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Research Article

Steady State Solution to Atmospheric Circulation Equations with Humidity Effect

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The steady state solution to atmospheric circulation equations with humidity effect is studied. A sufficient condition of existence of steady state solution to atmospheric circulation equations is obtained, and regularity of steady state solution is verified.

1. Introduction

This paper is concerned with steady state solution of the following initial-boundary problem of atmospheric circulation equations involving unknown functions (u,T,q,p) at $(x,t) = (x_1,x_2,t) \in \Omega \times (0,\infty)$ ($\Omega = (0,2\pi) \times (0,1)$ is a period of C^∞ field $(-\infty,+\infty) \times (0,1)$),

$$\frac{\partial u}{\partial t} = P_r(\Delta u - \nabla p - \sigma u) + P_r\left(RT - \widetilde{R}q\right)\vec{\kappa} - (u\cdot\nabla)u, \tag{1.1}$$

$$\frac{\partial T}{\partial t} = \Delta T + u_2 - (u\cdot\nabla)T + Q, \tag{1.2}$$

$$\frac{\partial q}{\partial t} = L_e\Delta q + u_2 - (u\cdot\nabla)q + G, \tag{1.3}$$

$$\operatorname{div} u = 0, \tag{1.4}$$

where $P_r > 0$, $R > 0$, $\widetilde{R}$, $L_e > 0$ are constants, $u = (u_1,u_2), T, q, p$ denote velocity field, temperature, humidity, and pressure, respectively, Q, G are known functions, and σ is a constant matrix

$$\sigma = \begin{pmatrix} \sigma_0 & \omega \\ \omega & \sigma_1 \end{pmatrix}. \tag{1.5}$$

The problems (1.1)–(1.4) are supplemented with the following Dirichlet boundary condition at $x_2 = 0, 1$ and periodic condition for x_1:

$$(u, T, q) = 0, \qquad x_2 = 0, 1,$$
$$(u, T, q)(0, x_2) = (u, T, q)(2\pi, x_2) \tag{1.6}$$

and initial value conditions:

$$(u, T, q) = (u_0, T_0, q_0), \quad t = 0. \tag{1.7}$$

The partial differential equations (1.1)–(1.7) were presented in atmospheric circulation with humidity effect. Atmospheric circulation is one of the main factors affecting the global climate, so it is very necessary to understand and master its mysteries and laws. Atmospheric circulation is an important mechanism to complete the transports and balance of atmospheric heat and moisture and the conversion between various energies. On the contrary, it is also the important result of these physical transports, balance, and conversion. Thus it is of necessity to study the characteristics, formation, preservation, change, and effects of the atmospheric circulation and master its evolution law, which is not only the essential part of human's understanding of nature, but also the helpful method of changing and improving the accuracy of weather forecasts, exploring global climate change, and making effective use of climate resources.

The atmosphere and ocean around the earth are rotating geophysical fluids, which are also two important components of the climate system. The phenomena of the atmosphere and ocean are extremely rich in their organization and complexity, and a lot of them cannot be produced by laboratory experiments. The atmosphere or the ocean or the couple atmosphere and ocean can be viewed as an initial- and boundary-value problem [1–4] or an infinite dimensional dynamical system [5–7]. We deduce atmospheric circulation models which are able to show features of atmospheric circulation and are easy to be studied from the very complex atmospheric circulation model based on the actual background and meteorological data, and we present global solutions of atmospheric circulation equations with the use of the T weakly continuous operator [8].

We investigate steady state solution of the atmospheric circulation equations in this paper. The steady state solution is a special state of evolution equations and the time-independent solution, which plays a very important role on understanding the dynamical behavior of the evolution equations and is the main directions and important content in studying evolution equations. Steady state solutions of some systems are studied [9–12]. The purpose to consider with the steady state solution of atmospheric circulation equations is to seek the conditions under which atmospheric circulation is stable and to understand structure of the circulation cell.

We discuss the existence and regularity of steady state solution to atmospheric circulation equations (1.1)–(1.4) with the boundary condition (1.6). In other words, we discuss the following equations:

$$\Delta u - \nabla p - \sigma u + \left(RT - \widetilde{R}q\right)\vec{\kappa} - \frac{1}{P_r}(u \cdot \nabla)u = 0, \quad x \in \Omega, \tag{1.8}$$

$$\Delta T + u_2 - (u \cdot \nabla)T + Q(x) = 0, \quad x \in \Omega, \tag{1.9}$$

$$L_e \Delta q + u_2 - (u \cdot \nabla)q + G(x) = 0, \quad x \in \Omega, \tag{1.10}$$

$$\operatorname{div} u = 0, \quad x \in \Omega, \tag{1.11}$$

$$(u, T, q) = 0, \qquad x_2 = 0, 1, \tag{1.12}$$

$$(u, T, q)(2\pi, x_2) = (u, T, q)(0, x_2). \tag{1.13}$$

The paper is organized as follows. In Section 2 we present preliminary results. In Section 3, we prove that the systems (1.8)–(1.13) possess steady state solutions in $W^{2,q}(\Omega, R^4) \times W^{1,q}(\Omega)$, $q \geq 2$ by using space sequence method. In Section 4, by using Sard-Smale Theorem and energy method, we obtain regularity of the solutions to the models (1.8)–(1.13).

Let $\widetilde{\sigma} = \min\{\sigma_0, \omega, \sigma_1\}$, and $\|\cdot\|_X$ denote norm of the space X.

2. Preliminaries

We introduce theory of linear elliptic equation and ADN theory of Stokes equation.

We consider with divergence form of linear elliptic equation:

$$Lu = -D_j\left(a_{ij} D_i u\right) + b_i D_i u + cu = f, \tag{2.1}$$

where $a_{ij}, b_i, c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $a_{ij} = a_{ji}$, (a_{ij}) is uniformly elliptic, that is, there exist constants $0 < \lambda_1 \leq \lambda_2$ such that

$$\lambda_1 |\xi|^2 \leq a_{ij}(x)\xi_i \xi_j \leq \lambda_2 |\xi|^2, \quad \forall \xi \in R^n, \ x \in \Omega. \tag{2.2}$$

The problem (1.1) is supplemented with the following Dirichlet boundary condition

$$u|_{\partial\Omega} = \varphi. \tag{2.3}$$

We define three classes of solutions of (2.1) and (2.3).

(1) Classical solution: if there is a function $u \in C^2(\Omega)$ satisfying (2.1), (2.3), we say u is a classical solution to (2.1) and (2.3).

(2) Strong solution: if there is a function $u \in W^{2,p}(\Omega)$ satisfying (2.1), (2.3) almost everywhere for some $p \geq 1$, we say u is a strong solution to (2.1) and (2.3).

(3) Weak solution: if there is a function $u \in W^{1,2}(\Omega)$ satisfying

$$\int_{\Omega}\left(a_{ij}D_i u D_j v + b_i D_i u v + c u v\right)dx = \int_{\Omega} f v dx, \quad \forall v \in W_0^{1,2}(\Omega), \tag{2.4}$$

and (2.3), we say u is a weak solution to (2.1) and (2.3).

Lemma 2.1 (see [13] (Schauder Theorem)). *Let $\Omega \subset R^n$ be a $C^{2,\alpha}$ field, $a_{ij}, b_i, c, f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}$. If $u \in C^{2,\alpha}$ is a solution to* (2.1) *and* (2.3)*, then*

$$\|u\|_{C^{2,\alpha}} \leq C\left(\|u\|_{C^0} + \|f\|_{C^{0,\alpha}} + \|\varphi\|_{C^{2,\alpha}}\right), \tag{2.5}$$

where $C > 0$ depends on n, α, λ, Ω and $C^{0,\alpha}$-norm of the coefficient functions a^{ij}, b^i, c.

Lemma 2.2 (see [13] (L^p Theorem)). *Let $\Omega \subset R^n$ be a C^2 field, $a_{ij} \in C^0(\Omega)$, $b_i, c \in L^\infty(\Omega)$, $f \in L^p(\Omega)$, $\varphi \in W^{2,p}(\Omega)$. If $u \in W^{2,p}$ is a solution of* (2.1) *and* (2.3)*, then*

$$\|u\|_{W^{2,p}} \leq C\left(\|u\|_{L^p} + \|f\|_{L^p} + \|\varphi\|_{W^{2,p}}\right), \tag{2.6}$$

where $C > 0$ depends on n, p, λ, Ω and $C^{0,\alpha}$-norm or L^∞-norm of the coefficient functions.

One considers with Stokes equation

$$\begin{aligned} -\mu\Delta u + \nabla p &= f(x), \\ \operatorname{div} u &= 0, \\ u|_{\partial\Omega} &= \varphi. \end{aligned} \tag{2.7}$$

Lemma 2.3 (see [14, 15] (ADN theory of Stokes equation)). *(1) Let $f \in C^{k,\alpha}(\Omega, R^n)$, $\varphi \in C^{k+2,\alpha}(\Omega, R^n)$, $k \geq 0$. If $(u,p) \in C^{2,\alpha}(\Omega, R^n) \times C^{1,\alpha}(\Omega)$ is a solution of* (2.7)*, then the solution $(u,p) \in C^{k+2,\alpha}(\Omega, R^n) \times C^{k+1,\alpha}(\Omega)$, and*

$$\|u\|_{C^{k+2,\alpha}} + \|p\|_{C^{k+1,\alpha}} \leq C\left(\|f\|_{C^{k,\alpha}} + \|(u,p)\|_{C^0} + \|\varphi\|_{C^{k+2,\alpha}}\right), \tag{2.8}$$

where $C > 0$ depends on μ, n, k, α, Ω.

(2) Let $f \in W^{k,p}(\Omega, R^n)$, $\varphi \in W^{k+2,p}(\Omega, R^n)$, $k \geq 0$. If $(u,p) \in W^{2,p}(\Omega, R^n) \times W^{1,p}(\Omega)(1 < p < \infty)$ is a solution of (2.7)*, then the solution $(u,p) \in W^{k+2,p}(\Omega, R^n) \times W^{k+1,p}(\Omega)$, and*

$$\|u\|_{W^{k+2,p}} + \|p\|_{W^{k+1,p}} \leq C\left(\|f\|_{W^{k,p}} + \|(u,p)\|_{L^p} + \|\varphi\|_{W^{k+2,p}}\right), \tag{2.9}$$

where $C > 0$ depends on μ, n, k, α, Ω.

Lemma 2.4. *The eigenvalue equation:*

$$\begin{aligned} \Delta T(x_1, x_2) = \lambda T(x_1, x_2), &\quad (x_1, x_2) \in (0, 2\pi) \times (0, 1) \\ T = 0, &\quad x_2 = 0, 1, \\ T(0, x_2) = T(2\pi, x_2), & \end{aligned} \tag{2.10}$$

has eigenvalue $\{\lambda_k\}_{k=1}^{\infty}$, *and*

$$0 < \lambda_1 \le \lambda_2 \le \cdots, \qquad \lambda_k \longrightarrow \infty, \quad \text{as } k \longrightarrow \infty. \tag{2.11}$$

Let X be a linear space, X_1, X_2 two Banach space, X_1 separable, and X_2 reflexive. Let $X \subset X_2$. There exists a linear mapping

$$L : X \longrightarrow X_1 \text{ is one to one and dense.} \tag{2.12}$$

Definition 2.5. A mapping $F : X_2 \to X_1^*$ is called weakly continuous provided

$$\lim_{n\to\infty} \langle F(u_n), v\rangle = \langle F(u_0), v\rangle, \quad \forall v \in X_1, \tag{2.13}$$

for all $\{u_n\} \subset X_2$, $u_n \rightharpoonup u_0$ in X_2.

Lemma 2.6 (see [2]). *If* $F : X_2 \to X_1^*$ *is weakly continuous,* $U \subset X_2$ *is bounded open set,* $0 \in U$, *and*

$$\langle F(u), Lu\rangle \ge 0, \quad \forall u \in \partial U \cap X, \tag{2.14}$$

then the equation $F(u) = 0$ *has a solution in* X_2.

One introduces the Sard-Smale Theorem of infinite dimensional operator. Let X, Y *be two separable Banach Spaces,* $F : X \to Y$ *be a* C^1 *mapping.* F *is called a Fredholm operator provided the derivative operator* $DF(x) : X \to Y$ *is a Fredholm operator for all* $x \in X$.

Lemma 2.7 (see [16, 17] (Sard-Smale Theorem)). *Let* $F : X \to Y$ *be a* C^1 *Fredholm operator with zero index. Then regular value of* F *is dense in* Y. *If* $p \in Y$ *is critical value of* F, *then* $F^{-1}(p)$ *is discrete set.*

3. Existence of Steady State Solution

Theorem 3.1. *If* $\tilde{\sigma}\lambda_1 \ge \max\{(R+1)^2, (\tilde{R}-1)^2/L_e\}$, *and* λ_1 *is the first eigenvalue of the elliptic equation* (2.10), *then for all* Q, $G \in L^q(\Omega)$, (1.8)–(1.13) *have a solution* $(u, T, q, p) \in W^{2,q}(\Omega, R^4) \times W^{1,q}(\Omega)$, $q \ge 2$.

Proof. Let $X = \{\phi = (u, T, q) \in C^{\infty}(\Omega, R^4) \mid \phi \text{ satisfy (1.11)–(1.13)}\}$, and $H_1 = \{\phi = (u, T, q) \in H^1(\Omega, R^4) \mid \phi \text{ satisfy (1.11)–(1.13)}\}$.

Define $F : H_1 \to H_1^*$, for all $\psi = (v, S, z) \in H_1$,

$$\langle F\phi, \psi \rangle = \int_\Omega \Big[\nabla u \nabla v + \sigma u \cdot v - \big(RT - \widetilde{R}q\big) v_2 + \frac{1}{P_r} (u \cdot \nabla) u \cdot v + \nabla T \nabla S \\ -u_2 S + (u \cdot \nabla) TS - QS + L_e \nabla q \nabla z - u_2 z + (u \cdot \nabla) qz - Gz \Big] dx. \tag{3.1}$$

Firstly, we prove the coercivity of F.

$$\begin{aligned}
\langle F\phi, \phi \rangle &= \int_\Omega \Big[|\nabla u|^2 + \sigma u \cdot u - \big(RT - \widetilde{R}q\big) u_2 + \frac{1}{P_r} (\nabla \cdot u) u \cdot u + |\nabla T|^2 \\
&\qquad -u_2 T + (u \cdot \nabla) TT - QT + L_e \big|\nabla q\big|^2 - u_2 q + (u \cdot \nabla) qq - Gq \Big] dx \\
&= \int_\Omega \Big[|\nabla u|^2 + \sigma u \cdot u - (R+1) u_2 T + \big(\widetilde{R} - 1\big) q u_2 + |\nabla T|^2 - QT + L_e \big|\nabla q\big|^2 - Gq \Big] dx \\
&\geq \int_\Omega \Big[|\nabla u|^2 + |\nabla T|^2 + L_e \big|\nabla q\big|^2 + \widetilde{\sigma} |u|^2 - |R+1| |u_2| |T| \\
&\qquad - \big|\widetilde{R} - 1\big| \big|q\big| |u_2| - |Q| |T| - |G| \big|q\big| \Big] dx \\
&\geq \int_\Omega \Big[|\nabla u|^2 + |\nabla T|^2 + L_e \big|\nabla q\big|^2 + \widetilde{\sigma} |u|^2 - \widetilde{\sigma} |u_2|^2 - \frac{1}{2\widetilde{\sigma}} \Big(|R+1|^2 |T|^2 + \big|\widetilde{R} - 1\big|^2 \big|q\big|^2 \Big) \\
&\qquad - \varepsilon |T|^2 - \frac{1}{\varepsilon} |Q|^2 - \varepsilon \big|q\big|^2 - \frac{1}{\varepsilon} |G|^2 \Big] dx \\
&\geq \int_\Omega \Big[|\nabla u|^2 + \frac{1}{2} |\nabla T|^2 + \frac{L_e}{2} \big|\nabla q\big|^2 \Big] dx + \int_\Omega \Big[\frac{1}{2} |\nabla T|^2 - \frac{|R+1|^2}{2\widetilde{\sigma}} |T|^2 \Big] dx \\
&\quad + \int_\Omega \Big[\frac{L_e}{2} |\nabla q|^2 - \frac{1}{2\widetilde{\sigma}} \big|\widetilde{R} - 1\big|^2 \big|q\big|^2 \Big] dx - \varepsilon \int_\Omega \Big[|T|^2 + \big|q\big|^2 \Big] dx - \frac{1}{\varepsilon} \int_\Omega \Big[|Q|^2 + |G|^2 \Big] dx \\
&\geq \int_\Omega \Big[|\nabla u|^2 + \frac{1}{2} |\nabla T|^2 + \frac{L_e}{2} \big|\nabla q\big|^2 \Big] dx - \varepsilon \int_\Omega \Big[|T|^2 + \big|q\big|^2 \Big] dx - \frac{1}{\varepsilon} \int_\Omega \Big[|Q|^2 + |G|^2 \Big] dx.
\end{aligned} \tag{3.2}$$

Let $\varepsilon > 0$ be appropriate small. Then

$$\langle F\phi, \phi \rangle \geq C_1 \int_\Omega \Big[|\nabla u|^2 + |\nabla T|^2 + \big|\nabla q\big|^2 \Big] dx - C_2 \int_\Omega \Big[|Q|^2 + |G|^2 \Big] dx. \tag{3.3}$$

From $Q,\ S \in L^q(\Omega)\ (q \geq 2)$, it follows that

$$\langle F\phi, \phi \rangle \geq C_1 \|\phi\|_{H_1} - C_3. \tag{3.4}$$

Then there exists an appropriate large constant M such that

$$\langle F\phi, \phi \rangle \geq 0, \quad \phi \in \partial B_M \cap X. \tag{3.5}$$

Furthermore, we verify that F is weakly continuous.
Let $\phi_k \rightharpoonup \phi$ in H_1, we have from the Sobolev imbedding Theorem

$$\phi_k \to \phi \quad \text{in } L^p\left(\Omega, R^4\right), \quad 1 \leq p < \infty. \tag{3.6}$$

By $u_0\, v \in L^2$, $u_k \rightharpoonup u_0$ in $H^1(\Omega, R^2)$, it follows that

$$\lim_{k\to\infty} \int_\Omega [(u_0 \cdot \nabla)u_k - (u_0 \cdot \nabla)u_0] \cdot v dx = 0. \tag{3.7}$$

Combining the general Hölder inequality and (3.6), we deduce

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega [(u_k \cdot \nabla)u_k - (u_0 \cdot \nabla)u_k] \cdot v dx \\
&\quad \leq \lim_{k\to\infty} \int_\Omega |u_k - u_0||\nabla u_k||v| dx \\
&\quad \leq \lim_{k\to\infty} \left(\int_\Omega |u_k - u_0|^4 dx\right)^{1/4} \left(\int_\Omega |\nabla u_k|^2 dx\right)^{1/2} \left(\int_\Omega |v|^4 dx\right)^{1/4} \\
&\quad = 0.
\end{aligned} \tag{3.8}$$

Then,

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega [(u_k \cdot \nabla)u_k - (u_0 \cdot \nabla)u_0] \cdot v dx \\
&\quad = \lim_{k\to\infty} \int_\Omega [(u_k \cdot \nabla)u_k - (u_0 \cdot \nabla)u_k] \cdot v dx + \lim_{k\to\infty} \int_\Omega [(u_0 \cdot \nabla)u_k - (u_0 \cdot \nabla)u_0] \cdot v dx \\
&\quad = 0.
\end{aligned} \tag{3.9}$$

Thus,

$$\lim_{k\to\infty} \int_\Omega [(u_k \cdot \nabla)u_k] \cdot v dx = \int_\Omega [(u_0 \cdot \nabla)u_0] \cdot v dx. \tag{3.10}$$

As $u_0\, S \in L^2$, $T_k \rightharpoonup T_0$ in H^1, we find

$$\lim_{k\to\infty} \int_\Omega [(u_0 \cdot \nabla)T_k - (u_0 \cdot \nabla)T_0] S dx = 0. \tag{3.11}$$

Combining the general Hölder inequality and (3.6), we deduce

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega[(u_k\cdot\nabla)T_k-(u_0\cdot\nabla)T_k]S\,dx \\
&\quad\le\lim_{k\to\infty}\int_\Omega|u_k-u_0||\nabla T_k||S|\,dx \\
&\quad\le\lim_{k\to\infty}\left(\int_\Omega|u_k-u_0|^4dx\right)^{1/4}\left(\int_\Omega|\nabla T_k|^2dx\right)^{1/2}\left(\int_\Omega|S|^4dx\right)^{1/4} \\
&\quad=0.
\end{aligned}\tag{3.12}$$

Then,

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega[(u_k\cdot\nabla)T_k-(u_0\cdot\nabla)T_0]S\,dx \\
&\quad=\lim_{k\to\infty}\int_\Omega[(u_k\cdot\nabla)T_k-(u_0\cdot\nabla)T_k]S\,dx \\
&\qquad+\lim_{k\to\infty}\int_\Omega[(u_0\cdot\nabla)T_k-(u_0\cdot\nabla)T_0]S\,dx \\
&\quad=0.
\end{aligned}\tag{3.13}$$

$$\lim_{k\to\infty}\int_\Omega[(u_k\cdot\nabla)T_k]S\,dx=\int_\Omega[(u_0\cdot\nabla)T_0]S\,dx.$$

By $u_0\,z\in L^2$, $q_k\rightharpoonup q_0$ in H^1, we have

$$\lim_{k\to\infty}\int_\Omega\left[(u_0\cdot\nabla)q_k-(u_0\cdot\nabla)q_0\right]z\,dx=0.\tag{3.14}$$

Combining the general Hölder inequality and (3.6), we deduce

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega\left[(u_k\cdot\nabla)q_k-(u_0\cdot\nabla)q_k\right]z\,dx \\
&\quad\le\lim_{k\to\infty}\int_\Omega|u_k-u_0|\left|\nabla q_k\right||z|\,dx \\
&\quad\le\lim_{k\to\infty}\left(\int_\Omega|u_k-u_0|^4dx\right)^{1/4}\left(\int_\Omega\left|\nabla q_k\right|^2dx\right)^{1/2}\left(\int_\Omega|z|^4dx\right)^{1/4} \\
&\quad=0.
\end{aligned}\tag{3.15}$$

Then,

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega\left[(u_k\cdot\nabla)q_k-(u_0\cdot\nabla)q_0\right]zdx\\
&\quad=\lim_{k\to\infty}\int_\Omega\left[(u_k\cdot\nabla)q_k-(u_0\cdot\nabla)q_k\right]zdx\\
&\qquad+\lim_{k\to\infty}\int_\Omega\left[(u_0\cdot\nabla)q_k-(u_0\cdot\nabla)q_0\right]zdx\\
&\quad=0.
\end{aligned}\tag{3.16}$$

Thus,

$$\lim_{k\to\infty}\int_\Omega\left[(u_k\cdot\nabla)q_k\right]zdx=\int_\Omega\left[(u_0\cdot\nabla)q_0\right]zdx.\tag{3.17}$$

Combining (3.10)–(3.17), we have

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega\langle F\phi_k,\psi\rangle dx\\
&\quad=\lim_{k\to\infty}\int_\Omega\Big[\nabla u_k\nabla v+\sigma u_k\cdot v-\left(RT_k-\widetilde{R}q_k\right)v_2+\frac{1}{P_r}(u_k\cdot\nabla)u_k\cdot v\\
&\qquad+\nabla T_k\nabla S-u_{k2}S+(u_k\cdot\nabla)T_kS-QS+L_e\nabla q_k\nabla z\\
&\qquad-u_{k2}z+(u_k\cdot\nabla)q_kz-Gz\Big]dx\\
&\quad=\int_\Omega\Big[\nabla u_0\nabla v+\sigma u_0\cdot v-\left(RT_0-\widetilde{R}q_0\right)v_2+\frac{1}{P_r}(u_0\cdot\nabla)u_0\cdot v+\nabla T_0\nabla S\\
&\qquad-u_{02}S+(u_0\cdot\nabla)T_0S-QS+L_e\nabla q_0\nabla z-u_{02}z+(u_0\cdot\nabla)q_0z-Gz\Big]dx\\
&\quad=\int_\Omega\langle F\phi_0,\psi\rangle dx,\quad\forall\psi\in H_1,
\end{aligned}\tag{3.18}$$

which imply that $F:H_1\to H_1^*$ is weakly continuous. According to Lemma 2.6, (1.8)–(1.13) have a solution $\phi=(u,T,q)\in H^1(\Omega,R^4)$.

Lastly we prove that $(\phi,p)\in W^{2,q}(\Omega,R^4)\times W^{1,q}(\Omega)$, $2\le q<\infty$.

From the Hölder inequality, we see

$$\begin{aligned}
\int_\Omega|(u\cdot\nabla)u|^{3/2}dx&\le\int_\Omega|u|^{3/2}|Du|^{3/2}dx\\
&\le\left(\int_\Omega|Du|^2dx\right)^{3/4}\left(\int_\Omega|u|^6dx\right)^{1/4}.
\end{aligned}\tag{3.19}$$

Then, $(u\cdot\nabla)u\in L^{3/2}(\Omega,R^2)$.

For the Stokes equation:

$$\begin{aligned} -\Delta u + \nabla p &= g, \\ \operatorname{div} u &= 0, \\ u = 0, &\qquad x_2 = 0, 1, \\ u(0, x_2) &= u(2\pi, x_2), \end{aligned} \tag{3.20}$$

since $g = -\sigma u + (RT - \widetilde{R}q)\vec{\kappa} - (1/P_r)(u \cdot \nabla)u \in L^{3/2}(\Omega)$, according to ADN theory, (3.20) has a solution:

$$(u, p) \in W^{2,3/2}\left(\Omega, R^2\right) \times W^{1,3/2}(\Omega). \tag{3.21}$$

By the Hölder inequality, we have

$$\begin{aligned} \int_\Omega |(u \cdot \nabla)T|^{3/2} dx &\le \int_\Omega |u||DT|^{3/2} dx \\ &\le \left(\int_\Omega |DT|^2 dx\right)^{3/4} \left(\int_\Omega |u|^6 dx\right)^{1/4} \le C \int_\Omega |DT|^2 dx, \end{aligned} \tag{3.22}$$

thus, $(u \cdot \nabla)T \in L^{3/2}(\Omega)$.

For the elliptic equation:

$$\begin{aligned} -\Delta T &= f_1, \\ T = 0, &\qquad x_2 = 0, 1, \\ T(0, x_2) &= T(2\pi, x_2), \end{aligned} \tag{3.23}$$

as $f_1 = u_2 - (u \cdot \nabla)T + Q \in L^{3/2}(\Omega)$, according to theory of linear elliptic equation, (3.23) has a solution

$$T \in W^{2,3/2}(\Omega). \tag{3.24}$$

From the Hölder inequality, we see

$$\int_\Omega \left|(u \cdot \nabla)q\right|^{3/2} dx \le \int_\Omega |u|^{3/2} \left|Dq\right|^{3/2} dx \le \left(\int_\Omega \left|\nabla q\right|^2 dx\right)^{3/4} \left(\int_\Omega |u|^6 dx\right)^{1/4}, \tag{3.25}$$

thus, $(u \cdot \nabla)q \in L^{3/2}(\Omega)$.

For the elliptic equation:

$$
\begin{aligned}
-\Delta q &= f_2, \\
q = 0, &\qquad x_2 = 0, 1, \\
q(0, x_2) &= q(2\pi, x_2),
\end{aligned}
\tag{3.26}
$$

as $f_2 = u_2 - (u \cdot \nabla)q + G \in L^{3/2}(\Omega)$, according to theory of linear elliptic equation, (3.26) has a solution

$$q \in W^{2,(3/2)}(\Omega). \tag{3.27}$$

By the Sobolev imbedding Theorem, we see

$$W^{2,(3/2)}(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow L^6(\Omega). \tag{3.28}$$

Then $T, q \in L^6$, and

$$\int_\Omega |(u \cdot \nabla)u|^3 dx \le \left(\int_\Omega |Du|^6 dx\right)^{1/2} \left(\int_\Omega |u|^6 dx\right)^{1/2} \le C\|u\|^6_{W^{2,(3/2)}}. \tag{3.29}$$

Thus, $(u \cdot \nabla)u \in L^3(\Omega, R^2)$. Consequently $g \in L^3(\Omega)$. According to ADN theory, (3.20) has a solution

$$(u, p) \in W^{2,3}\left(\Omega, R^2\right) \times W^{1,3}(\Omega). \tag{3.30}$$

Similarly, we deduce

$$(T, q) \in W^{2,3}\left(\Omega, R^2\right). \tag{3.31}$$

By doing the same procedures as above, (1.8)–(1.13) have a solution $(u, T, q, p) \in W^{2,q}(\Omega, R^4) \times W^{1,q}$, $q \ge 2$. □

4. Regularity of Steady State Solution

Theorem 4.1. *If $\tilde{\sigma}\lambda_1 \ge \max\{(R+1)^2, (\tilde{R}-1)^2/L_e\}$, and λ_1 is the first eigenvalue of elliptic equation* (2.10)*, then there exists a dense open set $\digamma \subset L^q(\Omega, R^2)(q \ge 2)$, the solution to* (1.8)–(1.13) *is finite for all $(Q, G) \in \digamma$.*

Proof. There are the following estimates for (1.8)–(1.13):

$$\|u\|_{W^{2,q}} + \|T\|_{W^{2,q}} + \|q\|_{W^{2,q}} + \|p\|_{W^{1,q}} \le C(\|Q\|_{L^q} + \|G\|_{L^q} + 1)^3, \quad q \ge 2. \tag{4.1}$$

As $\phi = (u, T, q)$ is a solution to (1.8)–(1.13), we have $\langle F\phi, \phi\rangle = 0$. Then

$$\int_\Omega \left[|\nabla u|^2 + \sigma u \cdot u - \left(RT - \widetilde{R}q\right)u_2 + |\nabla T|^2 - u_2 T - QT + L_e\left|\nabla q\right|^2 - u_2 q - Gq\right]dx = 0. \tag{4.2}$$

Thus,

$$\int_\Omega \left[|\nabla u|^2 + \sigma u \cdot u - (R+1)Tu_2 + \left(\widetilde{R} - 1\right)qu_2 + |\nabla T|^2 - QT + L_e\left|\nabla q\right|^2 - Gq\right]dx = 0. \tag{4.3}$$

Consequently,

$$\begin{aligned}
&\int_\Omega \left[|\nabla u|^2 + |\nabla T|^2 + L_e\left|\nabla q\right|^2 + \widetilde{\sigma}\left|u^2\right|\right]dx \\
&\quad\le \int_\Omega \left[|R+1||u_2||T| + \left|\widetilde{R} - 1\right|\left|q\right||u_2| + |Q||T| + |G|\left|q\right|\right]dx \\
&\quad\le \int_\Omega \left[\widetilde{\sigma}|u_2|^2 + \frac{|R+1|^2}{2\widetilde{\sigma}}|T|^2 + \frac{\left|\widetilde{R}-1\right|^2}{2\widetilde{\sigma}}\left|q\right|^2 + \varepsilon|T|^2 + \frac{1}{\varepsilon}|Q|^2 + \varepsilon\left|q\right|^2 + \frac{1}{\varepsilon}|G|^2\right]dx.
\end{aligned} \tag{4.4}$$

Choosing an appropriate constant ε, we see

$$\int_\Omega \left[|\nabla u|^2 + |\nabla T|^2 + \left|\nabla q\right|^2\right]dx \le C(\|Q\|_{L^q} + \|G\|_{L^q} + 1)^2. \tag{4.5}$$

According to the Sobolev imbedding Theorem and (4.5), we deduce

$$\|u\|_{L^q} \le C\|u\|_{L^{2q}} \le C\|Du\|_{L^2} \le C(\|Q\|_{L^q} + \|G\|_{L^q} + 1). \tag{4.6}$$

Using the Gagliardo-Nironberg inequality and Young inequality, we have

$$\|Du\|_{L^{2q}} \le C\left(\|u\|_{W^{2,q}}^{1/2}\|u\|_{W^{1,2}}^{1/2}\right) \le \varepsilon\left\|D^2 u\right\|_{L^q} + C\varepsilon^{-1}\|Du\|_{L^2}, \tag{4.7}$$

$$\|DT\|_{L^{2q}} \le C\left(\|T\|_{W^{2,q}}^{1/2}\|T\|_{W^{1,2}}^{1/2}\right) \le \varepsilon\left\|D^2 T\right\|_{L^q} + C\varepsilon^{-1}\|DT\|_{L^2}, \tag{4.8}$$

$$\left\|Dq\right\|_{L^{2q}} \le C\left(\left\|q\right\|_{W^{2,q}}^{1/2}\left\|q\right\|_{W^{1,2}}^{1/2}\right) \le \varepsilon\left\|D^2 q\right\|_{L^q} + C\varepsilon^{-1}\left\|Dq\right\|_{L^2}. \tag{4.9}$$

Combining the Hölder inequality and (4.6)–(4.9), we see

$$\begin{aligned}
&\|u\cdot\nabla u\|_{L^q}+\|u\cdot\nabla T\|_{L^q}+\left\|u\cdot\nabla q\right\|_{L^q}\\
&\quad\le \|u\|_{L^{2q}}\left(\|Du\|_{L^{2q}}+\|DT\|_{L^{2q}}+\left\|Dq\right\|_{L^{2q}}\right)\\
&\quad\le C(\|Q\|_{L^q}+\|G\|_{L^q}+1)\\
&\quad\times\left[\varepsilon\left(\left\|D^2u\right\|_{L^q}+\left\|D^2T\right\|_{L^q}+\left\|D^2q\right\|_{L^q}\right)+\varepsilon^{-1}\left(\|Du\|_{L^2}+\|DT\|_{L^2}+\left\|Dq\right\|_{L^2}\right)\right].
\end{aligned}\tag{4.10}$$

Since u, T, q are solutions to (3.20), (3.23), and (3.26), according to ADN theory and theory of linear elliptic equation, we have

$$\begin{aligned}
&\|u\|_{W^{2,q}}+\|T\|_{W^{2,q}}+\left\|q\right\|_{W^{2,q}}+\left\|p\right\|_{W^{1,q}}\\
&\quad\le C\left(\left\|g\right\|_{L^q}+\left\|f_1\right\|_{L^q}+\left\|f_2\right\|_{L^q}\right)\\
&\quad\le C\left(\|u\|_{L^q}+\|T\|_{L^q}+\left\|q\right\|_{L^q}+\|u\cdot\nabla u\|_{L^q}+\|u_2\|_{L^q}+\|u\cdot\nabla T\|_{L^q}\right.\\
&\qquad\quad\left.+\left\|u\cdot\nabla q\right\|_{L^q}+\|Q\|_{L^q}+\|G\|_{L^q}\right)\\
&\quad\le C\left(\|u\|_{L^q}+\|T\|_{L^q}+\left\|q\right\|_{L^q}\right)\\
&\qquad+C\left(\|u\cdot\nabla u\|_{L^q}+|u\cdot\nabla T\|_{L^q}+\left\|u\cdot\nabla q\right\|_{L^q}\right)+C(\|Q\|_{L^q}+\|G\|_{L^q}).
\end{aligned}\tag{4.11}$$

From (4.6) and (4.10), it follows that

$$\begin{aligned}
&\|u\|_{W^{2,q}}+\|T\|_{W^{2,q}}+\left\|q\right\|_{W^{2,q}}+\left\|p\right\|_{W^{1,q}}\\
&\quad\le C(\|Q\|_{L^q}+\|G\|_{L^q}+1)+C(\|Q\|_{L^q}+\|G\|_{L^q}+1)\left[\varepsilon\left(\left\|D^2u\right\|_{L^q}+\left\|D^2T\right\|_{L^q}+\left\|D^2q\right\|_{L^q}\right)\right]\\
&\qquad+C(\|Q\|_{L^q}+\|G\|_{L^q}+1)\varepsilon^{-1}\left(\|Du\|_{L^2}+\|DT\|_{L^2}+\left\|Dq\right\|_{L^2}\right)+C(\|Q\|_{L^q}+\|G\|_{L^q}).
\end{aligned}\tag{4.12}$$

Let $C\varepsilon(\|Q\|_{L^q}+\|G\|_{L^q}+1)=1/2$. Then

$$\begin{aligned}
&\|u\|_{W^{2,q}}+\|T\|_{W^{2,q}}+\left\|q\right\|_{W^{2,q}}+\left\|p\right\|_{W^{1,q}}\\
&\quad\le C(\|Q\|_{L^q}+\|G\|_{L^q}+1)+C(\|Q\|_{L^q}+\|G\|_{L^q}+1)^3\\
&\quad\le C(\|Q\|_{L^q}+\|G\|_{L^q}+1)^3,
\end{aligned}\tag{4.13}$$

which imply (4.1).

We introduce the mappings:

$$G = L + H : W^{2,q}\left(\Omega, R^4\right) \times W^{1,q}(\Omega) \longrightarrow L^q\left(\Omega, R^4\right),$$
$$L(u,p) = \begin{pmatrix} -\Delta u + \nabla p \\ -\Delta T \\ -L_e \Delta q \end{pmatrix},$$
$$H(u,p) = \begin{pmatrix} \sigma u - \left(RT - \widetilde{R}q\right)\vec{\kappa} + \dfrac{1}{P_r}(u \cdot \nabla)u \\ -u_2 + (u \cdot \nabla)T \\ -u_2 + (u \cdot \nabla)q \end{pmatrix}. \tag{4.14}$$

Let

$$f(x) = \begin{pmatrix} 0 \\ Q(x) \\ G(x) \end{pmatrix} \in L^q\left(\Omega, R^4\right). \tag{4.15}$$

Then, (1.8)–(1.13) can be rewrite as the following mapping

$$F(u,T,q,p) = f(x). \tag{4.16}$$

Clearly, $F : W^{2,q}(\Omega, R^4) \times W^{1,q}(\Omega) \to L^q(\Omega, R^4)$ is a completely continuous field. Thus F is a Fredholm operator with zero index. According to the Sard-Smale Theorem, the regular value of F is dense in $\mathcal{F} \subset L^q(\Omega, R^4)$, and $F^{-1}(f)$ is discrete in $W^{2,q}(\Omega, R^4) \times W^{1,q}(\Omega)$ for all $f \in \mathcal{F}$. $F^{-1}(f)$ in $W^{2,q}(\Omega, R^4) \times W^{1,q}(\Omega)$ is finite for $q \geq 2$ from (4.1). Consequently, $f \in \mathcal{F}$ is interior point and $\mathcal{F}$ is an open set. □

Theorem 4.2. *If* $\widetilde{\sigma}\lambda_1 \geq \max\{(R+1)^2, (\widetilde{R}-1)^2/L_e\}$, *and* λ_1 *is the first eigenvalue of elliptic equation* (2.10), *then*

(1) *the Equations* (1.8)–(1.13) *have a classical solution* $(u,T,q,p) \in C^{2,\alpha}(\Omega, R^4) \times C^{1,\alpha}(\Omega)$ *for all* $Q, G \in C^\alpha(\Omega)$,

(2) *there exists a tense open set* $\mathcal{F} \subset C^\alpha(\Omega, R^2)$, *such that the solution to* (1.8)–(1.13) *is finite for all* $(Q, G) \in \mathcal{F}$,

(3) *the solution* (u,T,q,p) *to* (1.8)–(1.13) *is in* $C^\infty(\Omega, R^5)$ *if* $Q, G \in C^\infty(\Omega)$.

Proof. We prove the assertion (1). As $C^\alpha(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \leq \infty$, $Q, G \in L^q(\Omega)$ for all $Q, G \in C^\alpha(\Omega)$. From Theorem 4.1, (1.8)–(1.13) have a strong solution $(u,T,q,p) \in W^{2,q}(\Omega, R^4) \times W^{1,q}$, $q \geq 2$.

When $q \geq 2$, $W^{2,q}(\Omega) \hookrightarrow C^{1,\alpha}(\Omega)$, $\alpha = 1 - (2/q)$ from the Sobolev imbedding theorem. Then

$$(u,T,q) \in C^{1,\alpha}\left(\Omega, R^4\right). \tag{4.17}$$

Thus,

$$(u \cdot \nabla)u, \qquad (u \cdot \nabla)T, \qquad (u \cdot \nabla)q \in C^{\alpha}(\Omega). \tag{4.18}$$

For Stokes equation:

$$\begin{aligned} -\Delta u + \nabla p &= g, \\ \operatorname{div} u &= 0, \\ u = 0, &\qquad x_2 = 0, 1, \\ u(0, x_2) &= u(2\pi, x_2), \end{aligned} \tag{4.19}$$

as $g = -\sigma u + (RT - \widetilde{R}q)\vec{\kappa} - (1/P_r)(u \cdot \nabla)u \in C^{\alpha}(\Omega)$, according to ADN theory, (4.19) has a solution:

$$(u, p) \in C^{2,\alpha}\left(\Omega, R^2\right) \times C^{1,\alpha}(\Omega). \tag{4.20}$$

For the elliptic equation:

$$\begin{aligned} -\Delta T &= f_1, \\ T = 0, &\qquad x_2 = 0, 1, \\ T(0, x_2) &= T(2\pi, x_2), \end{aligned} \tag{4.21}$$

as $f_1 = u_2 - (u \cdot \nabla)T + Q \in C^{\alpha}(\Omega)$, according to theory of linear elliptic equation, (4.21) has a solution:

$$T \in C^{2,\alpha}(\Omega). \tag{4.22}$$

For the elliptic equation:

$$\begin{aligned} -\Delta q &= f_2, \\ q = 0, &\qquad x_2 = 0, 1, \\ q(0, x_2) &= q(2\pi, x_2), \end{aligned} \tag{4.23}$$

as $f_2 = u_2 - (u \cdot \nabla)q + G \in C^{\alpha}(\Omega)$, according to theory of linear elliptic equation, (4.23) has a solution:

$$q \in C^{2,\alpha}(\Omega). \tag{4.24}$$

Thus, (1.8)–(1.13) have a solution $(u, T, q, p) \in C^{2,\alpha}(\Omega, R^4) \times C^{1,\alpha}(\Omega)$.

Secondly, we prove the assertion (2). Combining ADN theory, theory of linear elliptic equation, and (4.19)–(4.23), we have

$$\begin{aligned}
&\|u\|_{C^{2,\alpha}} + \|T\|_{C^{2,\alpha}} + \|q\|_{C^{2,\alpha}} + \|p\|_{C^{1,\alpha}} \\
&\quad \le C\left(\|g\|_{C^\alpha} + \|f_1\|_{C^\alpha} + \|f_2\|_{C^\alpha}\right) \\
&\quad \le C\left(\|u\|_{C^\alpha} + \|T\|_{C^\alpha} + \|q\|_{C^\alpha} + \|u\cdot\nabla u\|_{C^\alpha} + \|u_2\|_{C^\alpha}\right. \\
&\qquad \left. + \|u\cdot\nabla T\|_{C^\alpha} + \|u\cdot\nabla q\|_{C^\alpha} + \|Q\|_{C^\alpha} + \|G\|_{C^\alpha}\right) \\
&\quad \le C\left(\|u\|_{C^\alpha} + \|T\|_{C^\alpha} + \|q\|_{C^\alpha}\right) + C\left(\|u\cdot\nabla u\|_{C^\alpha} + |u\cdot\nabla T\|_{C^\alpha} + \|u\cdot\nabla q\|_{C^\alpha}\right) \\
&\qquad + C(\|Q\|_{C^\alpha} + \|G\|_{C^\alpha}) \\
&\quad \le C\left(\|u\|_{W^{1,q}} + \|T\|_{W^{1,q}} + \|q\|_{W^{1,q}}\right) \\
&\qquad + C\left(\|u\cdot\nabla u\|_{W^{1,q}} + |u\cdot\nabla T\|_{W^{1,q}} + \|u\cdot\nabla q\|_{W^{1,q}}\right) + C(\|Q\|_{C^\alpha} + \|G\|_{C^\alpha}) \\
&\quad \le C\left(\|u\|_{W^{2,q}} + \|T\|_{W^{2,q}} + \|q\|_{W^{2,q}}\right) \\
&\qquad + C\|u\|_{W^{2,q}}\left(\|u\|_{W^{2,q}} + \|T\|_{W^{2,q}} + \|q\|_{W^{2,q}}\right) + C(\|Q\|_{C^\alpha} + \|G\|_{C^\alpha}) \\
&\quad \le C(\|Q\|_{L^q} + \|G\|_{L^q} + 1) + C(\|Q\|_{L^q} + \|G\|_{L^q} + 1)^6 + C(\|Q\|_{C^\alpha} + \|G\|_{C^\alpha}) \\
&\quad \le C(\|Q\|_{C^\alpha} + \|G\|_{C^\alpha} + 1) + C(\|Q\|_{C^\alpha} + \|G\|_{C^\alpha} + 1)^6.
\end{aligned} \tag{4.25}$$

We introduce the mappings:

$$\begin{gathered}
F = L + H : C^{2,\alpha}\left(\Omega, R^4\right) \times C^{1,\alpha}(\Omega) \longrightarrow C^\alpha\left(\Omega, R^4\right), \\
L(u,p) = \begin{pmatrix} -\Delta u + \nabla p \\ -\Delta T \\ -L_e \Delta q \end{pmatrix}, \\
H(u,p) = \begin{pmatrix} \sigma u - \left(RT - \widetilde{R} q\right)\vec{\kappa} + \dfrac{1}{P_r}(u\cdot\nabla)u \\ -u_2 + (u\cdot\nabla)T \\ -u_2 + (u\cdot\nabla)q \end{pmatrix}.
\end{gathered} \tag{4.26}$$

Let

$$f(x) = \begin{pmatrix} 0 \\ Q(x) \\ G(x) \end{pmatrix} \in C^\alpha\left(\Omega, R^4\right), \tag{4.27}$$

then, (1.8)–(1.13) can be rewritten as

$$F(u,T,q,p) = f(x). \tag{4.28}$$

Clearly, $F : C^{2,\alpha}(\Omega, R^4) \times C^{1,\alpha}(\Omega) \to C^{\alpha}(\Omega, R^4)$ is complete continuous field. Then F is a Fredholm operator with zero index. The regular value $\mathcal{F} \subset C^{\alpha}(\Omega, R^4)$ is dense from Sard-Smale theorem, and $F^{-1}(f)$ is discrete in $C^{2,\alpha}(\Omega, R^4) \times C^{1,\alpha}(\Omega)$ for all $f \in \mathcal{F}$. From (4.25), we find that $F^{-1}(f)$ is finite in $C^{2,\alpha}(\Omega, R^4) \times C^{1,\alpha}(\Omega)$. Thus, $f \in \mathcal{F}$ is an interior point and $\mathcal{F}$ is an open set.

Finally we prove the assertion (3). Since $Q, G \in C^{\infty}(\Omega)$, it is true that $Q, G \in W^{k,q}(\Omega)$ (k is arbitrary integer). According to Theorem 4.1, we conclude that $(u, T, q, p) \in W^{k+2,q}(\Omega, R^4) \times W^{k+1,q}(\Omega, R)$ (k is arbitrary integer). From the Sobolev imbedding theorem, $(u, T, q) \in C^{k+1}(\Omega, R^4) \times C^{k}(\Omega, R)$ (k is arbitrary integer). Then $(u, T, q, p) \in C^{\infty}(\Omega, R^5)$. □

5. Remark

$\tilde{\sigma}\lambda_1 \geq \max\{(R+1)^2, (\tilde{R}-1)^2/L_e\}$ is a sufficient condition, not a necessary condition. In fact, if the condition does not hold, (1.8)–(1.13) may have not solution for some Q, G.

Returning to the problem of atmospheric circulation, as the temperature source and the moisture source are changed, the state of the atmospheric circulation changes, but there is still a corresponding steady state.

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