.8644e-4$	6	$7.1267e-5$	6	$1.7480e-5$	4.0193	4.0771
1.5	7	$2.6823e-4$	7	$6.7405e-5$	7	$1.7204e-5$	3.9794	3.9180
2.0	8	$2.5061e-4$	8	$6.2346e-5$	8	$1.5283e-5$	4.0197	4.0794
2.5	9	$2.3658e-4$	9	$5.9385e-5$	9	$1.5090e-5$	3.9838	3.9354

where

$$\tau = \frac{2}{(2 + b\|A\|)} \tag{4.16}$$

and $\varphi_h^{(k)}$ is computed by the formula

$$\varphi_h^{(k)} = (1 - \tau_h)\varphi_h^{(k-1)} - b\tau_h u_h^{(k-1)} \quad \text{on } \Omega_h, \tag{4.17}$$

τ_h being given by (4.13) and (4.12).

From (4.10)–(4.13), (4.16), and (3.51) it is possible to obtain the estimate

$$\tau_h = \tau + O\left(h^2\right). \tag{4.18}$$

Next, subtracting (4.15) from (4.17) and taking into account the above formula we get

$$\varphi_h^{(k)} - \varphi^{(k)} = (1 - \tau)\left(\varphi_h^{(k-1)} - \varphi^{(k-1)}\right) + \tau b\left(u_h^{(k-1)} - u^{(k-1)}\right) + O\left(h^2\right). \tag{4.19}$$

By the assumptions of the induction

$$\left\|u_h^{(k-1)} - u^{(k-1)}\right\| = O\left(h^2\right), \qquad \left\|\varphi_h^{(k-1)} - \varphi^{(k-1)}\right\| = O\left(h^2\right) \tag{4.20}$$

Table 4: Convergence of the method in Example 4.4 for $a = 0$.

b	Grid 65 × 65		Grid 129 × 129		Grid 257 × 257		$E1/E2$	$E2/E3$
	k	$E1$	k	$E2$	k	$E3$		
0.3	6	5.4815e – 4	6	1.3703e – 4	6	3.4247e – 5	4.0002	4.0012
0.7	7	5.0136e – 4	7	1.2555e – 4	7	3.1525e – 5	3.9933	3.9826
1.0	8	4.7101e – 4	8	1.1763e – 4	8	2.9257e – 5	4.0042	4.0206
1.5	9	4.2909e – 4	9	1.0770e – 4	9	2.7341e – 5	3.9841	3.9391
2.0	10	3.9191e – 4	10	9.7264e – 5	10	2.3582e – 5	4.0293	4.1245
2.5	11	3.6420e – 4	11	9.2141e – 5	11	2.4078e – 5	3.9526	3.8268

Table 5: Convergence of the method in Example 4.4 for $a = 0.5$.

b	Grid 65 × 65		Grid 129 × 129		Grid 257 × 257		$E1/E2$	$E2/E3$
	k	$E1$	k	$E2$	k	$E3$		
0.3	5	5.1875e – 4	5	1.2985e – 4	5	3.2589e – 5	3.9950	3.9845
0.7	7	4.8209e – 4	7	1.2061e – 4	7	3.0184e – 5	3.9971	3.9958
1.0	7	4.5846e – 4	7	1.1495e – 4	7	2.9067e – 5	3.9883	3.9547
1.5	8	4.2213e – 4	8	1.0498e – 4	8	2.5687e – 5	4.0211	4.0869
2.0	9	3.9362e – 4	9	9.9099e – 5	9	2.5458e – 5	3.9720	3.8926
2.5	10	3.6555e – 4	10	9.0709e – 5	10	2.1945e – 5	4.0299	4.1335

from (4.19) it follows $\|\varphi_h^{(k)} - \varphi^{(k)}\| = O(h^2)$. Now, having in mind this estimate due to the second-order approximation of the difference operators in (4.3) and (4.4) we get the second estimate in (4.14). Thus, the proof of the proposition is complete. □

In realization of the discrete iterative process (4.2)–(4.6) we shall stop iterations when $\|u_h^{(k)} - u_h^{(k-1)}\| < \mathrm{TOL}$, where TOL is a some given accuracy. Then for the total error of the discrete solution $u_h^{(k)}$ there there holds the following theorem.

Theorem 4.2. *For the total error of the discrete solution $u_h^{(k)}$ from the exact solution u of the original problem* (1.1)–(1.3) *there holds the estimate*

$$\left\|u_h^{(k)} - u\right\| \leq TOL + C_2 h^2 + C_1 \rho_0^{k-1}, \tag{4.21}$$

where C_1, C_2 are some constants and ρ_0 is the number in (3.36).

Proof. We have the following estimate

$$\left\|u_h^{(k)} - u\right\| \leq \left\|u_h^{(k)} - u_h^{(k-1)}\right\| + \left\|u_h^{(k-1)} - u^{(k-1)}\right\| + \left\|u^{(k-1)} - u\right\|. \tag{4.22}$$

Since the space $H^4(\Omega)$ is continuously embedded to the space $C(\Omega)$ (see [20]) from (3.36) we have

$$\left\|u^{(k-1)} - u\right\| \leq C_1 \rho_0^{k-1} \tag{4.23}$$

for some constant C_1. Now using Proposition 4.1 and the above estimate, from (4.22) we obtain (4.21). Thus, the theorem is proved. □

Table 6: Convergence of the method in Example 4.4 for $a = 1$.

b	Grid 65 × 65		Grid 129 × 129		Grid 257 × 257		$E1/E2$	$E2/E3$
	k	$E1$	k	$E2$	k	$E3$		
0.3	5	$4.9843e-4$	5	$1.2472e-4$	5	$3.1232e-5$	3.9964	3.9933
0.7	6	$4.6839e-4$	6	$1.1696e-4$	6	$2.9067e-5$	4.0047	4.0238
1.0	7	$4.4871e-4$	7	$1.1229e-4$	7	$2.8174e-5$	3.9960	3.9856
1.5	8	$4.1852e-4$	8	$1.0449e-4$	8	$2.5970e-5$	4.0054	4.0235
2.0	9	$3.9278e-4$	9	$9.8369e-5$	9	$2.4760e-5$	3.9929	3.9729
2.5	9	$3.7068e-4$	9	$9.3636e-5$	9	$2.4338e-5$	3.9587	3.8473

Table 7: Convergence of the method in Example 4.5 for grid 65 × 65.

b	$a = 0$		$a = 0.5$		$a = 1$	
	k	error	k	error	k	error
0.3	5	$7.8052e-9$	5	$3.7143e-9$	5	$1.9460e-9$
0.7	6	$1.7710e-8$	5	$2.3933e-7$	5	$1.2646e-7$
1.0	6	$1.3959e-7$	6	$5.8675e-8$	6	$2.7519e-8$
1.5	7	$1.0251e-7$	7	$3.8079e-8$	6	$2.8517e-7$
2.0	7	$6.6466e-7$	7	$2.5155e-7$	7	$1.0720e-7$
2.5	8	$3.2038e-7$	8	$1.0795e-7$	7	$4.5822e-7$

Below we report the results of some numerical examples for testing the convergence of the iterative method. In all examples we test the iterative method for some values of a and b with TOL $= 10^{-5}$. The results of convergence of the method are given in tables, where k is the number of iterations, E is the error of approximate solution $u_h^{(k)}$, $E = \|u_h^{(k)} - u\|$.

Example 4.3. We take an exact solution $u = \sin x \sin y$ in the rectangle $[0, \pi] \times [0, \pi]$ and corresponding boundary conditions. The right-hand side of (1.1) in this case is $f = (4 + 2a + b) \sin x \sin y$.

The results of convergence in the case of the uniform grids including 65×65, 129×129, and 257×257 nodes for $a = 0$, $a = 0.5$ and $a = 1$ are presented in Tables 1, 2, and 3, respectively.

Example 4.4. We take an exact solution $u = x \sin y + y \sin x$ in the rectangle $[-\pi, \pi] \times [-\pi, \pi]$ with corresponding boundary conditions. The the right-hand side of (1.1) is $f = (1 + a + b)(x \sin y + y \sin x)$.

The results of convergence in the case of the grids including 65 × 65, 129 × 129, and 257 × 257 nodes for $a = 0$, $a = 0.5$, and $a = 1$ are presented in Tables 4, 5, and 6, respectively.

In the two above examples the grid step sizes are $h = \pi/64$, $\pi/128$, $\pi/256$. Therefore, $h^2 = 0.0024$, $6.0239e-4$, $1.5060e-4$. According to the estimate (4.21) the total error of the discrete approximate solution depends on h^2. The columns $E1, E2, E3$ in Tables 1–6 show this fact. It is interesting to notice that in these tables $E1/E2, E2/E3 \approx 4$ and $(64 \times 64)/(128 \times 128) = (128 \times 128)/(256 \times 256) = 4$. It means that if the number of grid nodes increases in 4 times then it is expected that the accuracy of the approximate solution increases in the same times. From the tables we also observe that the number of iterations increases with the increase of the parameter b for fixed values of parameter a. This confirms Remark 3.4. We also remark that for each pair of the parameters a and b the number of iterations for achieving an accuracy corresponding to the grid step size (or density of grid) does not depend on the grid step size.

Table 8: Convergence of the method in Example 4.5 for grid 129 × 129.

b	$a = 0$		$a = 0.5$		$a = 1$	
	k	error	k	error	k	error
0.3	5	$7.7939e-9$	5	$3.7094e-9$	5	$1.9437e-9$
0.7	6	$1.7679e-8$	5	$2.3901e-7$	5	$1.2630e-7$
1.0	6	$1.3935e-7$	6	$5.8579e-8$	6	$2.7476e-8$
1.5	7	$1.0231e-7$	7	$3.8009e-8$	6	$2.8474e-7$
2.0	7	$6.6338e-7$	7	$2.5109e-7$	7	$1.0701e-7$
2.5	8	$3.1968e-7$	8	$1.0773e-7$	7	$4.5744e-7$

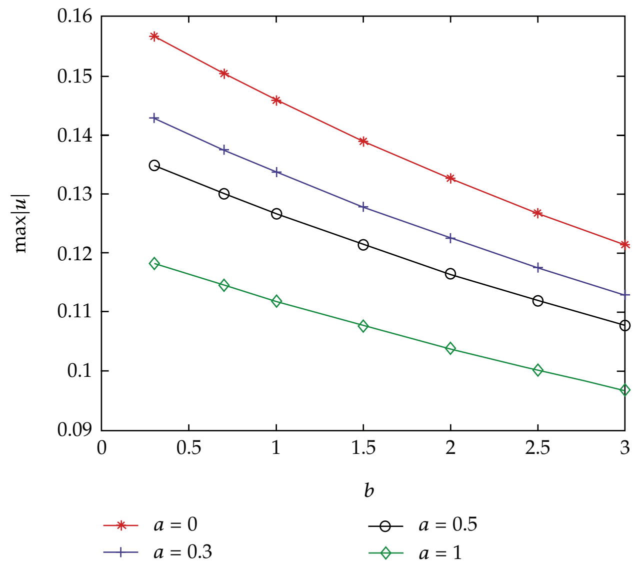

Figure 1: Variation of the value of u in the middle point with b for several values of a.

In general the error of discrete approximate solution strongly depends on the step size of grid. So, it is not expected to get an approximate solution of higher accuracy on a grid of low density. However, in some exceptional cases we can obtain very accurate approximate solution on sparse grid. Below is an example showing this fact.

Example 4.5. We take an exact solution $u = (x^2 - 4)(y^2 - 1)$ in the rectangle $[-2, 2] \times [-1, 1]$. The the right-hand side of (1.1) is $f = 8 - 2a(x^2 + y^2 - 5) + b(x^2 - 4)(y^2 - 1)$.

The results of convergence in the case of the grids including 65×65 and 129×129 nodes are presented in Tables 7 and 8, respectively.

From Tables 7 and 8 we see a high accuracy even on the grid 65×65. The reason of this fact is that for quadratic function the approximation error of the central difference scheme is zero for any grid step size.

The above three numerical examples demonstrate the fast convergence of the iterative method (3.4)–(3.7) for problem (1.1)–(1.3).

Below, we consider an example for examining the variation of the solution of Prob. (1.1)–(1.3) in dependence of the parameters a and b.

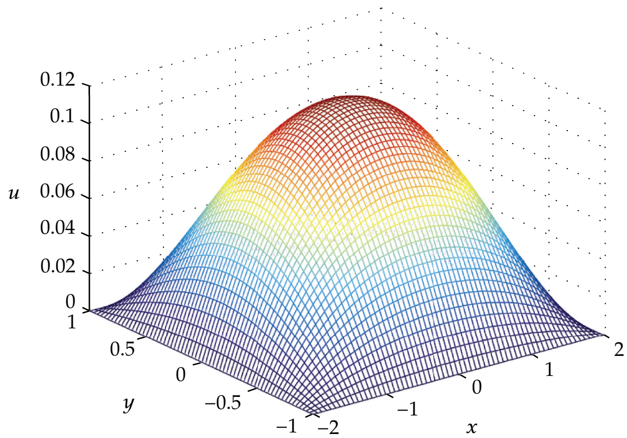

Figure 2: Solution in the case of $a = 0.5$ and $b = 1$.

Example 4.6. We take the the right-hand side of (1.1) $f = 1$ and the boundary conditions $g_1 = g_2 = 0$ and use the proposed method for finding approximate solution for different values of a and b. In all experiments we use the grid of 65×65 nodes in the domain $\Omega = [-2, 2] \times [-1, 1]$ and TOL $= 10^{-5}$. It turns out that the number of iterations in all cases of a and b does not exceed 7 and the value of the solution at any fixed point is decreasing with the growth of a and b. This fact is obvious from Figure 1. The graph of the solution in the case of $a = 0.5$ and $b = 1$ is given in Figure 2.

5. Concluding Remark

In the paper an iterative method was proposed for reducing the second problem for biharmonic-type equation to a sequence of Dirichlet problems for second-order equations. The convergence of the method was proved. In the case when the computational domain is a rectangle the optimal iterative parameter was given. Several numerical examples in this case show fast convergence of the method. When the computational domain consists of rectangles the proposed iterative method can be applied successfully if combining with the domain decomposition method.

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