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Research Article

Common Fixed Point Theorems in a New Fuzzy Metric Space

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We generalize the Hausdorff fuzzy metric in the sense of Rodríguez-López and Romaguera, and we introduce a new M_∞-fuzzy metric, where M_∞-fuzzy metric can be thought of as the degree of nearness between two fuzzy sets with respect to any positive real number. Moreover, under ϕ-contraction condition, in the fuzzy metric space, we give some common fixed point theorems for fuzzy mappings.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. After that, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and application [2, 3]. In the theory of fuzzy topological spaces, one of the main problems is to obtain an appropriate and consistent notion of fuzzy metric space. This problem was investigated by many authors [4–13] from different points of view. George and Veeramani's fuzzy metric space [6] has been widely accepted as an appropriate notion of metric fuzziness in the sense that it provides rich topological structures which can be obtained, in many cases, from classical theorems. Further, it is necessary to mention that this fuzzy metric space has very important application in studying fixed point theorems for contraction-type mappings [7, 14–16]. Besides that, a number of metrics are used on subspaces of fuzzy sets. For example, the sendograph metric [17–19] and the d_∞-metric for fuzzy sets [20–25] induced by the Hausdorff-Pompeiu metric have been studied most frequently, where d_∞-metric is an ordinary metric between two fuzzy sets. Combining fuzzy metric (in the sense of George and Veeramani) and Hausdorff-Pompeiu metric, Rodríguez-Lópezand Romaguera [26] construct a Hausdorff fuzzy metric, where Hausdorff fuzzy metric can be thought of as the degree of nearness between two crisp nonempty compact sets with respect to any positive real number.

In this present investigation, considering the Hausdorff-Pompeiu metric and theories on fuzzy metric spaces (in the sense of George and Veeramani) together, we study the degree of nearness between two fuzzy sets as a natural generalization of the degree of nearness between two crisp sets, in turn, it helps in studying new problems in fuzzy topology. Based on the Hausdorff fuzzy metric H_M, we introduce a suitable notion for the M_∞-fuzzy metric on the fuzzy sets whose λ-cut are nonempty compact for each $\lambda \in [0,1]$. In particular, we explore several properties of M_∞-fuzzy metric. Then, under ϕ-contraction condition, we give some common fixed point theorems in the fuzzy metric space on fuzzy sets.

2. Preliminaries

According to [27], a binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological semigroups with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 2.1 (see [6]). The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $t, s > 0$:

(i) $M(x,y,t) > 0$;

(ii) $M(x,y,t) = 1$ if and only if $x = y$;

(iii) $M(x,y,t) = M(y,x,t)$;

(iv) $M(x,z,t+s) \ge M(x,y,t) * M(z,y,s)$;

(v) $M(x,y,-) : (0,\infty) \to [0,1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, it will be said that $(M, *)$ is a fuzzy metric on X.

A simply but useful fact [7] is that $M(x,y,-)$ is nondecreasing for all $x, y \in X$. Let (X, d) be a metric space. Denote by $a \cdot b$ the usual multiplication for all $a, b \in [0,1]$, and let M_d be the fuzzy set defined on $X \times X \times (0,\infty)$ by

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}. \tag{2.1}$$

Then, $(X, M_d, \cdot)$ is a fuzzy metric space, and $(M_d, \cdot)$ is called the standard fuzzy metric induced by d [8].

George and Veeramani [6] proved that every fuzzy metric $(M, *)$ on X generates a topology τ_M on X which has a base the family of open sets of the form:

$$\{B_M(x,\varepsilon,t) : x \in X,\ 0 < \varepsilon < 1,\ t > 0\}, \tag{2.2}$$

where $B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1 - \varepsilon\}$ for all $\varepsilon \in (0,1)$ and $t > 0$. They proved that (X, τ_M) is a Hausdorff first countable topological space. Moreover, if (X, d) is a metric space, then the topology generated by d coincides with the topology τ_{M_d} generated by the induced fuzzy metric $(M_d, *)$ (see [8]).

Lemma 2.2 (see [6]). *Let $(X, M, *)$ be a fuzzy metric space and let τ be the topology induced by the fuzzy metric. Then, for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, $x_n \to x$ if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$.*

Definition 2.3 (see [6]). A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy sequence if and only if for each $1 > \varepsilon > 0$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. A fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Definition 2.4 (see [13]). Let A be a nonempty subset of a fuzzy metric space $(X, M, *)$. For $a \in X$ and $t > 0$, $M(a, A, t) = \sup\{M(a, y, t) \mid y \in A,\ t > 0\}$.

Lemma 2.5 (see [28]). *Let G be a set and let $\{G_\alpha : \alpha \in [0,1]\}$ be a family of subsets of G such that*

(1) $G_0 = G$;

(2) $\alpha \leq \beta$ *implies* $G_\beta \subseteq G_\alpha$;

(3) $\alpha_1 \leq \alpha_2 \leq \cdots$, $\lim_{n\to\infty}\alpha_n = \alpha$ *implies* $G_\alpha = \bigcap_{n=1}^{\infty} G_{\alpha_n}$.

Then, the function $\varphi : G \to [0,1]$ defined by $\varphi(x) = \sup\{\alpha \in [0,1] : x \in G_\alpha\}$ has the property that $\{x \in G : \varphi(x) \geq \alpha\} = G_\alpha$ for every $\alpha \in [0,1]$.

Next, we recall some pertinent concepts on Hausdorff fuzzy metric. Denote by $C_0(X)$ the set of nonempty closed and bounded subsets of a metric space (X, d). It is well known (see, e.g., [29]) that the function H_d defined on $C_0(X) \times C_0(X)$ by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\}, \tag{2.3}$$

for all $A, B \in C_0(X)$, is a metric on $C_0(X)$ called the Hausdorff-Pompeiu metric. In [30], it is proved that the metric $(C(X), H)$ is complete provided X is complete.

Let $C(X)$ be the set of all nonempty compact subsets of a fuzzy metric space $(X, M, *)$, $A, B \in C(X)$, $t > 0$, according to [26], the Hausdorff fuzzy metric H_M on $C(X) \times C(X) \times (0, \infty)$ is defined as

$$\begin{aligned} H_M(A, B, t) &= \min\left\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(A, b, t)\right\} \\ &= \min\{\rho(A, B, t), \rho(B, A, t)\}, \end{aligned} \tag{2.4}$$

where $\rho(A, B, t) = \inf_{a\in A} M(a, B, t)$, and $(H_M, *)$ is a fuzzy metric on $C(X)$. It is shown that $\rho(A, B, t) = 1$ if and only if $A \subseteq B$, and $H_M(A, B, t) = 1$ if and only if $A = B$.

Lemma 2.6 (see [26]). *Let $(X, M, *)$ be a fuzzy metric space. Then, $(C(X), H_M, *)$ is complete if and only if $(X, M, *)$ is complete.*

Lemma 2.7 (see [26]). *Let (X, d) be a metric space. Then, the Hausdorff fuzzy metric $(H_{M_d}, \cdot)$ of the standard fuzzy metric $(M_d, \cdot)$ coincides with standard fuzzy metric $(M_{H_d}, \cdot)$ of the Hausdorff metric H_d on $C(X)$.*

3. On M_∞-Fuzzy Metric

Let $(X, M, *)$ be a fuzzy metric space. Denote by $\mathcal{C}(X)$ the totality of fuzzy sets:

$$\mu : X \longrightarrow [0,1] = I, \tag{3.1}$$

which satisfy that, for each $\lambda \in I$, the λ-cut of μ,

$$[\mu]_\lambda = \{x \in X : \mu(x) \geq \lambda\}, \tag{3.2}$$

is nonempty compact in X.

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space. The M_∞-fuzzy metric between two fuzzy sets is induced by the Hausdorff fuzzy metric H_M as

$$M_\infty(\mu_1, \mu_2, t) = \min\{\rho_\infty(\mu_1, \mu_2, t), \rho_\infty(\mu_2, \mu_1, t)\}, \tag{3.3}$$

where $\mu_1, \mu_2 \in \mathcal{C}(X)$, $t > 0$, and

$$\rho_\infty(\mu_1, \mu_2, t) = \inf_{0\leq\lambda\leq1} \rho([\mu_1]_\lambda, [\mu_2]_\lambda, t) \tag{3.4}$$

is the fuzzy separation of μ_1 from μ_2.

Lemma 3.2. *Let $(X, M, *)$ be a fuzzy metric space, $\mu_1, \mu_2, \mu_3 \in \mathcal{C}(X)$, $s, t > 0$. Then one has*

(1) $M_\infty(\mu_1, \mu_2, t) \in (0,1]$,

(2) $M_\infty(\mu_1, \mu_2, t) = M_\infty(\mu_2, \mu_1, t)$,

(3) $\rho_\infty(\mu_1, \mu_2, t) = 1$ *if and only if* $\mu_1 \subseteq \mu_2$,

(4) $M_\infty(\mu_1, \mu_2, t) = 1$ *if and only if* $\mu_1 = \mu_2$,

(5) *if* $\mu_1 \subseteq \mu_2$, *then* $\rho_\infty(\mu_1, \mu_3, t + s) \geq M_\infty(\mu_2, \mu_3, t)$,

(6) $\rho_\infty(\mu_1, \mu_3, t + s) \geq M_\infty(\mu_1, \mu_2, t) * \rho_\infty(\mu_2, \mu_3, s)$,

(7) $M_\infty(\mu_1, \mu_3, t + s) \geq M_\infty(\mu_1, \mu_2, t) * M_\infty(\mu_2, \mu_3, s)$,

(8) $M_\infty(\mu_1, \mu_2, -) : (0, \infty) \to [0,1]$ *is continuous.*

Proof. For (1), by the definition of the λ-cut $[\mu_1]_\lambda$, for every $\lambda \in I$, $[\mu_1]_\lambda$ is nonempty compact in X. By the theorem of nested intervals, there exists a point a_0 in $[\mu_1]_\lambda$ for every $\lambda \in I$, likewise, there exists a points b_0 in $[\mu_2]_\lambda$ for every $\lambda \in I$. Thus, $M_\infty(\mu_1, \mu_2, t) > 0$. Moreover, it is clear that $A = B \Leftrightarrow H_M(A, B, t) = 1 \Leftrightarrow M_\infty(\mu_1, \mu_2, t) = 1$.

For (2), it is clear that $M_\infty(\mu_1, \mu_2, t) = M_\infty(\mu_2, \mu_1, t)$.

For (3), since $\rho_\infty(\mu_1, \mu_2, t) = 1$ if and only if $\rho([\mu_1]_\lambda, [\mu_2]_\lambda, t) = 1$ for all $\lambda \in I$, which implies $[\mu_1]_\lambda \subseteq [\mu_2]_\lambda$ for all $\lambda \in I$, we have that $\rho_\infty(\mu_1, \mu_2, t) = 1$ if and only if $\mu_1 \subseteq \mu_2$.

For (4), it follows from (3).

For (5), for every $\lambda \in I$, any $x \in [\mu_1]_\lambda$, $y \in [\mu_2]_\lambda$ and $z \in [\mu_3]_\lambda$, by the proof of Theorem 1 in [26], we have

$$M(x, z, t+s) \geq M(x, y, t) * M(y, z, s) \tag{3.5}$$

with all $x, y, z \in X$, which implies

$$M(x, [\mu_3]_\lambda, t+s) \geq M(x, y, t) * M(y, [\mu_3]_\lambda, s) \tag{3.6}$$

for all $x \in [\mu_1]_\lambda$ and all $y \in [\mu_2]_\lambda$. Since $\mu_1 \subseteq \mu_2$, then $\rho([\mu_1]_\lambda, [\mu_2]_\lambda, s) = 1$. By (iv) of Definition 2.1 and the arbitrariness of x and y, we have

$$\begin{aligned}
\rho([\mu_1]_\lambda, [\mu_3]_\lambda, t+s) &= \inf_{x \in [\mu_1]_\lambda} M(x, [\mu_3]_\lambda, t+s) \\
&\geq \inf_{x \in [\mu_1]_\lambda} M(x, [\mu_2]_\lambda, s) * \inf_{y \in [\mu_2]_\lambda} M(y, [\mu_3]_\lambda, t) \\
&= \rho([\mu_1]_\lambda, [\mu_2]_\lambda, s) * \rho([\mu_2]_\lambda, [\mu_3]_\lambda, t) \\
&= \rho([\mu_2]_\lambda, [\mu_3]_\lambda, t) \\
&\geq H_M([\mu_2]_\lambda, [\mu_3]_\lambda, t),
\end{aligned} \tag{3.7}$$

which implies

$$\inf_{0 \leq \lambda \leq 1} \rho([\mu_1]_\lambda, [\mu_3]_\lambda, t+s) \geq \inf_{0 \leq \lambda \leq 1} H_M([\mu_2]_\lambda, [\mu_3]_\lambda, t). \tag{3.8}$$

Consequently, $\rho_\infty(\mu_1, \mu_3, t+s) \geq M_\infty(\mu_2, \mu_3, t)$.

For (6), for every $\lambda \in I$, by the proof of (5) and (iv) of Definition 2.1, we have

$$\begin{aligned}
\inf_{0 \leq \lambda \leq 1} \rho([\mu_1]_\lambda, [\mu_3]_\lambda, t+s) &\geq \inf_{0 \leq \lambda \leq 1} \{\rho([\mu_1]_\lambda, [\mu_2]_\lambda, t) * \rho([\mu_2]_\lambda, [\mu_3]_\lambda, s)\} \\
&\geq \inf_{0 \leq \lambda \leq 1} \{H_M([\mu_1]_\lambda, [\mu_2]_\lambda, t) * \rho([\mu_2]_\lambda, [\mu_3]_\lambda, s)\}.
\end{aligned} \tag{3.9}$$

Consequently, $\rho_\infty(\mu_1, \mu_3, t+s) \geq M_\infty(\mu_1, \mu_2, t) * \rho_\infty(\mu_2, \mu_3, s)$.

For (7), for every $\lambda \in I$, by the proof of (6), we have

$$\begin{aligned}
\inf_{0 \leq \lambda \leq 1} \rho([\mu_1]_\lambda, [\mu_3]_\lambda, t+s) &\geq \inf_{0 \leq \lambda \leq 1} \{\rho([\mu_1]_\lambda, [\mu_2]_\lambda, t) * \rho([\mu_2]_\lambda, [\mu_3]_\lambda, s)\} \\
&\geq \left\{\inf_{0 \leq \lambda \leq 1} \rho([\mu_1]_\lambda, [\mu_2]_\lambda, t)\right\} * \left\{\inf_{0 \leq \lambda \leq 1} \rho([\mu_2]_\lambda, [\mu_3]_\lambda, s)\right\}.
\end{aligned} \tag{3.10}$$

Similarly, it can be shown that

$$\inf_{0 \leq \lambda \leq 1} \rho([\mu_3]_\lambda, [\mu_1]_\lambda, t+s) \geq \left\{\inf_{0 \leq \lambda \leq 1} \rho([\mu_3]_\lambda, [\mu_2]_\lambda, s)\right\} * \left\{\inf_{0 \leq \lambda \leq 1} \rho([\mu_2]_\lambda, [\mu_1]_\lambda, t)\right\}. \tag{3.11}$$

Hence, $M_\infty(\mu_1, \mu_3, t+s) \geq M_\infty(\mu_1, \mu_2, t) * M_\infty(\mu_2, \mu_3, s)$.

For (8), by the continuity on $(0,\infty)$ of the function $t \mapsto H_M(A,B,t)$, it is clear that $M_\infty(\mu_1,\mu_2,-):(0,\infty) \to [0,1]$ is continuous. □

Theorem 3.3. *Let $(X,M,*)$ be a fuzzy metric space. Then, $(\mathcal{C}(X),M_\infty,*)$ is a fuzzy metric space, where M_∞ is a fuzzy set on the $\mathcal{C}(X)\times\mathcal{C}(X)\times(0,+\infty)$.*

Proof. It is easily proved by Lemma 3.2. □

Example 3.4. Let d be the Euclidean metric on $\mathbb{R}$, and let $A=[a_1,a_2]$ and let $B=[b_1,b_2]$ be two compact intervals. Then, $H_d(A,B)=\max\{|a_1-b_1|,|a_2-b_2|\}$. Let $(\mathbb{R},M_d,*)$ be a fuzzy metric space, where $a*b$ the usual multiplication for all $a,b\in[0,1]$, and M_d is defined on $\mathbb{R}\times\mathbb{R}\times(0,\infty)$ by

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}. \tag{3.12}$$

Denote by $\mathcal{C}(\mathbb{R})$ the totality of fuzzy sets $\mu:\mathbb{R}\to[0,1]$ which satisfy that for each $\lambda\in I$, the λ-cut of $\mu[\mu]_\lambda=\{x\in\mathbb{R}:\mu(x)\geq\lambda\}$ is a nonempty compact interval. For any λ-cuts of fuzzy sets $\mu_1,\mu_2\in\mathcal{C}(\mathbb{R})$ and for all $t>0$, by a simple calculation, we have

$$H_M([\mu_1]_\lambda,[\mu_2]_\lambda,t)=\frac{t}{t+H_d([\mu_1]_\lambda,[\mu_2]_\lambda)}. \tag{3.13}$$

So by Definition 3.1, we get

$$M_\infty(\mu_1,\mu_2,t)=\inf_{0\leq\lambda\leq1}\frac{t}{t+H_d([\mu_1]_\lambda,[\mu_2]_\lambda)}. \tag{3.14}$$

4. Properties of the M_∞-Fuzzy Metric

Definition 4.1. Let $(\mathcal{C}(X),M_\infty,*)$ be a fuzzy metric space. For $t\in(0,+\infty)$, define $B(\mu,r,t)$ with center a fuzzy set $\mu\in\mathcal{C}(X)$ and radius $r, 0<r<1, t>0$ as

$$B(\mu,r,t)=\{\gamma\in\mathcal{C}(X)\mid M_\infty(\mu,\gamma,t)>1-r\}. \tag{4.1}$$

Proposition 4.2. *Every $B(\mu,r,t)$ is an open set.*

Proof. It is identical with the proof in [6]. □

Proposition 4.3. *Let $(\mathcal{C}(X),M_\infty,*)$ be a fuzzy metric space. Define $\tau_{M_\infty}=\{\mathcal{A}\subset\mathcal{C}(X)\mid\mu\in\mathcal{A}$ if and only if there exist $t>0$ and r, $0<r<1$ such that $B(\mu,r,t)\subset\mathcal{A}\}$.*

Then, τ_{M_∞} is a topology on $\mathcal{C}(X)$.

Proof. It is identical with the proof in [6]. □

Definition 4.4. A sequence $\{\mu_n\}$ in a fuzzy metric space $(\mathcal{C}(X),M_\infty,*)$ is a Cauchy sequence if and only if for each $\varepsilon>0$, $t>0$, there exists $n_0\in\mathbb{N}$ such that $M_\infty(\mu_n,\mu_m,t)>1-\varepsilon$ for all $n,m\geq n_0$.

Lemma 4.5. *Let $(\mathcal{C}(X), M_\infty, *)$ be a fuzzy metric space on fuzzy metric M_∞ and let τ be the topology induced by the fuzzy metric M_∞. Then, for a sequence $\{\mu_n\}$ in $\mathcal{C}(X)$, $\mu_n \to \mu$ if and only if $M_\infty(\mu, \mu_n, t) \to 1$ as $n \to \infty$.*

Proof. It is identical with the proof of Theorem 3.11 in [6]. □

Theorem 4.6. *The fuzzy metric space $(\mathcal{C}(X), M_\infty, *)$ is complete provided $(X, M, *)$ is complete.*

Proof. Let $(X, M, *)$ be a complete fuzzy metric space and let a sequence $\{\mu_n, n \geq 1\}$ be a Cauchy sequence in $\mathcal{C}(X)$. Consider a fixed $0 < \lambda < 1$. Then, $\{[\mu_n]_\lambda, n \geq 1\}$ is a Cauchy sequence in $(C(X), H_M, *)$, where $C(X)$ denotes all nonempty compact subsets of $(X, M, *)$.

Since $(C(X), H_M, *)$ is complete by Lemma 2.6, it follows that $[\mu_n]_\lambda \to \mu_\lambda \in C(X)$. Actually, from the definition of M_∞ and the continuity of H_M, it is easy to see that $[\mu_n]_\lambda \to \mu_\lambda$, uniformly in $\lambda \in [0,1]$.

Now, consider the family $\{\mu_\lambda : \lambda \in [0,1]\}$, where $\mu_0 = X$. Take $\lambda \leq \beta$, we have

$$\rho(\mu_\beta, \mu_\lambda, t) \geq \rho\left(\mu_\beta, [\mu_n]_\beta, \frac{t}{3}\right) * \rho\left([\mu_n]_\beta, [\mu_n]_\lambda, \frac{t}{3}\right) * \rho\left([\mu_n]_\lambda, \mu_\lambda, \frac{t}{3}\right). \tag{4.2}$$

Since $[\mu_n]_\beta \subseteq [\mu_n]_\lambda$, it follows that $\rho([\mu_n]_\beta, [\mu_n]_\lambda, t/3) = 1$. Thus, for each $0 < \varepsilon < 1$, $\rho(\mu_\beta, \mu_\lambda, t) \geq \rho(\mu_\beta, [\mu_n]_\beta, t/3) * \rho([\mu_n]_\lambda, \mu_\lambda, t/3)$ if n is large enough. Hence, $\rho(\mu_\beta, \mu_\lambda, t) = 1$, and by Lemma 3.2, we have $\mu_\beta \subseteq \mu_\lambda$.

Now, take $\lambda_n \uparrow$ and $\lim_{n\to\infty}\lambda_n = \lambda$. We have to show that $\mu_\lambda = \bigcap_{n=1}^{\infty} \mu_{\lambda_n}$. It is clear that

$$\mu_\lambda \subseteq \bigcap_{n=1}^{\infty} \mu_{\lambda_n}. \tag{$*$}$$

On the other hand, we have

$$\rho\left(\bigcap_{n=1}^{\infty} \mu_{\lambda_n}, \mu_\lambda, t\right) \geq \rho\left(\bigcap_{n=1}^{\infty} \mu_{\lambda_n}, \bigcap_{n=1}^{\infty} [\mu_j]_{\lambda_n}, \frac{t}{3}\right) * \rho\left(\bigcap_{n=1}^{\infty} [\mu_j]_{\lambda_n}, [\mu_j]_\lambda, \frac{t}{3}\right) * \rho\left([\mu_j]_\lambda, \mu_\lambda, \frac{t}{3}\right), \tag{4.3}$$

for fixed j. However,

$$\rho\left(\bigcap_{n=1}^{\infty} [\mu_j]_{\lambda_n}, [\mu_j]_\lambda, \frac{t}{3}\right) = 1. \tag{4.4}$$

Consequently, for every $0 < \varepsilon < 1$, there exists $0 < \varepsilon_0 < \varepsilon < 1$ such that $(1-\varepsilon_0)*(1-\varepsilon_0)*(1-\varepsilon_0) > 1 - \varepsilon$. For given ε_0, since $[\mu_j]_\lambda \to \mu_\lambda$, there exists j_{ε_0} such that

$$\rho\left(\bigcap_{n=1}^{\infty} \mu_{\lambda_n}, \mu_\lambda, t\right) \geq \rho\left(\bigcap_{n=1}^{\infty} \mu_{\lambda_n}, \bigcap_{n=1}^{\infty} [\mu_j]_{\lambda_n}, \frac{t}{3}\right) * (1 - \varepsilon_0), \tag{4.5}$$

for $j \geq j_{\varepsilon_0}$. Now,

$$\rho\left(\bigcap_{n=1}^{\infty}\mu_{\lambda_n}, \bigcap_{n=1}^{\infty}[\mu_j]_{\lambda_n}, \frac{t}{3}\right) \geq \rho\left(\bigcap_{n=1}^{\infty}\mu_{\lambda_n}, \mu_{\lambda_p}, \frac{t}{9}\right) * \rho\left(\mu_{\lambda_p}, [\mu_j]_{\lambda_n}, \frac{t}{9}\right) * \rho\left([\mu_j]_{\lambda_p}, \bigcap_{n=1}^{\infty}[\mu_j]_{\lambda_n}, \frac{t}{9}\right), \tag{4.6}$$

for any $p \geq 1$. Since $\bigcap_{n=1}^{\infty}\mu_{\lambda_n} \subseteq \mu_{\lambda_p}$, we obtain

$$\rho\left(\bigcap_{n=1}^{\infty}\mu_{\lambda_n}, \bigcap_{n=1}^{\infty}[\mu_j]_{\lambda_n}, t\right) \geq \rho\left(\mu_{\lambda_p}, [\mu_j]_{\lambda_p}, \frac{t}{2}\right) * \rho\left([\mu_j]_{\lambda_p}, \bigcap_{n=1}^{\infty}[\mu_j]_{\lambda_n}, \frac{t}{2}\right). \tag{4.7}$$

Now, $\rho(\mu_{\lambda_p}, [\mu_j]_{\lambda_p}, t/2) > 1-\varepsilon_0$ for $j \geq j_0$ and all $t > 0$. Note that (since the convergence $[\mu_j]_\lambda \to \mu_\lambda$ is uniform in λ) j_0 does not depend on p. Since $\{[\mu_j]_{\lambda_p}, p \geq 1\}$ decreases to $\bigcap_{n=1}^{\infty}[\mu_j]_{\lambda_n}$, if follows that $\rho([\mu_j]_{\lambda_{p_0}}, \bigcap_{n=1}^{\infty}[\mu_j]_{\lambda_n}, t/2) > 1-\varepsilon_0$ for some p_0 (depending on j).

Thus, $\rho(\bigcap_{n=1}^{\infty}\mu_{\lambda_n}, \bigcap_{n=1}^{\infty}[\mu_j]_{\lambda_n}, t/3) \geq (1-\varepsilon_0) * (1-\varepsilon_0)$, if j is large.

Finally, by taking j large enough, we obtain

$$\rho\left(\bigcap_{n=1}^{\infty}\mu_{\lambda_n}, \mu_\lambda, t\right) \geq (1-\varepsilon_0) * (1-\varepsilon_0) * (1-\varepsilon_0) \geq 1-\varepsilon, \tag{4.8}$$

that is,

$$\bigcap_{n=1}^{\infty}\mu_{\lambda_n} \subseteq \mu_\lambda. \tag{$**$}$$

From (4.3) and (4.9), it yields $\bigcap_{n=1}^{\infty}\mu_{\lambda_n} = \mu_\lambda$. Thus, Lemma 2.5 is applicable and there exists $\mu \in \mathcal{C}(X)$ for every $\lambda \in [0,1]$ such that $[\mu_n]_\lambda \to \mu_\lambda$. It remains to show that $\mu_n \to \mu$ in $(\mathcal{C}(X), M_\infty, *)$.

Let $\varepsilon > 0$. Then, since $\{\mu_n\}$ is a Cauchy sequence, there exists n_ε such that $n, m > n_\varepsilon$ implies $M_\infty(\mu_n, \mu_m, t) > 1-\varepsilon$.

Let $n(> n_\varepsilon)$ be fixed. Then,

$$\begin{aligned} H_M([\mu_n]_\lambda, [\mu]_\lambda, t) &= \lim_{m\to\infty} H_M([\mu_n]_\lambda, [\mu_m]_\lambda, t) \geq \overline{\lim_{m\to\infty}} \inf_{0\leq\lambda\leq 1} H_M([\mu_n]_\lambda, [\mu_m]_\lambda, t) \\ &= \overline{\lim_{m\to\infty}} M_\infty(\mu_n, \mu_m, t) > 1-\varepsilon. \end{aligned} \tag{4.9}$$

Thus, $\mu_n \to \mu$ in the M_∞-fuzzy metric. The proof is completed. □

Lemma 4.7. *Let $(X, M, *)$ be a compact fuzzy metric space and compact subsets $A, B \in C(X)$. Then, for each $x \in A$ and $t > 0$, there exists a $y \in B$ such that $M(x, y, t) \geq H_M(A, B, t)$.*

Proof. Suppose there exists a $x_0 \in A$ such that $M(x_0, y, t) < H_M(A, B, t)$ for any $y \in B$ and $t > 0$. Then,

$$\sup_{y\in B} M(x_0, y, t) < H_M(A, B, t), \tag{4.10}$$

that is,

$$\sup_{y\in B} M(x_0, y, t) < \min\left\{\inf_{x\in B}\sup_{y\in A} M(x, y, t), \inf_{x\in A}\sup_{y\in B} M(x, y, t)\right\}. \tag{4.11}$$

So,

$$\sup_{y\in B} M(x_0, y, t) < \inf_{x\in A}\sup_{y\in B} M(x, y, t). \tag{4.12}$$

This is a contradiction with $x \in A$. □

Lemma 4.8. *Let $(X, M, *)$ be a compact fuzzy metric space, $t > 0$ and $A, B \in C(X)$. Then, for any compact set $A_1 \subseteq A$, there exists a compact set $B_1 \subseteq B$ such that $H_M(A_1, B_1, t) \geq H_M(A, B, t)$.*

Proof. Let $C = \{y :$ there exists a $x \in A_1$ such that $M(x, y, t) \geq H_M(A, B, t)\}$ and let $B_1 = C \bigcap B$. For any $x \in A_1 \subseteq A,\ t > 0$, by Lemma 4.7, there exists a $y \in B$ such that

$$M(x, y, t) \geq H_M(A, B, t). \tag{4.13}$$

Thus, $B_1 \neq \emptyset$, moreover, B_1 is compact since it is closed in X and $B_1 \subseteq B$.

Now, for any $x \in A_1$, $t > 0$, there exists a $y \in B_1$ such that

$$M(x, y, t) \geq H_M(A, B, t). \tag{4.14}$$

Thus, we have $M(x, B_1, t) \geq H_M(A, B, t)$, which implies that

$$\rho(A_1, B_1, t) = \inf_{x\in A_1} M(x, B_1, t) \geq H_M(A, B, t). \tag{4.15}$$

Similarly, it can be shown that $\rho(A_1, B_1, t) \geq H_M(A, B, t)$.

Hence, $H_M(A_1, B_1, t) \geq H_M(A, B, t)$. This completes the proof. □

Theorem 4.9. *Let $(X, M, *)$ be a compact fuzzy metric space and $\mu_1, \mu_2 \in \mathcal{C}(X)$, $t > 0$. Then, for any $\mu_3 \in \mathcal{C}(X)$ satisfying $\mu_3 \subseteq \mu_1$, there exists a $\mu_4 \in \mathcal{C}(X)$ such that $\mu_4 \subseteq \mu_2$ and*

$$M_\infty(\mu_3, \mu_4, t) \geq M_\infty(\mu_1, \mu_2, t). \tag{4.16}$$

Proof. Since μ_1, μ_2, and μ_3 are normal, we have $\emptyset \neq [\mu_3]_\lambda \subseteq [\mu_1]_\lambda$ and $\emptyset \neq [\mu_2]_\lambda$ for all $\lambda \in I$. Let

$$C_\lambda = \{y : \text{there exists a } x \in [\mu_3]_\lambda \text{ such that } M(x, y, t) \geq M_\infty(\mu_1, \mu_2, t)\}, \tag{4.17}$$

and let $B_\lambda = C_\lambda \bigcap [\mu_2]_\lambda$. For any $x \in [\mu_3]_\lambda \subseteq [\mu_1]_\lambda$, by Lemma 4.7, there exists a $y \in [\mu_2]_\lambda$ such that

$$M(x, y, t) \geq H_M([\mu_1]_\lambda, [\mu_2]_\lambda, t) \geq M_\infty(\mu_1, \mu_2, t). \tag{4.18}$$

Thus, B_λ is nonempty compact in X, moreover, $B_\lambda \subseteq B\gamma$ if $0 \leq \gamma \leq \lambda \leq 1$.

From the proof of Lemma 4.8, we have

$$H_M([\mu_3]_\lambda, B_\lambda, t) \geq M_\infty(\mu_1, \mu_2, t). \tag{4.19}$$

By Lemma 3.1 in [28], there exists a fuzzy set μ_4 with the property that $[\mu_4]_\lambda = B_\lambda$ for $\lambda \in I$. Since B_λ are nonempty compact for all $\lambda \in I$, we have $\mu_4 \in C(X)$. Consequently,

$$M_\infty(\mu_3, \mu_4, t) \geq M_\infty(\mu_1, \mu_2, t). \tag{4.20}$$

This completes the proof. □

Definition 4.10 (see [24]). Let X, Y be any fuzzy metric space. $\mathcal{F}$ is said to be a fuzzy mapping if and only if $\mathcal{F}$ is a mapping from the space $C(X)$ into $C(Y)$, that is, $\mathcal{F}(\mu) \in C(Y)$ for each $\mu \in C(X)$.

5. Common Fixed Point Theorems in the Fuzzy Metric Space on Fuzzy Sets

Theorem 5.1. *Let $(X, M, *)$ be a compact fuzzy metric space and let $\{\mathcal{F}_i\}_{i=1}^\infty$ be a sequence of fuzzy self-mappings of $C(X)$. Let $\phi : [0,1] \to [0,1]$ be a nondecreasing function satisfying the following condition: ϕ is continuous from the left and*

$$\phi(h) * \phi^2(h) * \cdots * \phi^n(h) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty,\ \forall h \in (0,1], \tag{5.1}$$

where ϕ^n denote the nth iterative function of ϕ. Suppose that for each $\mu_1, \mu_2 \in C(X)$, and for arbitrary positive integers i and j, $i \neq j$, $t > 0$,

$$M_\infty(\mathcal{F}_i(\mu_1), \mathcal{F}_j(\mu_2), t) \geq \phi\Bigg(\inf\bigg\{M_\infty(\mu_1, \mu_2, t), \rho_\infty(\mu_1, \mathcal{F}_i(\mu_1), 2t), \rho_\infty(\mu_2, \mathcal{F}_j(\mu_2), 2t), \frac{1}{2}[\rho_\infty(\mu_2, \mathcal{F}_i(\mu_1), 4t) + \rho_\infty(\mu_1, \mathcal{F}_j(\mu_2), 4t)]\bigg\}\Bigg), \tag{5.2}$$

then there exists $\mu_* \in C(X)$ such that $\mu_* \subseteq \mathcal{F}_i(\mu_*)$ for all $i \in Z_+$.

Proof. Let $\mu_0 \in C(X)$ and $\mu_1 \subseteq \mathcal{F}_1(\mu_0)$. By Theorem 4.9, for any $t > 0$, there exists $\mu_2 \in C(X)$ such that $\mu_2 \subseteq \mathcal{F}_2(\mu_1)$ and

$$M_\infty(\mu_1, \mu_2, t) \geq M_\infty(\mathcal{F}_1(\mu_0), \mathcal{F}_2(\mu_1), t). \tag{5.3}$$

Again by Theorem 4.9, for any $t > 0$, we can find $\mu_3 \in \mathcal{C}(X)$ such that $\mu_3 \subseteq \mathcal{F}_3(\mu_2)$ and

$$M_\infty(\mu_2, \mu_3, t) \geq M_\infty(\mathcal{F}_2(\mu_1), \mathcal{F}_3(\mu_2), t). \tag{5.4}$$

By induction, we produce a sequence $\{\mu_n\}$ of points of $\mathcal{C}(X)$ such that

$$\begin{aligned} &\mu_{n+1} \subseteq \mathcal{F}_{n+1}(\mu_n), \qquad n = 0, 1, 2, \ldots; \\ &M_\infty(\mu_n, \mu_{n+1}, t) \geq M_\infty(\mathcal{F}_n(\mu_{n-1}), \mathcal{F}_{n+1}(\mu_n), t). \end{aligned} \tag{5.5}$$

Now, we prove that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{C}(X)$. In fact, for arbitrary positive integer n, by the inequality (5.2), Lemma 3.2, and the formula (5.5), we have

$$\begin{aligned} M_\infty(\mu_n, \mu_{n+1}, t) &\geq M_\infty(\mathcal{F}_n(\mu_{n-1}), \mathcal{F}_{n+1}(\mu_n), t) \\ &\geq \phi\Bigg(\inf\Bigg\{M_\infty(\mu_{n-1}, \mu_n, t), \rho_\infty(\mu_{n-1}, \mathcal{F}_n(\mu_{n-1}), 2t), \rho_\infty(\mu_n, \mathcal{F}_{n+1}(\mu_2), 2t), \\ &\qquad \frac{1}{2}[\rho_\infty(\mu_{n-1}, \mathcal{F}_{n+1}(\mu_n), 4t) + \rho_\infty(\mu_n, \mathcal{F}_n(\mu_{n-1}), 4t)]\Bigg\}\Bigg), \\ &\geq \phi\Bigg(\inf\Bigg\{M_\infty(\mu_{n-1}, \mu_n, t), M_\infty(\mu_{n-1}, \mu_n, t), \\ &\qquad M_\infty(\mu_n, \mu_{n+1}, 2t), \frac{1}{2}[M_\infty(\mu_{n-1}, \mu_{n+1}, 2t) + 1]\Bigg\}\Bigg), \\ &\geq \phi\Bigg(\inf\Bigg\{M_\infty(\mu_{n-1}, \mu_n, t), M_\infty(\mu_n, \mu_{n+1}, t), \\ &\qquad M_\infty(\mu_n, \mu_{n+1}, 2t), \frac{1}{2}[M_\infty(\mu_{n-1}, \mu_n, t) + 1 * M_\infty(\mu_n, \mu_{n+1}, t) + 1]\Bigg\}\Bigg), \end{aligned} \tag{5.6}$$

where $\mu_n \subseteq \mathcal{F}_n(\mu_{n-1})$ implies $\rho_\infty(\mu_n, \mathcal{F}_n(\mu_{n-1}), 2t) = 1$, by (3) of Lemma 3.2. In addition, it is easy to get that $\phi(h) > h$ for all $h \in (0,1)$. In fact, suppose that there exists some $t_0 \in (0,1)$ such that $\phi(h_0) \leq h_0$. Since ϕ is nondecreasing, we have

$$\phi^n(h_0) \leq \phi^{n-1}(h_0) \leq \cdots \leq \phi(h_0) \leq h_0. \tag{5.7}$$

Since $\phi(h) * \phi^2(h) * \cdots * \phi^n(h) \to 1$ as $n \to \infty$, for all $h \in (0,1)$, then we have $\phi^n(h_0) \to 1$ as $n \to \infty$. From the inequality (5.7), we have $1 \leq h_0$. This is a contradiction which implies $\phi(h) > h$ for all $h \in (0,1)$. We can prove that $M_\infty(\mu_{n-1}, \mu_n, t) \leq M_\infty(\mu_n, \mu_{n+1}, t)$. In fact, if $M_\infty(\mu_{n-1}, \mu_n, t) > M_\infty(\mu_n, \mu_{n+1}, t)$, then from the inequality (5.6), we get

$$M_\infty(\mu_n, \mu_{n+1}, t) \geq \phi(M_\infty(\mu_n, \mu_{n+1}, t)) > M_\infty(\mu_n, \mu_{n+1}, t), \tag{5.8}$$

which is a contradiction. Thus, from the inequality (5.6), we have

$$M_\infty(\mu_n, \mu_{n+1}, t) \geq \phi(M_\infty(\mu_{n-1}, \mu_n, t)) \geq \cdots \geq \phi^n(M_\infty(\mu_0, \mu_1, t)). \tag{5.9}$$

Furthermore, for arbitrary positive integers m and k, we have

$$\begin{aligned}1 &\geq M_\infty(\mu_k, \mu_{k+m}, t)\\ &\geq M_\infty\left(\mu_k, \mu_{k+1}, \frac{t}{m}\right) * M_\infty\left(\mu_{k+1}, \mu_{k+2}, \frac{t}{m}\right) * \cdots * M_\infty\left(\mu_{k+m-1}, \mu_{k+m}, \frac{t}{m}\right)\\ &\geq \phi^k\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) * \phi^{k+1}\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) * \cdots * \phi^{k+m-1}\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right),\end{aligned} \tag{5.10}$$

and $\phi(h) * \phi^2(h) * \cdots * \phi^n(h) \to 1$ as $n \to \infty$, for all $h \in (0,1)$, it follows that

$$\phi^k\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) * \phi^{k+1}\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) * \cdots * \phi^{k+m-1}\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) \tag{5.11}$$

is convergent, which implies that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{C}(X)$. Since X is a compact fuzzy metric space, it follows X is complete. By Theorem 4.6, $\mathcal{C}(X)$ is complete. Let $\mu_n \to \mu_*$. Next, we show that $\mu_* \subseteq \mathcal{F}_i(\mu_*)$ for all $i \in Z_+$. In fact, for arbitrary positive integers i and j, $i \neq j$, by Theorem 4.9, we have

$$\begin{aligned}&\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t)\\ &\quad\geq M_\infty\left(\mu_*, \mu_j, \frac{t}{4}\right) * \rho_\infty\left(\mu_j, \mathcal{F}_i(\mu_*), \frac{3t}{4}\right)\\ &\quad\geq M_\infty\left(\mu_*, \mu_j, \frac{t}{4}\right) * M_\infty\left(\mathcal{F}_j(\mu_{j-1}), \mathcal{F}_i(\mu_*), \frac{t}{2}\right)\\ &\quad\geq M_\infty\left(\mu_*, \mu_j, \frac{t}{4}\right) * \phi\left(\inf\left\{M_\infty\left(\mu_{j-1}, \mu_*, \frac{t}{2}\right), \rho_\infty(\mu_{j-1}, \mathcal{F}_j(\mu_{j-1}), t), \rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t),\right.\right.\\ &\qquad\qquad\left.\left.\frac{1}{2}\left[\rho_\infty(\mu_*, \mathcal{F}_j(\mu_{j-1}), 2t) + \rho_\infty(\mu_{j-1}, \mathcal{F}_i(\mu_*), 2t)\right]\right\}\right)\\ &\quad\geq M_\infty\left(\mu_*, \mu_j, \frac{t}{4}\right) * \phi\left(\inf\left\{M_\infty\left(\mu_{j-1}, \mu_*, \frac{t}{2}\right), M_\infty\left(\mu_{j-1}, \mu_j, \frac{t}{2}\right), \rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t),\right.\right.\\ &\qquad\qquad\left.\left.\frac{1}{2}\left[M_\infty(\mu_*, \mu_j, t) + M_\infty(\mu_*, \mu_{j-1}, t) * \rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t)\right]\right\}\right),\end{aligned} \tag{5.12}$$

where $\mu_j \subseteq \mathcal{F}_j(\mu_{j-1})$ implies $\rho_\infty(\mu_j, \mathcal{F}_j(\mu_{j-1}), t) = 1$. Letting $n \to \infty, M_\infty(\mu_n, \mu_*, t) = 1$, and using the left continuity of ϕ, we have

$$\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t) \geq \phi(\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t)), \tag{5.13}$$

which implies $\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t) = 1$. Hence, by Lemma 3.2, it follows that $\mu_* \subseteq \mathcal{F}_i(\mu_*)$. Then, the proof is completed. □

Theorem 5.2. *Let $(X, M, *)$ be a compact fuzzy metric space and let $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be a sequence of fuzzy self-mappings of $\mathcal{C}(X)$. Suppose that for each $\mu_1, \mu_2 \in \mathcal{C}(X)$, and for arbitrary positive integers i and j, $i \neq j$, $t > 0$,*

$$\begin{aligned} M_\infty(\mathcal{F}_i(\mu_1), \mathcal{F}_j(\mu_2), t) \geq \phi(&M_\infty(\mu_1, \mu_2, t), \rho_\infty(\mu_1, \mathcal{F}_i(\mu_1), 2t), \\ &\rho_\infty(\mu_2, \mathcal{F}_j(\mu_2), 2t), \rho_\infty(\mu_1, \mathcal{F}_j(\mu_2), 4t), \rho_\infty(\mu_2, \mathcal{F}_i(\mu_1), t)), \end{aligned} \tag{5.14}$$

where $\phi(h_1, h_2, h_3, h_4, h_5) : (0,1]^5 \to [0,1]$ is nondecreasing and continuous from the left for each variable. Denote $\gamma(h) = \phi(h, h, h, a, b)$, where $(a, b) \in \{(h * h, 1), (1, h * h)\}$. If

$$\gamma(h) * \gamma^2(h) * \cdots * \gamma^n(h) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \ \forall h \in (0,1], \tag{5.15}$$

where γ^n denote the nth iterative function of γ, then there exists $\mu_* \in \mathcal{C}(X)$ such that $\mu_* \subseteq \mathcal{F}_i(\mu_*)$ for all $i \in Z_+$.

Proof. Let $\mu_0 \in \mathcal{C}(X)$ and $\mu_1 \subseteq \mathcal{F}_1(\mu_0)$. By Theorem 4.9, for any $t > 0$, there exists $\mu_2 \in \mathcal{C}(X)$ such that $\mu_2 \subseteq \mathcal{F}_2(\mu_1)$ and

$$M_\infty(\mu_1, \mu_2, t) \geq M_\infty(\mathcal{F}_1(\mu_0), \mathcal{F}_2(\mu_1), t). \tag{5.16}$$

Again by Theorem 4.9, for any $t > 0$, we can find $\mu_3 \in \mathcal{C}(X)$ such that $\mu_3 \subseteq \mathcal{F}_3(\mu_2)$ and

$$M_\infty(\mu_2, \mu_3, t) \geq M_\infty(\mathcal{F}_2(\mu_1), \mathcal{F}_3(\mu_2), t). \tag{5.17}$$

By induction, we produce a sequence $\{\mu_n\}$ of points of $\mathcal{C}(X)$ such that

$$\begin{aligned} &\mu_{n+1} \subseteq \mathcal{F}_{n+1}(\mu_n), \quad n = 0, 1, 2, \ldots; \\ &M_\infty(\mu_n, \mu_{n+1}, t) \geq M_\infty(\mathcal{F}_n(\mu_{n-1}), \mathcal{F}_{n+1}(\mu_n), t). \end{aligned} \tag{5.18}$$

Now, we prove that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{C}(X)$. In fact, for arbitrary positive integer n, by the inequality (5.14), Lemma 3.2, and the formula (5.18), we have

$$\begin{aligned} M_\infty(\mu_n, \mu_{n+1}, t) &\geq M_\infty(\mathcal{F}_n(\mu_{n-1}), \mathcal{F}_{n+1}(\mu_n), t) \\ &\geq \phi(M_\infty(\mu_{n-1}, \mu_n, t), \rho_\infty(\mu_{n-1}, \mathcal{F}_n(\mu_{n-1}), 2t), \\ &\quad \rho_\infty(\mu_n, \mathcal{F}_{n+1}(\mu_n), 2t), \rho_\infty(\mu_{n-1}, \mathcal{F}_{n+1}(\mu_n), 4t), \rho_\infty(\mu_n, \mathcal{F}_n(\mu_{n-1}), t)) \\ &\geq \phi(M_\infty(\mu_{n-1}, \mu_n, t), M_\infty(\mu_{n-1}, \mu_n, t), M_\infty(\mu_n, \mu_{n+1}, t), M_\infty(\mu_{n-1}, \mu_{n+1}, 2t), 1) \\ &\geq \phi(M_\infty(\mu_{n-1}, \mu_n, t), M_\infty(\mu_{n-1}, \mu_n, t), \\ &\quad M_\infty(\mu_n, \mu_{n+1}, t), M_\infty(\mu_{n-1}, \mu_n, t) * M_\infty(\mu_n, \mu_{n+1}, t), 1), \end{aligned} \tag{5.19}$$

where $\mu_n \subseteq \mathcal{F}_n(\mu_{n-1})$ implies $\rho_\infty(\mu_n, \mathcal{F}_n(\mu_{n-1}), 2t) = 1$ by (3) in Lemma 3.2 Likewise, we have $\gamma(h) > h$ for all $h \in (0,1)$, $t > 0$. If $M_\infty(\mu_{n-1}, \mu_n, t) > M_\infty(\mu_n, \mu_{n+1}, t)$, then from the inequality (5.19), we obtain

$$M_\infty(\mu_n, \mu_{n+1}, t) \geq \gamma(M_\infty(\mu_n, \mu_{n+1}, t)) > M_\infty(\mu_n, \mu_{n+1}, t), \tag{5.20}$$

which is a contradiction. Thus, from the inequality (5.19), we have

$$M_\infty(\mu_n, \mu_{n+1}, t) \geq \gamma(M_\infty(\mu_{n-1}, \mu_n, t)) \geq \cdots \geq \gamma^n(M_\infty(\mu_0, \mu_1, t)). \tag{5.21}$$

Furthermore, for arbitrary positive integers m and k, we have

$$\begin{aligned}
M_\infty(\mu_n, \mu_{n+1}, t) &\geq \phi(M_\infty(\mu_{n-1}, \mu_n, t), M_\infty(\mu_{n-1}, \mu_n, t), \\
&\quad M_\infty(\mu_n, \mu_{n+1}, t), M_\infty(\mu_{n-1}, \mu_n, t) * M_\infty(\mu_n, \mu_{n+1}, t), 1) \\
&\geq \phi(M_\infty(\mu_{n-1}, \mu_n, t), M_\infty(\mu_{n-1}, \mu_n, t), \\
&\quad M_\infty(\mu_{n-1}, \mu_n, t), M_\infty(\mu_{n-1}, \mu_n, t) * M_\infty(\mu_{n-1}, \mu_n, t), 1) \\
&= \gamma(M_\infty(\mu_{n-1}, \mu_n, t)), \\
M_\infty(\mu_n, \mu_{n+1}, t) &\geq \gamma(M_\infty(\mu_{n-1}, \mu_n, t)) \geq \cdots \geq \gamma(M_\infty(\mu_0, \mu_1, t)).
\end{aligned} \tag{5.22}$$

Furthermore, for arbitrary positive integers m and k, we have

$$\begin{aligned}
1 &\geq M_\infty(\mu_k, \mu_{k+m}, t) \\
&\geq M_\infty\left(\mu_k, \mu_{k+1}, \frac{t}{m}\right) * M_\infty\left(\mu_{k+1}, \mu_{k+2}, \frac{t}{m}\right) * \cdots * M_\infty\left(\mu_{k+m-1}, \mu_{k+m}, \frac{t}{m}\right) \\
&\geq \gamma^k\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) * \gamma^{k+1}\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) * \cdots * \gamma^{k+m-1}\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right).
\end{aligned} \tag{5.23}$$

Since $\phi(h) * \phi^2(h) * \cdots * \phi^n(h) \to 1$ as $n \to \infty$, for all $h \in (0, 1)$, it follows that

$$\gamma^k\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) * \gamma^{k+1}\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) * \cdots * \gamma^{k+m-1}\left(M_\infty\left(\mu_0, \mu_1, \frac{t}{m}\right)\right) \tag{5.24}$$

is convergent, this implies that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{C}(X)$. Since X is a compact fuzzy metric space, it follows that X is complete. By Theorem 4.6, $\mathcal{C}(X)$ is complete. Let $\mu_n \to \mu_*$.

Now, we show that $\mu_* \subseteq \mathcal{F}_i(\mu_*)$ for all $i \in Z_+$. In fact, for arbitrary positive integers i and j, $i \neq j$, by Theorem 4.9, we have

$$\begin{aligned}
&\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t) \\
&\quad \geq M_\infty\left(\mu_*, \mu_j, \frac{t}{4}\right) * \rho_\infty\left(\mu_j, \mathcal{F}_i(\mu_*), \frac{3t}{4}\right) \\
&\quad \geq M_\infty\left(\mu_*, \mu_j, \frac{t}{4}\right) * M_\infty\left(\mathcal{F}_j(\mu_{j-1}), \mathcal{F}_i(\mu_*), \frac{t}{2}\right) \\
&\quad \geq M_\infty\left(\mu_*, \mu_j, \frac{t}{4}\right) * \phi\left(M_\infty\left(\mu_{j-1}, \mu_*, \frac{t}{2}\right), \rho_\infty(\mu_{j-1}, \mathcal{F}_j(\mu_{j-1}), t),\right. \\
&\qquad\qquad \left.\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t), \rho_\infty(\mu_{j-1}, \mathcal{F}_i(\mu_*), 2t), \rho_\infty(\mu_*, \mathcal{F}_j(\mu_{j-1}), t)\right) \\
&\quad \geq M_\infty\left(\mu_*, \mu_j, \frac{t}{4}\right) * \phi\left(M_\infty\left(\mu_{j-1}, \mu_*, \frac{t}{2}\right), M_\infty\left(\mu_{j-1}, \mu_j, \frac{t}{2}\right), \rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t),\right. \\
&\qquad\qquad \left. M_\infty(\mu_{j-1}, \mu_*, t) * \rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t), M_\infty\left(\mu_*, \mu_j, \frac{t}{2}\right)\right),
\end{aligned} \tag{5.25}$$

where $\mu_j \subseteq \mathcal{F}_j(\mu_{j-1})$ implies $\rho_\infty(\mu_j, \mathcal{F}_j(\mu_{j-1}), t) = 1$. Letting $n \to \infty, M_\infty(\mu_n, \mu_*, t) = 1$, and using the left continuity of ϕ, we have

$$\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t) \geq \phi(1, 1, \rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t), \rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t), 1) \geq \gamma(\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t)), \tag{5.26}$$

which implies $\rho_\infty(\mu_*, \mathcal{F}_i(\mu_*), t) = 1$. Hence, by Lemma 3.2, it follows that $\mu_* \subseteq \mathcal{F}_i(\mu_*)$, then the proof is completed. □

Now, we give an example to illustrate the validity of the results in fixed point theory. For simplicity, we only exemplify Theorem 5.1, while the example may be similarly constructed for Theorem 5.2.

Example 5.3. Let $(\mathcal{C}(X), M_\infty, *)$ be a fuzzy metric space, where $X = [-1, 1]$, M_d, H_M, and M_∞ are the same as in Example 3.4. Then, $(\mathcal{C}(X), M_\infty, *)$ is a compact metric space.

Now, define $\phi : [0,1] \to [0,1]$ as $\phi(x) = \sqrt{x}$, and define $\{\mathcal{F}_i\}_{i=1}^{\infty}$ a sequence of fuzzy self-mappings of $\mathcal{C}(X)$ as

$$\mathcal{F}_i(\mu) = \frac{1}{2^i}\mu, \quad \text{for any } \mu \in \mathcal{C}(X). \tag{5.27}$$

For arbitrary positive integers i and j, without loss of generality, suppose $i < j$. For each $\mu_1, \mu_2 \in \mathcal{C}(X)$, by a routine calculation, we have

$$\begin{aligned}
M_\infty(\mathcal{F}_i(\mu_1), \mathcal{F}_j(\mu_2), t) &= M_\infty\left(\frac{1}{2^i}\mu_1, \frac{1}{2^j}\mu_2, t\right) \\
&= M_\infty\left(\mu_1, \frac{1}{2^{j-i}}\mu_2, 2^i t\right)
\end{aligned}$$

$$\begin{aligned}
&\geq M_\infty\left(\mu_1,\mu_2,2^i t\right)\\
&\geq \phi(M_\infty(\mu_1,\mu_2,t))\\
&\geq \phi\Bigg(\inf\bigg\{M_\infty(\mu_1,\mu_2),\rho_\infty(\mu_1,\mathcal{F}_i(\mu_1),2t),\rho_\infty(\mu_2,\mathcal{F}_j(\mu_2),2t),\\
&\qquad\qquad \frac{1}{2}\left[\rho_\infty(\mu_2,\mathcal{F}_i(\mu_1),4t)+\rho_\infty(\mu_1,\mathcal{F}_j(\mu_2),4t)\right]\bigg\}\Bigg).
\end{aligned} \tag{5.28}$$

Therefore, by Theorem 5.1, we assert that the sequence of fuzzy self-mappings $\{\mathcal{F}_i\}_{i=1}^{\infty}$ has a common fixed point μ_* in $\mathcal{C}(X)$. In fact, it is easy to check that

$$\mu_*(x)=\begin{cases}1, & \text{if } x=(0,0,\ldots),\\ 0, & \text{otherwise.}\end{cases} \tag{5.29}$$

6. Conclusion

So far many authors have made a great deal of work in the Hausdorff-Pompeiu metric [20–25]. To describe the degree of nearness between two crisp sets, Rodrguez-López and Romaguera have defined Hausdorff fuzzy metric. In this paper, we define a new M_∞-fuzzy metric, which describes the degree of nearness between two fuzzy sets. Then, some properties on M_∞-fuzzy metric are discussed. In addition, in this new circumstances, we give some fixed point theorems which are the important generalizations of contraction mapping principle in functional analysis.

The results of the present paper may be applied in different settings. In terms of topology, one can make use of topology in data analysis and knowledge acquisition [31]. For another, topologies corresponding to fuzzy sets are used to detect dependencies of attributes in information systems with respect to gradual rules as in [32]. Furthermore, fuzzy fixed point theory can be used in existence and continuity theorems for dynamical systems with some vague parameters [33, 34]. In addition, this work offers a new tool for the description and analysis of fuzzy metric spaces. It would be possible to obtain more topological properties on the new fuzzy metric space. So, we hope our results contribute to dealing with some problems in practical applications for future study.

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