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Research Article

Hidden and Not So Hidden Symmetries

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Hidden symmetries entered the literature in the late Eighties when it was observed that there could be gain of Lie point symmetry in the reduction of order of an ordinary differential equation. Subsequently the reverse process was also observed. Such symmetries were termed "hidden". In each case the source of the "new" symmetry was a contact symmetry or a nonlocal symmetry, that is, a symmetry with one or more of the coefficient functions containing an integral. Recent work by Abraham-Shrauner and Govinder (2006) on the reduction of partial differential equations demonstrates that it is possible for these "hidden" symmetries to have a point origin. In this paper we show that the same phenomenon can be observed in the reduction of ordinary differential equations and in a sense loosen the interpretation of hidden symmetries.

1. Introduction

The concept of hidden symmetries (According to Abraham-Shrauner the original observation was due to Peter Olver (private communication).) was largely developed by Barbara Abraham-Shrauner, partially in association with her student, Ann Guo, in the early Nineties. There was a number of papers [1–6] chronicling their occurrence and characteristics. Further papers [7–11], to list a few, devoted to the subject have appeared over the years. A hidden symmetry is a Lie point symmetry which appears in the target differential equation after a change of order using a nonlocal transformation and which does not have a point counterpart in the source equation. If the symmetry appears on an increase of order, it is termed a Type I hidden symmetry and, if the symmetry appears on a decrease of order, it is termed a Type II hidden symmetry. The potential for the origin of a hidden symmetry is simply explained. Suppose that a differential equation has two Lie point symmetries, Γ_1 and Γ_2, with the Lie

Bracket $[\Gamma_1, \Gamma_2]_{LB} = \lambda\Gamma_2$, where λ may be zero. Then reduction of order using Γ_1, the so-called nonnormal subgroup, results in Γ_2 becoming a nonlocal symmetry (In the case of an equation having more than two Lie point symmetries the situation is the same if the Lie Bracket gives a third symmetry. The point-like nature of a symmetry is preserved only if reduction is performed using the normal subgroup. See also [12].) of the reduced equation. Reduction of order is a nonlocal transformation, and the variables of the reduced equation are expressed in terms of the two invariants of the once-extended symmetry. If one increases the order of an equation by means of a nonlocal transformation, this transformation is due to the presence of a symmetry in the higher-order equation. If it happens that the higher-order equation has a symmetry with the "wrong" Lie Bracket with this symmetry, then it becomes a nonlocal symmetry of the equation of lower order. Consequently in the process of the increase of order this symmetry is not a consequence of the existence of a Lie point symmetry of the lower-order equation. The symmetry appears, as it were, from nowhere which one can easily accept as a sufficient reason to term it a "hidden symmetry," and is classified as a Type I hidden symmetry. In the reverse process, that is, in the reduction of order, the presence of an unexpected Lie point symmetry in the reduced equation gives rise to a Type II hidden symmetry.

Abraham-Shrauner and Govinder [13] have recently shown a new potential source of hidden symmetries for partial differential equations. The symmetries do not come from nonlocal symmetries but are a result of the possibility that several partial differential equations could lead to the same partial differential equation on reduction using one of the Lie point symmetries of the original equation. (Subsequently, other sources of hidden symmetries for partial differential equations have been indicated in the literature [14, 15].)

An immediate question which comes to mind is whether a similar result holds for reduction of order of different ordinary differential equations to the same equation. Equally one can think of increase of order from a variety of lower-order equations to the same higher-order equation. However, the latter is a somewhat more trivial matter [16].

The purpose of this paper is to show how different equations which can be reduced to the same equation provide point sources for each of the Lie point symmetries of the reduced equation even though any particular of the higher-order equations may not provide the full complement of Lie point symmetries. Thus we see that the set of Lie point symmetries of the reduced equation can be viewed as having two sources. Firstly the Lie point symmetries of the reduced equation can be obtained from a combination of point and nonlocal symmetries of a given higher-order equation. Secondly the same set of Lie point symmetries are obtained from the point symmetries of a variety of higher-order equations. In so doing we do not wish to detract from the originality of the work of Abraham-Shrauner and Govinder [13]. Rather we wish to demonstrate that the same effect can be found in the consideration of ordinary differential equations. It often happens that new ideas, concepts, and objects originate in esoteric contexts, perhaps by way of example arising accidentally, as it were, in an investigation requiring deep thought. The commonplace does not invite deep thinking since the solution of problems there proceeds via methods upon which the experienced practitioner need not dwell for their execution. This does not mean that new ideas, concepts, and objects cannot be found in the commonplace. One is sometimes pushed to think uncommonly in the context of the commonplace to see that the new has been under our noses since the beginnings of time, if not earlier. In a pedagogical context the elimination of the esoteric origin is essential to lead the neophyte to understanding. So it is with hidden symmetries. Indeed we wish to stress the fact that a point symmetry can be both hidden and not hidden depending upon one's approach to its determination through reduction of order.

2. The Case for a Partial Differential Equation

We illustrate the method with a simple example [13, equation (2.1)],

$$u_{xxx} + u(u_t + cu_x) = 0, \tag{2.1}$$

which possesses the Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_t, \\ \Gamma_2 &= \partial_x, \\ \Gamma_3 &= 3t\partial_t + (x + 2ct)\partial_x, \\ \Gamma_4 &= t\partial_t + ct\partial_x + u\partial_u. \end{aligned} \tag{2.2}$$

We reduce (2.1) to an ordinary differential equation using the symmetry $c\Gamma_2 + \Gamma_1$ for which the invariants are $w = u$ and $y = x - ct$; that is, we seek a travelling-wave solution. Note that this is not an invertible point transformation, and so preservation of point symmetries is not guaranteed. The reduced equation is simply the ordinary differential equation

$$w_{yyy} = 0, \tag{2.3}$$

which has the seven Lie point symmetries

$$\begin{aligned} &\Upsilon_1 = \partial_y, && \Upsilon_5 = y\partial_w, \\ &\Upsilon_2 = \partial_w, && \Upsilon_6 = w\partial_w, \\ &\Upsilon_3 = y^2\partial_w, && \Upsilon_7 = \frac{1}{2}y^2\partial_y + yw\partial_w, \\ &\Upsilon_4 = y\partial_y. \end{aligned} \tag{2.4}$$

Equation (2.1) is not the only source of (2.3) under reduction. Equally it can be obtained from

$$u_{xxx} = 0, \qquad u_{ttt} = 0, \qquad u_{xxt} = 0, \qquad u_{xtt} = 0, \tag{2.5}$$

where u is still a function of t and x, by means of the same invariants. For example, the first of (2.5) has an eightfold infinity of Lie point symmetries. They are

$$\begin{aligned} &\Delta_1 = F_1(t)\partial_x, && \Delta_5 = F_5(t)x\partial_x, \\ &\Delta_2 = F_2(t)\partial_u, && \Delta_6 = F_1(t)x\partial_u, \\ &\Delta_3 = F_3(t)\partial_t, && \Delta_7 = F_7(t)u\partial_u, \\ &\Delta_4 = F_4(t)x^2\partial_u, && \Delta_8 = F_8(t)\left(\frac{1}{2}x^2\partial_x + xu\partial_u\right), \end{aligned} \tag{2.6}$$

where the $F_i(t)$, $i = 1, 8$, are arbitrary functions. A subset of these symmetries is obtained by making specific choices for the arbitrary functions, and in suitable combinations we have

$$\begin{aligned} &\Sigma_1 = \partial_x, && \Sigma_5 = (x - ct)\partial_x, \\ &\Sigma_2 = \partial_u, && \Sigma_6 = (x - ct)\partial_u, \\ &\Sigma_3 = \partial_t, && \Sigma_7 = u\partial_u, \\ &\Sigma_4 = (x - ct)^2\partial_u, && \Sigma_8 = \frac{1}{2}(x - ct)^2\partial_x + (x - ct)u\partial_u, \end{aligned} \tag{2.7}$$

which reduce to the seven Lie point symmetries of (2.3). Thus, the hidden symmetries of (2.3) could have arisen from point symmetries of

$$u_{xxx} = 0. \tag{2.8}$$

In this example the invariants used for the reduction of order were the same. There is no requirement for this to be the case, and Abraham-Shrauner and Govinder [13] discuss the procedure for this more general case and show that the same type of result is found.

3. The Case for an Ordinary Differential Equation

We firstly show that the principle of the idea of Abraham-Shrauner and Govinder [13] applied to ordinary differential equations is commonplace [16]—it is a simple matter to find reductions of different nth-order ordinary differential equations to the same $(n - 1)$th-order ordinary differential equation.

The two third-order equations

$$y''' = 0, \tag{3.1}$$

$$2y'y''' - 3y''^2 = 0, \tag{3.2}$$

can be reduced to the same second-order equation

$$Y'' = 0, \tag{3.3}$$

(now the prime denotes differentiation with respect to the transformed independent variable, X, which, in this case, happens to be the same as the original independent variable) by means of the transformations

$$X = x, \qquad Y = y', \tag{3.4}$$

$$X = x, \qquad Y = y'^{-1/2}, \tag{3.5}$$

respectively.

For (3.1) the symmetry generating the transformation (3.4) is $\Gamma_1 = \partial_y$. The remaining six Lie point symmetries are transformed as

$$\begin{aligned}
&\Gamma_2 = x\partial_y, && \Lambda_2 = \partial_Y,\\
&\Gamma_3 = \frac{1}{2}x^2\partial_y, && \Lambda_3 = X\partial_Y,\\
&\Gamma_4 = y\partial_y, && \Lambda_4 = Y\partial_Y,\\
&\Gamma_5 = \partial_x, && \Lambda_5 = \partial_X,\\
&\Gamma_6 = x\partial_x + y\partial_y, && \Lambda_6 = \partial_X,\\
&\Gamma_7 = x^2\partial_x + 2xy\partial_y, && \Lambda_7 = X^2\partial_X + \left(2\int Y\,\mathrm{d}X\right)\partial_Y,
\end{aligned} \tag{3.6}$$

from which it is evident that we are missing three of the Lie point symmetries of (3.3). The missing three are

$$\begin{aligned}
\Sigma_1 &= X^2\partial_X + XY\partial_Y,\\
\Sigma_2 &= Y\partial_X,\\
\Sigma_3 &= XY\partial_X + Y^2\partial_Y,
\end{aligned} \tag{3.7}$$

and it is a simple calculation to show that they have their origins from the symmetries

$$\begin{aligned}
\Delta_1 &= x^2\partial_x + 3\left[\frac{1}{2}x^2y' - \frac{1}{6}x^3y''\right]\partial_y,\\
\Delta_2 &= y'\partial_x + \frac{1}{2}y'^2\partial_y,\\
\Delta_3 &= xy'\partial_x + \left[2xy'^2 - \frac{3}{2}x^2y'y'' + \frac{1}{2}x^3y''^2\right]\partial_y
\end{aligned} \tag{3.8}$$

of (3.1). The symmetry Δ_2 is one of the contact symmetries of (3.1). The other two are generalised symmetries and have been written as such instead of the nonlocal version since the integration of (3.1) is trivial.

In the case of (3.2) and the reduction (3.5) the Lie point symmetries of the former and their expression as symmetries of (3.3) are

$$\begin{aligned}
&\Gamma_1 = \partial_x, && \Lambda_1 = \partial_X,\\
&\Gamma_2 = x\partial_x, && \Lambda_2 = X\partial_X + \frac{1}{2}Y\partial_Y,\\
&\Gamma_3 = x^2\partial_x, && \Lambda_3 = X^2\partial_X + XY\partial_Y,\\
&\Gamma_5 = y\partial_y, && \Lambda_5 = Y\partial_Y,\\
&\Gamma_6 = y^2\partial_y, && \Lambda_6 = Y\int Y^2\,\mathrm{d}X\partial_Y.
\end{aligned} \tag{3.9}$$

The symmetry $\Gamma_4 = \partial_y$ is the symmetry used for the transformation (3.5).

The missing Lie point symmetries are

$$\begin{aligned} \Sigma_1 &= \partial_Y, \\ \Sigma_2 &= X\partial_Y, \\ \Sigma_3 &= Y\partial_X, \\ \Sigma_4 &= XY\partial_X + Y^2\partial_Y. \end{aligned} \tag{3.10}$$

We note that in both cases the noncartan symmetries of (3.3) are absent in the reduction of the point symmetries of the third-order equations.

What we do wish to emphasise is that the two reductions gave us a different selection of the Lie point symmetries of (3.3) which is precisely the same effect reported by Abraham-Shrauner and Govinder [13] for partial differential equations. The main difference is that their effect was as result of a point transformation of the original variables. Here we have a nonlocal transformation of the original variables.

4. An Equation Arising in Cosmology

The equation

$$yy'''' + \frac{5}{2}y'y''' = y^{-3}, \tag{4.1}$$

arises in the analysis of shear-free spherically symmetric spacetimes [17]. It has the two Lie point symmetries

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = x\partial_x + \frac{4}{5}y\partial_y \quad \text{with } [\Gamma_1, \Gamma_2] = \Gamma_1. \tag{4.2}$$

One reduces (4.1) by Γ_1 (with characteristics $u = y,\ v = y'$) to obtain the third-order ordinary differential equation

$$u\left(v^3v''' + 4v^2v'v'' + vv'^3\right) + \frac{5}{2}v\left(v^2v'' + vv'^2\right) = u^{-3} \tag{4.3}$$

with

$$\begin{aligned} \Delta_2 &= \frac{4}{5}u\partial_u - \frac{1}{5}v\partial_v \ \text{(inherited)}, \\ \Delta_3 &= 2u^2\partial_u + uv\partial_v, \end{aligned} \tag{4.4}$$

which we now interpret as coming from a point symmetry of a different fourth-order ordinary differential equation.

We note that

$$[\Delta_2, \Delta_3] = \frac{4}{5}\Delta_3 \tag{4.5}$$

and look for a fourth-order ordinary differential equation with Σ_1, Σ_2 and Σ_3 such that reduction using Σ_1 leads to a Σ_2 and Σ_3 with a property as in (4.5), that is, the Lie algebra A_2. There are two realisations of A_2, but it is clear that we should use the realisation in which the symmetries are unconnected. (If the two symmetries of the higher-order equation are connected, the two symmetries of the lower-order equation are connected. This is not the case for the reduced third-order equation (4.3), and so we use the unconnected realisation of A_2.) In addition, we realise that the most direct route to the Lie algebra A_2 is if we start with three point symmetries that form the Lie algebra $A_1 \oplus A_2$. Reduction using the symmetry representing A_1 results in the remaining symmetries as symmetries of the reduced equation with the Lie algebra A_2.

Putting all of this together results in the following realisation of $A_1 \oplus A_2$:

$$\Sigma_1 = t\partial_q, \qquad \Sigma_2 = -t\partial_t - q\partial_q, \qquad \Sigma_3 = \partial_q \tag{4.6}$$

with

$$[\Sigma_1, \Sigma_2] = 0, \qquad [\Sigma_1, \Sigma_3] = 0, \qquad [\Sigma_2, \Sigma_3] = \Sigma_3. \tag{4.7}$$

A general fourth-order ordinary differential equation is

$$\ddddot{q} = f\left(t, q, \dot{q}, \ddot{q}, \dddot{q}\right). \tag{4.8}$$

Invariance under Σ_3 gives

$$\ddddot{q} = f\left(t, \dot{q}, \ddot{q}, \dddot{q}\right), \tag{4.9}$$

under Σ_1 gives

$$\ddddot{q} = f\left(t, \ddot{q}, \dddot{q}\right), \tag{4.10}$$

and under Σ_2 one has the form

$$\ddddot{q} = t^{-3} f\left(t\ddot{q}, t^2\,\dddot{q}\right). \tag{4.11}$$

The associated Lagrange's system for reduction by Σ_1 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}q}{t} = \frac{\mathrm{d}\dot{q}}{1}, \tag{4.12}$$

for which $x = t$ and $y = t\dot{q} - q$. The differential consequences are

$$\begin{aligned} \frac{dy}{dx} &= t\ddot{q}, \\ \frac{d^2y}{dx^2} &= t\dddot{q} + \ddot{q}, \\ \frac{d^3y}{dx^3} &= t\ddddot{q} + 2\dddot{q}, \end{aligned} \tag{4.13}$$

so that the structure of the third-order differential equation is

$$x^2y''' - 2xy'' + 2y' = f\left(y', xy'' - y'\right). \tag{4.14}$$

The remaining symmetries are

$$\begin{aligned} \Sigma_2 &= -t\partial_t - q\partial_q + 0\partial_{\dot{q}} = -x\partial_x - t\dot{q}\partial_y + q\partial_y = -x\partial_x - y\partial_y, \\ \Sigma_3 &= \partial_q = -\partial_y. \end{aligned} \tag{4.15}$$

We see that $[\Sigma_2, \Sigma_3] = \Sigma_3$.

To maintain this structure one writes the given symmetries as

$$\Delta_2 = u\partial_u - \frac{1}{4}v\partial_v, \qquad \Delta_3 = 2u^2\partial_u + uv\partial_v, \tag{4.16}$$

so that we also have $[\Delta_2, \Delta_3] = \Delta_3$.

We need the transformation between

$$\begin{aligned} \Sigma_2 &= -x\partial_x - y\partial_y, \qquad & \Delta_2 &= u\partial_u - \frac{1}{4}v\partial_v, \\ \Sigma_3 &= -\partial_y, & \Delta_3 &= 2u^2\partial_u + uv\partial_v. \end{aligned} \tag{4.17}$$

Let

$$x = F(u, v), \qquad y = G(u, v). \tag{4.18}$$

Then

$$\begin{aligned} u\frac{\partial F}{\partial u} - \frac{1}{4}v\frac{\partial F}{\partial v} &= -F, \qquad & u\frac{\partial G}{\partial u} - \frac{1}{4}v\frac{\partial G}{\partial v} &= -G, \\ 2u^2\frac{\partial F}{\partial u} + uv\frac{\partial F}{\partial v} &= 0, & 2u^2\frac{\partial G}{\partial u} + uv\frac{\partial G}{\partial v} &= -1, \end{aligned} \tag{4.19}$$

from which it is easily seen that one has

$$x = v^{4/3}u^{-2/3}, \qquad y = \frac{1}{2u}. \tag{4.20}$$

If we compare (4.3) and (4.14) it is evident that f in (4.14) must take the form

$$f\left(y', xy'' - y'\right) = f(\alpha, \beta) = \frac{32\alpha^4}{3} + \frac{3\beta^2}{\alpha} + \frac{7\alpha}{8} + \frac{7\beta}{4}. \tag{4.21}$$

5. Summary

We can now summarise our result as follows. In the reduction of the fourth-order equation

$$yy'''' + \frac{5}{2}y'y''' = y^{-3} \tag{5.1}$$

(which admits the symmetries Γ_1 and Γ_2) via $u = y,\ v = y'$ we obtain the third-order equation

$$u\left(v^3v''' + 4v^2v'v'' + vv'^3\right) + \frac{5}{2}v\left(v^2v'' + vv'^2\right) = u^{-3} \tag{5.2}$$

which admits

$$\Gamma_3 = 2u^2\partial_u + uv\partial_v \tag{5.3}$$

as a hidden symmetry. In the past [4, 5] Γ_3 was only interpreted as arising from a nonlocal symmetry of (5.1). However, we now have the result that the fourth-order equation

$$t^2\ddddot{q} = \frac{7}{4}t\,\dddot{q} + \frac{3t^2}{\ddot{q}}\dddot{q}^2 + \frac{7}{8}\ddot{q} + \frac{32}{3}t^3\ddot{q}^4 \tag{5.4}$$

(which admits the symmetries $\Sigma_1,\ \Sigma_2$ and Σ_3) can be reduced to (5.2) via the reduction

$$u = \frac{1}{2(t\dot{q} - q)}, \qquad v = \frac{t^{3/4}}{\sqrt{2}}(t\dot{q} - q)^{-1/2}. \tag{5.5}$$

As a result a new interpretation of Γ_3 is that it arises from the *point* symmetry Σ_3 of (5.4).

We observe that this result relates (5.1) and (5.4) via the nonlocal transformation

$$y = \frac{1}{2(t\dot{q} - q)}, \qquad y' = \frac{t^{3/4}}{\sqrt{2}}(t\dot{q} - q)^{-1/2}. \tag{5.6}$$

It is obvious that (5.1) and (5.4) are not related via a point transformation. It is suspected that many equations are related via nonlocal transformations. However, no method is available to find these transformations in general. By investigating the point origins of Type II hidden symmetries we have been able to provide a method to determine these nonlocal relations between equations not previously thought to be related.

We have thus been able to show that hidden symmetries of ordinary differential equations can also have a point origin—they are no longer confined to arise only as contact or nonlocal symmetries of the original equation, but can arise as a point symmetry of another equation of the same order as the original equation. In a sense, if one could prove the rather strong result that there always exists a point origin for any so-thought hidden symmetry (except for second-order equations), then the need for nonlocal symmetries for ordinary differential equations falls away. The exception for second-order equations is due to the fact that the maximal number of Lie point symmetries is eight while the maximal number for third-order equations is only seven. It is not possible to relate every hidden symmetry of second-order equations to point symmetries of third-order equations.

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