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*Research Article*

# Fixed Point Theorems for Various Classes of Cyclic Mappings

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We introduce new classes of cyclic mappings and we study the existence and uniqueness of fixed points for such mappings. The presented theorems generalize and improve several existing results in the literature.

## 1. Introduction

The Banach contraction principle is one of the most important results on fixed point theory. Several extensions and generalizations of this result have appeared in the literature and references therein, see [1–11]. One of the most interesting extensions was given by Kirk et al. in [12].

**Theorem 1.1** (see [12]). *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$. Suppose that $T : A \cup B \to A \cup B$ satisfies*

(i) $T(A) \subseteq B$, $T(B) \subseteq A$;

(ii) *there exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq k d(x, y), \quad \forall x \in A,\ y \in B. \tag{1.1}$$

*Then $T$ has a unique fixed point that belongs to $A \cap B$.*

A mapping satisfying (i) is called cyclic. Recently, many results dealing with cyclic contractions have appeared in several works (see, e.g., [13–30]). Karpagam and Agrawal [31, 32] introduced the notion of a cyclic orbital contraction, and obtained a unique fixed point theorem for such class of mappings (see also [33, 34]).

In what follows, $\mathbb{N}$ is the set of positive integers.

*Definition 1.2* (see [31]). Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$, let $T : A \cup B \to A \cup B$ be a cyclic mapping such that for some $x \in A$, there exists a $k_x \in (0,1)$ such that

$$d\left(T^{2n}x, Ty\right) \leq k_x d\left(T^{2n-1}x, y\right), \tag{1.2}$$

for all $n \in \mathbb{N}$ and $y \in A$. Then $T$ is called a cyclic orbital contraction.

**Theorem 1.3** (see [31]). *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$, and let $T : A \cup B \to A \cup B$ be a cyclic orbital contraction. Then $T$ has a unique fixed point that belongs to $A \cap B$.*

Very recently, Chen [19] introduced the class of cyclic orbital stronger Meir-Keeler contraction.

*Definition 1.4* (see [19]). Let $(X, d)$ be a metric space. A mapping $\psi : [0, \infty) \to [0, 1)$ is called a stronger Meir-Keeler type mapping in $X$ if the following condition holds for all $\varepsilon \geq 0$, there exist $\delta_\varepsilon > 0$ and $\gamma_\varepsilon \in (0,1)$ such that for all $x, y \in X$, we have

$$\varepsilon \leq d(x, y) < \varepsilon + \delta_\varepsilon \Longrightarrow \psi(d(x, y)) < \gamma_\varepsilon. \tag{1.3}$$

*Definition 1.5* (see [19]). Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. Suppose that $T : A \cup B \to A \cup B$ is a cyclic map such that for some $x \in A$, there exists a stronger Meir-Keeler type mapping $\psi_x : [0, \infty) \to [0, 1)$ such that

$$d\left(T^{2n}x, Ty\right) \leq \psi_x\left(d\left(T^{2n-1}x, y\right)\right) d\left(T^{2n-1}x, y\right), \tag{1.4}$$

for all $n \in \mathbb{N}$ and $y \in A$. Then $T$ is called a a cyclic orbital stronger Meir-Keeler $\psi_x$-contraction.

Clearly, if $T : A \cup B \to A \cup B$ is a cyclic orbital contraction, then $T$ is a cyclic orbital stronger Meir-Keeler $\psi_x$-contraction, where $\psi_x(t) = k_x$ for all $t \geq 0$.

**Theorem 1.6** (see [19]). *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$, and let $T : A \cup B \to A \cup B$ be a cyclic orbital stronger Meir-Keeler $\psi_x$-contraction, for some $x \in A$. Then $T$ has a unique fixed point that belongs to $A \cap B$.*

In this work, we introduce new classes of cyclic contractive mappings and we study the existence and uniqueness of fixed points for such mappings. Our obtained results extend and generalize several existing fixed point theorems in the literature, including Theorems 1.3 and 1.6.

## 2. Main Results

We denote by $\Psi$ the class of stronger Meir-Keeler type mappings (see Definition 1.4).

The following lemma will be useful later.

**Lemma 2.1.** *Let $(X, d)$ be a metric space, $\psi \in \Psi$, $\{x_n\}, \{y_n\} \subset X$. If $d(x_n, y_n) \to r$ as $n \to \infty$, then there exist $\gamma = \gamma_r \in (0,1)$ and $N \in \mathbb{N}$ such that*

$$\psi(d(x_n, y_n)) < \gamma, \quad \forall n \geq N. \tag{2.1}$$

*Proof.* Since $\psi \in \Psi$, there exist $\delta_r > 0$ and $\gamma = \gamma_r \in (0,1)$ such that for all $x, y \in X$, we have

$$r \leq d(x, y) < r + \delta_r \Longrightarrow \psi(d(x, y)) < \gamma_r. \tag{2.2}$$

Since $d(x_n, y_n) \to r^+$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that

$$r \leq d(x_n, y_n) < r + \delta_r, \quad \forall n \geq N. \tag{2.3}$$

It follows from (2.2) that

$$\psi(d(x_n, y_n)) < \gamma_r, \quad \forall n \geq N. \tag{2.4}$$

This makes end to the proof. □

Our first result is the following.

**Theorem 2.2.** *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that for some $x \in A$ there exists $\psi_x \in \Psi$ such that*

$$d\left(T^{2n}x, Ty\right) \leq \psi_x\left(d\left(T^{2n-1}x, y\right)\right) \frac{d(T^{2n-1}x, T^{2n}x) + d(Ty, y)}{2}, \tag{2.5}$$

*for all $n \in \mathbb{N}$ and $y \in A$. Then $T$ has a unique fixed point that belongs to $A \cap B$.*

*Proof.* At first, since $T$ is a cyclic mapping and $x \in A$, we have $\{T^{2n}x : n \in \mathbb{N}\} \subset A$. So, taking $y = T^{2n}x$ in (2.5), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d\left(T^{2n}x, T^{2n+1}x\right) &\leq \psi_x\left(d\left(T^{2n-1}x, T^{2n}x\right)\right) \frac{d(T^{2n-1}x, T^{2n}x) + d(T^{2n}x, T^{2n+1}x)}{2} \\ &\leq \frac{d(T^{2n-1}x, T^{2n}x) + d(T^{2n}x, T^{2n+1}x)}{2}. \end{aligned} \tag{2.6}$$

Thus for all $n \in \mathbb{N}$, we have

$$d\left(T^{2n}x, T^{2n+1}x\right) \leq d\left(T^{2n-1}x, T^{2n}x\right). \tag{2.7}$$

Again, taking $y = T^{2n}x$ in (2.5), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d\left(T^{2n+1}x, T^{2n+2}x\right) &= d\left(T^{2n+2}x, T^{2n+1}x\right) \\ &\leq \varphi_x\left(d\left(T^{2n+1}x, T^{2n}x\right)\right)\frac{d(T^{2n+1}x, T^{2n+2}x) + d(T^{2n+1}x, T^{2n}x)}{2} \\ &\leq \frac{d(T^{2n+1}x, T^{2n+2}x) + d(T^{2n+1}x, T^{2n}x)}{2}. \end{aligned} \tag{2.8}$$

Thus for all $n \in \mathbb{N}$, we have

$$d\left(T^{2n+1}x, T^{2n+2}x\right) \leq d\left(T^{2n}x, T^{2n+1}x\right). \tag{2.9}$$

It follows from (2.7) and (2.9) that $\{d(T^n x, T^{n+1}x)\}$ is a decreasing sequence. Then there exists $\varepsilon \geq 0$ such that

$$\lim_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = \varepsilon. \tag{2.10}$$

Applying Lemma 2.1 with $r = \varepsilon$, we obtain that there exist $n_0 \in \mathbb{N}$ and $\gamma_\varepsilon \in (0,1)$ such that

$$\varphi_x\left(d\left(T^n x, T^{n+1}x\right)\right) < \gamma_\varepsilon, \quad \forall n \geq n_0. \tag{2.11}$$

Denote $k_0 = [(n_0+1)/2] + 1$, where $[(n_0+1)/2]$ is the integer part of $(n_0+1)/2$. By (2.11) and (2.5), for all $k \geq k_0$, we have

$$\begin{aligned} d\left(T^{2k}x, T^{2k+1}x\right) &\leq \varphi_x\left(d\left(T^{2k-1}x, T^{2k}x\right)\right)\frac{d(T^{2k-1}x, T^{2k}x) + d(T^{2k+1}x, T^{2k}x)}{2} \\ &\leq \gamma_\varepsilon \frac{d(T^{2k-1}x, T^{2k}x) + d(T^{2k+1}x, T^{2k}x)}{2}. \end{aligned} \tag{2.12}$$

This implies that

$$d\left(T^{2k}x, T^{2k+1}x\right) \leq \left(\frac{\gamma_\varepsilon}{2-\gamma_\varepsilon}\right) d\left(T^{2k-1}x, T^{2k}x\right), \quad \forall k \geq k_0. \tag{2.13}$$

Note that $\gamma_\varepsilon/(2-\gamma_\varepsilon) \in (0,1)$. Similarly, from (2.11) and (2.5), for all $k \geq k_0$, we have

$$\begin{aligned} d\left(T^{2k+2}x, T^{2k+1}x\right) &\leq \varphi_x\left(d\left(T^{2k+1}x, T^{2k}x\right)\right)\frac{d(T^{2k+1}x, T^{2k+2}x) + d(T^{2k}x, T^{2k+1}x)}{2} \\ &\leq \gamma_\varepsilon \frac{d(T^{2k+1}x, T^{2k+2}x) + d(T^{2k}x, T^{2k+1}x)}{2}. \end{aligned} \tag{2.14}$$

This implies that

$$d\left(T^{2k+1}x, T^{2k+2}x\right) \leq \left(\frac{\gamma_\varepsilon}{2-\gamma_\varepsilon}\right) d\left(T^{2k}x, T^{2k+1}x\right), \quad \forall k \geq k_0. \tag{2.15}$$

Now, it follows from (2.13) and (2.15) that

$$d\left(T^{2k_0+n}x, T^{2k_0+n+1}x\right) \leq \left(\frac{\gamma_\varepsilon}{2-\gamma_\varepsilon}\right)^n d\left(T^{2k_0-1}x, T^{2k_0}x\right), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{2.16}$$

Denote $q_\varepsilon = \gamma_\varepsilon / (2-\gamma_\varepsilon)$. From (2.16), for all $p \in \mathbb{N}$, we have

$$\begin{aligned} d\left(T^{2k_0+n}x, T^{2k_0+n+p}x\right) &\leq \sum_{i=n}^{n+p-1} d\left(T^{2k_0+i}x, T^{2k_0+i+1}x\right) \\ &\leq \frac{q_\varepsilon^n}{1-q_\varepsilon} d\left(T^{2k_0-1}x, T^{2k_0}x\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.17}$$

This implies that $\{T^n x\}$ is a Cauchy sequence. Since $(X, d)$ is complete, $A$ and $B$ are closed, and $\{T^n x\} \subset A \cup B$, there exists a $z \in A \cup B$ such that $T^n x \to z$ as $n \to \infty$. On the other hand, since $T$ is a cyclic mapping and $x \in A$, we have $\{T^{2n}x\} \subset A$ and $\{T^{2n+1}x\} \subset B$. Since $A$ and $B$ are closed, this implies that $z \in A \cap B$. On the other hand, since $d(T^{2n-1}x, z) \to 0$ as $n \to \infty$, by Lemma 2.1, there exist $N \in \mathbb{N}$ and $\gamma_0 \in (0,1)$ such that

$$\psi\left(d\left(T^{2n-1}x, z\right)\right) < \gamma_0, \quad \forall n \geq N. \tag{2.18}$$

From (2.5) and (2.18), for all $n \geq N$, we have

$$\begin{aligned} d\left(T^{2n}x, Tz\right) &\leq \psi_x\left(d\left(T^{2n-1}x, z\right)\right) \frac{d(T^{2n-1}x, T^{2n}x) + d(z, Tz)}{2} \\ &\leq \gamma_0 \frac{d(T^{2n-1}x, T^{2n}x) + d(z, Tz)}{2}. \end{aligned} \tag{2.19}$$

Letting $n \to \infty$ in the above inequality, we get

$$d(z, Tz) \leq \gamma_0 \frac{d(z, Tz)}{2}, \tag{2.20}$$

which implies (since $\gamma_0 \in (0,1)$) that $d(z,Tz)=0$, that is, $z$ is a fixed point of $T$. To show the uniqueness of the fixed point, suppose that $w \in A \cup B$ is also a fixed point of $T$. Clearly, since $T$ is a cyclic mapping, we have $w = Tw \in A \cap B$. Now, from (2.5), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d\left(T^{2n}x, w\right) &= d\left(T^{2n}x, Tw\right) \\ &\leq \varphi_x\left(d\left(T^{2n-1}x, w\right)\right)\frac{d(T^{2n-1}x, T^{2n}x) + d(Tw, w)}{2} \\ &\leq \frac{d(T^{2n-1}x, T^{2n}x)}{2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.21}$$

Thus we have $T^{2n}x \to w$ as $n \to \infty$. From the uniqueness of the limit, we have $z = w$. Thus $z$ is the unique fixed point of $T$. □

*Example 2.3.* Let $X = [0,\infty)$ with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. Consider $A = [0,1/2]$ and $B = [1/2,1]$. We have $A \cap B = 1/2$. Define $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} \dfrac{1}{2} & \text{if } x \in [0,1), \\ 0 & \text{if } x = 1. \end{cases} \tag{2.22}$$

Fix any $x$ in $A$. For all $n \in \mathbb{N}$ and $y \in A$, we have

$$d\left(T^{2n}x, Ty\right) = 0, \tag{2.23}$$

that is, (2.49) holds. Applying Theorem 2.2, the mapping $T$ has a unique fixed point which is $z = 1/2$.

Our second main result is the following.

**Theorem 2.4.** *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X,d)$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that for some $x \in A$ there exists $\varphi_x \in \Psi$ such that*

$$d\left(T^{2n}x, Ty\right) \leq \varphi_x\left(d\left(T^{2n-1}x, y\right)\right)\frac{d(T^{2n-1}x, Ty) + d(T^{2n}x, y)}{2}, \tag{2.24}$$

*for all $n \in \mathbb{N}$ and $y \in A$. Then $T$ has a unique fixed point that belongs to $A \cap B$.*

*Proof.* Taking $y = T^{2n}x$ in (2.24), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d\left(T^{2n}x, T^{2n+1}x\right) &\leq \frac{\varphi_x(d(T^{2n-1}x, T^{2n}x))(d(T^{2n-1}x, T^{2n+1}x) + d(T^{2n}x, T^{2n}x))}{2} \\ &\leq \frac{d(T^{2n-1}x, T^{2n+1}x)}{2} \leq \frac{d(T^{2n-1}x, T^{2n}x) + d(T^{2n}x, T^{2n+1}x)}{2}. \end{aligned} \tag{2.25}$$

Thus for all $n \in \mathbb{N}$, we have

$$d\left(T^{2n}x, T^{2n+1}x\right) \le d\left(T^{2n-1}x, T^{2n}x\right). \tag{2.26}$$

Again, taking $y = T^{2n}x$ in (2.24), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d\left(T^{2n+1}x, T^{2n+2}x\right) &= d\left(T^{2n+2}x, T^{2n+1}x\right) \\ &\le \frac{\varphi_x\left(d\left(T^{2n+1}x, T^{2n}x\right)\right)\left(d\left(T^{2n+1}x, T^{2n+1}x\right) + d\left(T^{2n+2}x, T^{2n}x\right)\right)}{2} \\ &\le \frac{d\left(T^{2n+2}x, T^{2n}x\right)}{2} \le \frac{d\left(T^{2n+2}x, T^{2n+1}x\right) + d\left(T^{2n+1}x, T^{2n}x\right)}{2}. \end{aligned} \tag{2.27}$$

Thus for all $n \in \mathbb{N}$, we have

$$d\left(T^{2n+1}x, T^{2n+2}x\right) \le d\left(T^{2n}x, T^{2n+1}x\right). \tag{2.28}$$

It follows from (2.26) and (2.28) that $\{d(T^n x, T^{n+1}x)\}$ is a decreasing sequence. Then there exists $\varepsilon \ge 0$ such that

$$\lim_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = \varepsilon. \tag{2.29}$$

Applying Lemma 2.1 with $r = \varepsilon$, we obtain that there exist $n_0 \in \mathbb{N}$ and $\gamma_\varepsilon \in (0,1)$ such that

$$\varphi_x\left(d\left(T^n x, T^{n+1}x\right)\right) < \gamma_\varepsilon, \quad \forall n \ge n_0. \tag{2.30}$$

Denote $k_0 = [(n_0+1)/2] + 1$. By (2.30) and (2.24), for all $k \ge k_0$, we have

$$\begin{aligned} d\left(T^{2k}x, T^{2k+1}x\right) &\le \frac{\varphi_x\left(d\left(T^{2k-1}x, T^{2k}x\right)\right)\left(d\left(T^{2k-1}x, T^{2k+1}x\right) + d\left(T^{2k}x, T^{2k}x\right)\right)}{2} \\ &\le \gamma_\varepsilon \frac{d\left(T^{2k-1}x, T^{2k+1}x\right)}{2} \le \gamma_\varepsilon \frac{d\left(T^{2k-1}x, T^{2k}x\right) + d\left(T^{2k}x, T^{2k+1}x\right)}{2}. \end{aligned} \tag{2.31}$$

This implies that

$$d\left(T^{2k}x, T^{2k+1}x\right) \le q_\varepsilon d\left(T^{2k-1}x, T^{2k}x\right), \quad \forall k \ge k_0, \tag{2.32}$$

where $q_\varepsilon = \gamma_\varepsilon/(2-\gamma_\varepsilon) \in (0,1)$. Similarly, we can show that

$$d\left(T^{2k+1}x, T^{2k+2}x\right) \le q_\varepsilon d\left(T^{2k}x, T^{2k+1}x\right), \quad \forall k \ge k_0. \tag{2.33}$$

Now, it follows from (2.32) and (2.33) that

$$d\left(T^{2k_0+n}x, T^{2k_0+n+1}x\right) \leq q_\varepsilon^n d\left(T^{2k_0-1}x, T^{2k_0}x\right), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{2.34}$$

As in the proof of Theorem 2.2, we obtain from the above inequality that $\{T^n x\}$ is a Cauchy sequence in $(X, d)$, which implies that there exists a $z \in A \cap B$ such that $T^n x \to z$ as $n \to \infty$. On the other hand, by Lemma 2.1, there exist $N \in \mathbb{N}$ and $\gamma_0 \in (0,1)$ such that

$$\varphi\left(d\left(T^{2n-1}x, z\right)\right) < \gamma_0, \quad \forall n \geq N. \tag{2.35}$$

From (2.24) and (2.35), for all $n \geq N$, we have

$$\begin{aligned} d\left(T^{2n}x, Tz\right) &\leq \varphi_x\left(d\left(T^{2n-1}x, z\right)\right) \frac{d(T^{2n-1}x, Tz) + d(T^{2n}x, z)}{2} \\ &\leq \gamma_0 \frac{d(T^{2n-1}x, Tz) + d(T^{2n}x, z)}{2}. \end{aligned} \tag{2.36}$$

Letting $n \to \infty$ in the above inequality, we get

$$d(z, Tz) \leq \gamma_0 \frac{d(z, Tz)}{2}, \tag{2.37}$$

which implies (since $\gamma_0 \in (0,1)$) that $d(z, Tz) = 0$, that is, $z$ is a fixed point of $T$. Suppose now that $w \in A \cap B$ is also a fixed point of $T$. From (2.24), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d\left(T^{2n}x, w\right) &= d\left(T^{2n}x, Tw\right) \\ &\leq \varphi_x\left(d\left(T^{2n-1}x, w\right)\right) \frac{d(T^{2n-1}x, Tw) + d(T^{2n}x, w)}{2} \\ &\leq \frac{d(T^{2n-1}x, w) + d(T^{2n}x, w)}{2}. \end{aligned} \tag{2.38}$$

This implies that

$$d\left(T^{2n}x, w\right) \leq d\left(T^{2n-1}x, w\right), \quad \forall n \in \mathbb{N}. \tag{2.39}$$

Then there exists $a \geq 0$ such that $d(T^{2n}x, w) \to a$ as $n \to \infty$. On the other hand, we have

$$\begin{aligned} \left| d\left(T^{2n}x, w\right) - d\left(T^{2n-1}x, w\right) \right| &= d\left(T^{2n-1}x, w\right) - d\left(T^{2n}x, w\right) \\ &\leq d\left(T^{2n}x, T^{2n-1}x\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.40}$$

Thus we have $d(T^{2n-1}x, w) \to a$ as $n \to \infty$. By Lemma 2.1, there exist $q \in \mathbb{N}$ and $\gamma_a \in (0,1)$ such that

$$\psi\left(d\left(T^{2n-1}x, w\right)\right) < \gamma_a, \quad \forall n \geq q. \tag{2.41}$$

From (2.41) and (2.24), for all $n \geq q$, we have

$$\begin{aligned} d\left(T^{2n}x, w\right) &= d\left(T^{2n}x, Tw\right) \\ &\leq \psi_x\left(d\left(T^{2n-1}x, w\right)\right)\frac{d(T^{2n-1}x, Tw) + d(T^{2n}x, w)}{2} \\ &\leq \gamma_a \frac{d(T^{2n-1}x, w) + d(T^{2n}x, w)}{2}. \end{aligned} \tag{2.42}$$

Letting $n \to \infty$, we get

$$a \leq \gamma_a a, \tag{2.43}$$

which implies (since $\gamma_a \in (0,1)$) that $a = 0$. Thus we have $d(T^{2n}x, w) \to 0$ as $n \to \infty$. By the uniqueness of the limit, we get that $w = z$. Thus $z$ is the unique fixed point of $T$. □

The following result is an immediate consequence of Theorem 2.2.

**Theorem 2.5.** *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that for some $x \in A$ there exists $k_x \in (0,1)$ such that*

$$d\left(T^{2n}x, Ty\right) \leq k_x \frac{d(T^{2n-1}x, T^{2n}x) + d(Ty, y)}{2}, \tag{2.44}$$

*for all $n \in \mathbb{N}$ and $y \in A$. Then $T$ has a unique fixed point that belongs to $A \cap B$.*

The following result is an immediate consequence of Theorem 2.4.

**Theorem 2.6.** *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that for some $x \in A$ there exists $k_x \in (0,1)$ such that*

$$d\left(T^{2n}x, Ty\right) \leq k_x \frac{d(T^{2n-1}x, Ty) + d(T^{2n}x, y)}{2}, \tag{2.45}$$

*for all $n \in \mathbb{N}$ and $y \in A$. Then $T$ has a unique fixed point that belongs to $A \cap B$.*

Now, we present a data dependence result.

**Theorem 2.7.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X,d)$ and $T,S : A\cup B \to A\cup B$ are two mappings satisfying*

(i) *$T$ is a cyclic mapping;*

(ii) *$T$ satisfies* (2.5)*;*

(iii) *$S$ has a fixed point $x_S \in A$;*

(iv) *there exists $\eta > 0$ such that $d(Tu,Su) \le \eta$ for all $u \in A$.*

*Then $d(x_T,x_S) \le 3/2\eta$, where $x_T$ is the unique fixed point of $T$.*

*Proof.* From Theorem 2.2, conditions (i) and (ii) imply that $T$ has a unique fixed point $x_T \in A\cap B$. Using (2.5), we have

$$\begin{aligned} d(x_T,x_S) &= d(Tx_T,Sx_S)\\ &\le d\left(T^{2n}x,Tx_T\right) + d\left(T^{2n}x,Tx_S\right) + d(Tx_S,Sx_S)\\ &\le \frac{\varphi_x\left(d\left(T^{2n-1}x,x_T\right)\right)\left(d\left(T^{2n-1}x,T^{2n}x\right)+d(Tx_T,x_T)\right)}{2}\\ &\quad + \frac{\varphi_x\left(d\left(T^{2n-1}x,x_S\right)\right)\left(d\left(T^{2n-1}x,T^{2n}x\right)+d(Tx_S,x_S)\right)}{2} + d(Tx_S,Sx_S)\\ &\le \frac{d\left(T^{2n-1}x,T^{2n}x\right)+d(Tx_T,x_T)}{2} + \frac{d\left(T^{2n-1}x,T^{2n}x\right)+d(Tx_S,x_S)}{2} + d(Tx_S,Sx_S). \end{aligned} \tag{2.46}$$

Letting $n \to \infty$ and using (iv), we get that

$$d(x_T,x_S) \le \frac{3}{2}d(Tx_S,Sx_S) \le \frac{3}{2}\eta. \tag{2.47}$$

□

We introduce now the following class of cyclic contractive mappings. At first define the following set Let $\Phi$ be the set of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions

($\Phi_1$) $\varphi$ is lower semicontinuous;

($\Phi_2$) $\varphi(t) = 0$ if and only if $t = 0$.

*Definition 2.8.* Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. A cyclic map $T : A\cup B \to A\cup B$ is said to be a cyclic weakly orbital contraction if for some $x \in A$, there exists $\varphi_x \in \Phi$ such that

$$d\left(T^{2n}x,Ty\right) \le d\left(T^{2n-1}x,y\right) - \varphi_x\left(d\left(T^{2n-1}x,y\right)\right), \quad \forall y \in A,\ n \in \mathbb{N}. \tag{2.48}$$

We have the following result.

**Theorem 2.9.** *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$. Let $T : A \cup B \to A \cup B$ be a cyclic weakly orbital contraction. Then $T$ has a unique fixed point that belongs to $A \cap B$.*

*Proof.* Since $T$ is a cyclic weakly orbital contraction, there exist $x \in A$ and $\varphi_x \in \Phi$ such that (2.48) is satisfied. Taking $y = T^{2n}x$ in (2.48), for all $n \in \mathbb{N}$, we get

$$d\left(T^{2n}x, T^{2n+1}x\right) \le d\left(T^{2n-1}x, T^{2n}x\right) - \varphi_x\left(d\left(T^{2n-1}x, T^{2n}x\right)\right) \le d\left(T^{2n-1}x, T^{2n}x\right). \tag{2.49}$$

Thus for all $n \in \mathbb{N}$, we have

$$d\left(T^{2n}x, T^{2n+1}x\right) \le d\left(T^{2n-1}x, T^{2n}x\right). \tag{2.50}$$

Again, taking $y = T^{2n}x$ in (2.48), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d\left(T^{2n+1}x, T^{2n+2}x\right) &= d\left(T^{2n+2}x, T^{2n+1}x\right) \\ &\le d\left(T^{2n+1}x, T^{2n}x\right) - \varphi_x\left(d\left(T^{2n+1}x, T^{2n}x\right)\right) \\ &\le d\left(T^{2n}x, T^{2n+1}x\right). \end{aligned} \tag{2.51}$$

Thus for all $n \in \mathbb{N}$, we have

$$d\left(T^{2n+1}x, T^{2n+2}x\right) \le d\left(T^{2n}x, T^{2n+1}x\right). \tag{2.52}$$

Then $\{d(T^n x, T^{n+1}x)\}$ is a decreasing sequence, and there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = r. \tag{2.53}$$

Letting $n \to \infty$ in (2.49), using (2.53) and the lower semicontinuity of $\varphi_x$, we obtain

$$r \le r - \varphi_x(r), \tag{2.54}$$

which implies from condition $(\Phi_2)$ that $r = 0$. Then we have

$$\lim_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = 0. \tag{2.55}$$

Now, we shall prove that $\{T^n x\}$ is Cauchy. From (2.55), it is sufficient to show that $\{T^{2n}x\}$ is Cauchy. We argue by contradiction, suppose that $\{T^{2n}x\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and two subsequences $\{T^{2n(k)}x\}$ and $\{T^{2m(k)}x\}$ of $\{T^{2n}x\}$ such that for all $k$, $n(k) > m(k) > k$ and

$$d\left(T^{2n(k)}x, T^{2m(k)}x\right) \geq \varepsilon. \tag{2.56}$$

We can take $n(k)$ the smallest integer such that the above inequality is satisfied. So, for all $k$, we have

$$d\left(T^{2n(k)-2}x, T^{2m(k)}x\right) < \varepsilon. \tag{2.57}$$

Using (2.56) and (2.57), we obtain

$$\begin{aligned}
\varepsilon &\leq d\left(T^{2n(k)}x, T^{2m(k)}x\right) \\
&\leq d\left(T^{2n(k)}x, T^{2n(k)-1}x\right) + d\left(T^{2n(k)-1}x, T^{2n(k)-2}x\right) + d\left(T^{2n(k)-2}x, T^{2m(k)}x\right) \\
&< d\left(T^{2n(k)}x, T^{2n(k)-1}x\right) + d\left(T^{2n(k)-1}x, T^{2n(k)-2}x\right) + \varepsilon.
\end{aligned} \tag{2.58}$$

Letting $k \to \infty$ and using (2.55), we have

$$\lim_{k\to\infty} d\left(T^{2n(k)}x, T^{2m(k)}x\right) = \varepsilon. \tag{2.59}$$

On the other hand, for all $k$, we have

$$\left| d\left(T^{2n(k)}x, T^{2m(k)+1}x\right) - d\left(T^{2n(k)}x, T^{2m(k)}x\right) \right| \leq d\left(T^{2m(k)+1}x, T^{2m(k)}x\right). \tag{2.60}$$

Letting $k \to \infty$, using (2.55) and (2.59), we have

$$\lim_{k\to\infty} d\left(T^{2n(k)}x, T^{2m(k)+1}x\right) = \varepsilon. \tag{2.61}$$

Similarly, for all $k$, we have

$$\left| d\left(T^{2n(k)-1}x, T^{2m(k)}x\right) - d\left(T^{2n(k)}x, T^{2m(k)}x\right) \right| \leq d\left(T^{2n(k)-1}x, T^{2n(k)}x\right). \tag{2.62}$$

Letting $k \to \infty$, using (2.55) and (2.59), we have

$$\lim_{k\to\infty} d\left(T^{2n(k)-1}x, T^{2m(k)}x\right) = \varepsilon. \tag{2.63}$$

Now, applying (2.48) with $y = T^{2m(k)}x$, for all $k$, we have

$$d\left(T^{2n(k)}x, T^{2m(k)+1}x\right) \leq d\left(T^{2n(k)-1}x, T^{2m(k)}x\right) - \varphi_x\left(d\left(T^{2n(k)-1}x, T^{2m(k)}x\right)\right). \tag{2.64}$$

Letting $k \to \infty$ in the above inequality, using (2.59), (2.63), and the lower semi-continuity of $\varphi_x$, we obtain

$$\varepsilon \leq \varepsilon - \varphi_x(\varepsilon), \tag{2.65}$$

which implies from $(\Phi_2)$ that $\varepsilon = 0$, a contradiction. Thus we proved that $\{T^n x\}$ is a Cauchy sequence. Since $(X, d)$ is complete, $A$ and $B$ are closed, and $\{T^n x\} \subset A \cup B$, there exists a $z \in A \cup B$ such that $T^n x \to z$ as $n \to \infty$. On the other hand, since $T$ is a cyclic mapping and $x \in A$, we have $\{T^{2n} x\} \subset A$ and $\{T^{2n+1} x\} \subset B$. Since $A$ and $B$ are closed, this implies that $z \in A \cap B$. Now, we will prove that $z$ is a fixed point of $T$. From (2.48), for all $n \in \mathbb{N}$, we have

$$d\left(T^{2n} x, Tz\right) \leq d\left(T^{2n-1} x, z\right) - \varphi_x\left(d\left(T^{2n-1} x, z\right)\right). \tag{2.66}$$

Letting $n \to \infty$ and using the properties of $\varphi_x$, we get

$$d(z, Tz) \leq d(z, z) - \varphi_x(d(z, z)) = 0, \tag{2.67}$$

which implies that $d(z, Tz) = 0$, that is, $z = Tz$. Then $z$ is a fixed point of $T$. Suppose now that $w \in A \cap B$ is a fixed point of $T$. From (2.48), for all $n \in \mathbb{N}$, we have

$$d(z, w) = d\left(T^{2n} z, Tw\right) \leq d\left(T^{2n-1} x, w\right) - \varphi_x\left(d\left(T^{2n-1} x, w\right)\right). \tag{2.68}$$

Letting $n \to \infty$, we get

$$d(z, w) \leq d(z, w) - \varphi_x(d(z, w)), \tag{2.69}$$

which implies that $d(z, w) = 0$, that is, $z = w$. Then $T$ has a unique fixed point. □

Denote now by $\Lambda$ the set of functions $\alpha : [0, \infty) \to [0, \infty)$ satisfying

$(\Lambda_1)$ $\alpha$ is locally integrable on $[0, \infty)$;

$(\Lambda_2)$ for all $\varepsilon > 0$, we have $\int_0^{\varepsilon} \alpha(s)\, ds > 0$.

The following result is an immediate consequence of Theorem 2.9.

**Theorem 2.10.** *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that for some $x \in A$ there exists $\alpha_x \in \Lambda$ such that*

$$d\left(T^{2n} x, Ty\right) \leq d\left(T^{2n-1} x, y\right) - \int_0^{d(T^{2n-1} x, y)} \alpha_x(s)\, ds, \tag{2.70}$$

*for all $n \in \mathbb{N}$ and $y \in A$. Then $T$ has a unique fixed point that belongs to $A \cap B$.*

*Remark 2.11.* Clearly, any cyclic orbital contraction is a cyclic weakly orbital contraction. So Theorem 1.3 is a particular case of our Theorem 2.9.

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