Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 864186, 14 pages
doi:10.1155/2012/864186

Research Article

Uniqueness of Weak Solutions to an Electrohydrodynamics Model

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Received 11 October 2011; Accepted 3 March 2012

Academic Editor: Narcisa C. Apreutesei

This paper studies uniqueness of weak solutions to an electrohydrodynamics model in $\mathbb{R}^d (d = 2, 3)$. When $d = 2$, we prove a uniqueness without any condition on the velocity. For $d = 3$, we prove a weak-strong uniqueness result with a condition on the vorticity in the homogeneous Besov space.

1. Introduction

We consider the following model of electrokinetic fluid in $\mathbb{R}^d \times (0, \infty)$ [1, 2]:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \Delta \phi \nabla \phi, \tag{1.1}$$

$$\text{div } u = 0, \tag{1.2}$$

$$\partial_t n + u \cdot \nabla n = \nabla \cdot (\nabla n - n \nabla \phi), \tag{1.3}$$

$$\partial_t p + u \cdot \nabla p = \nabla \cdot (\nabla p + p \nabla \phi), \tag{1.4}$$

$$\Delta \phi = n - p, \tag{1.5}$$

$$(u, n, p)(x, 0) = (u_0, n_0, p_0)(x), \quad x \in \mathbb{R}^d \ (d = 2, 3). \tag{1.6}$$

The unknowns u, π, ϕ, n, and p denote the velocity, pressure, electric potential, anion concentration, and cation concentration, respectively.

Equations (1.3)–(1.5) are known as the electrochemical equations [3] or semiconductor equations [4, 5], and electro-rheological systems [2, 6] when formally setting $u = 0$. (1.1) and (1.2) are Navier-Stokes equations with the Lorentz force $\Delta\phi\nabla\phi$.

The uniqueness of weak solutions to the Navier-Stokes equations is still open. In 1962, Serrin [7] gave the first uniqueness condition:

$$u \in L^r\left(0,T;L^s\left(\mathbb{R}^3\right)\right) \text{ with } \frac{2}{r}+\frac{3}{s}=1, \quad 3<s\leq\infty. \tag{1.7}$$

Kozono and Taniuchi [8] proved the following uniqueness criterion:

$$u \in L^2\left(0,T;\mathrm{BMO}\left(\mathbb{R}^3\right)\right). \tag{1.8}$$

Here BMO denotes the functions of bounded mean oscillation. Ogawa and Taniuchi [9] obtained the uniqueness criterion:

$$\nabla u \in L\,\mathrm{Log}\,L\left(0,T;\dot{B}^0_{\infty,\infty}\left(\mathbb{R}^3\right)\right) \tag{1.9}$$

with

$$L\,\mathrm{Log}\,L\left(0,T;\dot{B}^0_{\infty,\infty}\right) := \left\{f;\int_0^T \|f\|_{\dot{B}^0_{\infty,\infty}} \log\left(e+\|f\|_{\dot{B}^0_{\infty,\infty}}\right)dt<\infty\right\}. \tag{1.10}$$

Here it should be noted that Kozono et al. [10] proved that u is smooth if

$$\nabla u \in L^1\left(0,T;\dot{B}^0_{\infty,\infty}\left(\mathbb{R}^3\right)\right). \tag{1.11}$$

Here $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space.

Kurokiba and Ogawa [4] considered the semiconductor equations (1.3)–(1.5) when $u = 0$ and proved that the existence and uniqueness of weak solutions with L^p initial data (n_0, p_0) when $p = (d/2)(d \geq 3)$ and $1 < p < 2 \ \ (d = 2)$.

Note that the system (1.1)–(1.5) holds its form under the scaling $(u,\pi,\phi,n,p) \to (u_\lambda,\pi_\lambda,\phi_\lambda,n_\lambda,p_\lambda) := (\lambda u,\lambda^2\pi,\phi,\lambda^2 n,\lambda^2 p)(\lambda^2 t,\lambda x)$. Under this scaling, the space $L^r(0,T;L^s)$ is invariant for u when $2/r + d/s = 1$ and the space $L^r(0,T;L^s)$ is invariant for (n,p) when $2/r + d/s = 2$. Furthermore, L^d for u_0 and $L^{d/2}$ for (n_0,p_0) are invariant spaces under this scaling. Fan and Gao [11], Ryham [12], and Schmuck [13] proved the existence, uniqueness, and regularity of global weak solutions to system (1.1)–(1.6) in a bounded domain Ω. When $\Omega = \mathbb{R}^d$, Jerome [14] established the first existence result in Kato's semigroup framework. Zhao et al. [15] obtained global well-posedness for small initial data in Besov spaces with negative index.

The aim of this paper is to generalize the results of [4, 9]. We will prove the following results.

Theorem 1.1. *Let $(n_0, p_0) \in L^1(\mathbb{R}^2) \cap L\log L(\mathbb{R}^2)$, $n_0, p_0 \geq 0$ in $\mathbb{R}^2$, $\int n_0\,dx = \int p_0\,dx$, $\nabla\phi_0 \in L^2$, and $u_0 \in L^2$. Then there exists a unique weak solution (u, n, p, ϕ) to the problem* (1.1)–(1.6) *satisfying*

$$
\begin{gathered}
(n,p) \in L^\infty(0,T;L^1 \cap L\log L) \cap L^2(0,T;L^2) \cap L^{4/3}(0,T;W^{1,4/3}), \quad n,p \geq 0 \ \textit{in}\ \mathbb{R}^2 \times (0,T) \\
(\partial_t n, \partial_t p) \in L^{4/3}(0,T;W^{-1,4/3}), \\
\nabla\phi \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \cap L^4(0,T;L^4), \\
u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \cap L^4(0,T;L^4), \\
\partial_t u \in L^{4/3}(0,T;H^{-1}) \quad \textit{for any}\ T > 0.
\end{gathered}
\tag{1.12}
$$

Remark 1.2. We can assume $n_0 - p_0 \in \mathscr{H}^1$ (Hardy space) and $\Delta\phi_0 = n_0 - p_0$ gives $\nabla\phi_0 \in L^2(\mathbb{R}^2)$.

Theorem 1.3 ($d = 3$). *Let $(n_0, p_0) \in L^{3/2}$, $n_0, p_0 \geq 0$ in $\mathbb{R}^3$, $\int n_0\,dx = \int p_0\,dx$, and $u_0 \in L^2$. Suppose that* (1.9) *holds true, then there exists a unique weak solution (u, n, p, ϕ) to the problem* (1.1)–(1.6) *satisfying*

$$
\begin{gathered}
(n^{3/4}, p^{3/4}) \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \quad n,p \geq 0 \ \textit{in}\ \mathbb{R}^3 \times (0,T), \\
(n,p) \in L^\infty(0,T;L^{3/2}) \cap L^{5/2}(0,T;L^{5/2}) \cap L^{5/3}(0,T;W^{1,5/3}) \cap L^4(0,T;L^2), \\
(\partial_t n, \partial_t p) \in L^{5/3}(0,T;W^{-1,3/2}), \\
\nabla\phi \in L^\infty(0,T;W^{1,3/2}) \cap L^{5/2}(0,T;W^{1,5/2}), \\
\nabla\phi \in L^\infty(0,T;L^3) \cap L^{5/2}(0,T;L^{15}), \\
u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \qquad \partial_t u \in L^2(0,T;W^{-1,3/2})
\end{gathered}
\tag{1.13}
$$

for any $T > 0$.

Let η_j, $j = 0, \pm1, \pm2, \pm3, \ldots$, be the Littlewood-Paley dyadic decomposition of unity that satisfies $\widehat{\eta} \in C_0^\infty(B_2 \setminus B_{1/2})$, $\widehat{\eta}_j(\xi) = \widehat{\eta}(2^{-j}\xi)$, and $\sum_{j=-\infty}^{\infty} \widehat{\eta}_j(\xi) = 1$ except $\xi = 0$. To fill the origin, we put a smooth cut off $\varphi \in \mathcal{S}(\mathbb{R}^3)$ with $\widehat{\varphi}(\xi) \in C_0^\infty(B_1)$ such that

$$
\widehat{\varphi} + \sum_{j=0}^{\infty} \widehat{\eta}_j(\xi) = 1.
\tag{1.14}
$$

The homogeneous Besov space $\dot{B}^s_{p,q} := \{f \in \mathcal{S}' : \|f\|_{\dot{B}^s_{p,q}} < \infty\}$ is introduced by the norm

$$
\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j=-\infty}^{\infty} \left\| 2^{js} \eta_j * f \right\|_{L^p}^q \right)^{1/q},
\tag{1.15}
$$

for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$.

It is easy to prove the existence of weak solutions [14] and thus we omit the details here; we only need to derive the estimates (1.12) and (1.13) and prove the uniqueness.

2. Proof of Theorem 1.1

First, by the maximum principle, it is easy to prove that

$$n, p \geq 0 \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{2.1}$$

Testing (1.3) by $1 + \log n$ and testing (1.4) by $1 + \log p$, respectively, using (1.2), summing up the resulting equality, we obtain

$$\begin{aligned} &\int n \log n + p \log p \, dx + 4 \int_0^T \int \left|\nabla \sqrt{n}\right|^2 + \left|\nabla \sqrt{p}\right|^2 dx\, dt + \int_0^T \int \left|\Delta \phi\right|^2 dx\, dt \\ &\quad = \int n_0 \log n_0 + p_0 \log p_0 dx. \end{aligned} \tag{2.2}$$

Substracting (1.4) from (1.3), we see that

$$\partial_t (n - p) + u \cdot \nabla (n - p) = \nabla \cdot (\nabla (n - p) - (n + p) \nabla \phi). \tag{2.3}$$

Testing the above equation by $-\phi$, using (1.5) and (2.1), we see that

$$\frac{1}{2} \frac{d}{dt} \int \left|\nabla \phi\right|^2 dx - \int u \cdot \nabla \Delta \phi \cdot \phi \, dx + \int \left|\Delta \phi\right|^2 dx + \int (n + p) \left|\nabla \phi\right|^2 dx = 0. \tag{2.4}$$

Whence

$$\frac{1}{2} \frac{d}{dt} \int \left|\nabla \phi\right|^2 dx + \int u \Delta \phi \nabla \phi \, dx + \int \left|\Delta \phi\right|^2 dx + \int (n + p) \left|\nabla \phi\right|^2 dx = 0. \tag{2.5}$$

Testing (1.1) by u, using (1.2), we find that

$$\frac{1}{2} \frac{d}{dt} \int u^2 dx + \int |\nabla u|^2 dx = \int u \Delta \phi \nabla \phi \, dx. \tag{2.6}$$

Summing up (2.5) and (2.6), we get

$$\frac{1}{2} \frac{d}{dt} \int u^2 + \left|\nabla \phi\right|^2 dx + \int |\nabla u|^2 + \left|\Delta \phi\right|^2 + (n + p) \left|\nabla \phi\right|^2 dx = 0, \tag{2.7}$$

whence

$$\frac{1}{2} \int u^2 + \left|\nabla \phi\right|^2 dx + \int_0^T \int |\nabla u|^2 + \left|\Delta \phi\right|^2 + (n + p) \left|\nabla \phi\right|^2 dx\, dt \leq \frac{1}{2} \int u_0^2 + \left|\nabla \phi_0\right|^2 dx. \tag{2.8}$$

Integrating (1.3) and (1.4), we have

$$\int n\,dx = \int n_0\,dx = \int p_0\,dx = \int p\,dx. \tag{2.9}$$

Using the Gagliardo-Nirenberg inequality,

$$\|u\|_{L^4}^2 \le C\|u\|_{L^2}\|\nabla u\|_{L^2}, \tag{2.10}$$

we deduce that

$$\|u\|_{L^4(0,T;L^4)} \le C, \tag{2.11}$$

$$\|\nabla\phi\|_{L^4(0,T;L^4)} \le C, \tag{2.12}$$

$$\|(n,p)\|_{L^2(0,T;L^2)} \le C. \tag{2.13}$$

Since $\nabla n = 2\nabla\sqrt{n}\cdot\sqrt{n}$, $\nabla\sqrt{n} \in L^2(0,T;L^2)$, $\sqrt{n} \in L^4(0,T;L^4)$, we easily infer that

$$\nabla n \in L^{4/3}\left(0,T;L^{4/3}\right), \tag{2.14}$$

by the Hölder inequality. Similarly, we have

$$\nabla p \in L^{4/3}\left(0,T;L^{4/3}\right). \tag{2.15}$$

It is easy to show that

$$(\partial_t n, \partial_t p) \in L^{4/3}\left(0,T;W^{-1,4/3}\right), \qquad \partial_t u \in L^{4/3}\left(0,T;H^{-1}\right). \tag{2.16}$$

Now we are in a position to prove the uniqueness. Let $(u_i, \pi_i, n_i, p_i, \phi_i)$ $(i = 1,2)$ be two weak solutions to the problem (1.1)–(1.6). Also let us denote

$$u := u_1 - u_2, \qquad \pi := \pi_1 - \pi_2, \qquad n := n_1 - n_2, \qquad p := p_1 - p_2, \qquad \phi := \phi_1 - \phi_2. \tag{2.17}$$

We define N and P satisfying the following equations:

$$-\Delta N + N = n \text{ in } \mathbb{R}^d \times (0,\infty), \tag{2.18}$$

$$-\Delta P + P = p \text{ in } \mathbb{R}^d \times (0,\infty). \tag{2.19}$$

It is easy to verify that

$$\partial_t n + \nabla\cdot(u_1 n + u n_2) = \Delta n - \nabla\cdot(n\nabla\phi_1 + n_2\nabla\phi), \tag{2.20}$$

$$\partial_t p + \nabla\cdot(u_1 p + u p_2) = \Delta p + \nabla\cdot(p\nabla\phi_1 + p_2\nabla\phi), \tag{2.21}$$

$$\partial_t(n-p) + \nabla\cdot(u_1(n-p) + u(n_2-p_2)) = \Delta(n-p) - \nabla\cdot((n+p)\nabla\phi_1 + (n_2+p_2)\nabla\phi). \tag{2.22}$$

Testing (2.20) by N, we derive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int N^2 + |\nabla N|^2\,dx + \int |\nabla N|^2 + |\Delta N|^2\,dx\\ &\quad= \int n\nabla\phi_1\cdot\nabla N + n_2\nabla\phi\nabla N + u_1 n\nabla N + u n_2\nabla N\,dx =: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{2.23}$$

Using (2.10), (2.18) and (2.19), each term I_i $(i = 1,2,3,4)$ can be bounded as follows:

$$\begin{aligned}
I_1 &\le \|\Delta N\|_{L^2}\|\nabla\phi_1\|_{L^4}\|\nabla N\|_{L^4} + \|N\|_{L^4}\|\nabla\phi_1\|_{L^4}\|\nabla N\|_{L^2}\\
&\le C\|\Delta N\|_{L^2}\|\nabla\phi_1\|_{L^4}\|\nabla N\|_{L^2}^{1/2}\|\Delta N\|_{L^2}^{1/2} + C\|\nabla\phi_1\|_{L^4}\|N\|_{H^1}^2\\
&\le \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|\nabla\phi_1\|_{L^4}^4\|\nabla N\|_{L^2}^2 + C\|\nabla\phi_1\|_{L^4}\|N\|_{H^1}^2,\\
I_2 &\le \|n_2\|_{L^2}\|\nabla\phi\|_{L^4}\|\nabla N\|_{L^4}\\
&\le \|n_2\|_{L^2}\|\nabla\phi\|_{L^4}^2 + \|n_2\|_{L^2}\|\nabla N\|_{L^4}^2\\
&\le C\|n_2\|_{L^2}\|\nabla\phi\|_{L^2}\|\Delta\phi\|_{L^2} + C\|n_2\|_{L^2}\|\nabla N\|_{L^2}\|\Delta N\|_{L^2}\\
&\le \frac{1}{18}\|\Delta\phi\|_{L^2}^2 + \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|n_2\|_{L^2}^2\left(\|\nabla\phi\|_{L^2}^2 + \|\nabla N\|_{L^2}^2\right),\\
I_3 &\le \|u_1\|_{L^4}\|\Delta N\|_{L^2}\|\nabla N\|_{L^4}\\
&\le C\|u_1\|_{L^4}\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}^{1/2}\|\Delta N\|_{L^2}^{1/2}\\
&\le \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|u_1\|_{L^4}^4\|\nabla N\|_{L^2}^2,\\
I_4 &\le \|n_2\|_{L^2}\|u\|_{L^4}\|\nabla N\|_{L^4}\\
&\le \|n_2\|_{L^2}\|u\|_{L^4}^2 + \|n_2\|_{L^2}\|\nabla N\|_{L^4}^2\\
&\le C\|n_2\|_{L^2}\|u\|_{L^2}\|\nabla u\|_{L^2} + C\|n_2\|_{L^2}\|\nabla N\|_{L^2}\|\Delta N\|_{L^2}\\
&\le \frac{1}{18}\|\nabla u\|_{L^2}^2 + \frac{1}{18}\|\Delta N\|_{L^2}^2 + C\|n_2\|_{L^2}^2\left(\|u\|_{L^2}^2 + \|\nabla N\|_{L^2}^2\right).
\end{aligned} \tag{2.24}$$

Substituting these estimates into (2.23), we obtain

$$\frac{1}{2}\frac{d}{dt}\int N^2+|\nabla N|^2\,dx+\frac{1}{2}\int|\nabla N|^2+|\Delta N|^2\,dx$$
$$\le C\left(\|\nabla\phi_1\|_{L^4}^4+\|n_2\|_{L^2}^2+\|u_1\|_{L^4}^4+1\right)\left(\|N\|_{L^2}^2+\|\nabla N\|_{L^2}^2+\|u\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)\tag{2.25}$$
$$+\frac{1}{18}\|\Delta\phi\|_{L^2}^2+\frac{1}{18}\|\nabla u\|_{L^2}^2.$$

Similarly for the p-equation, we get

$$\frac{1}{2}\frac{d}{dt}\int p^2+|\nabla p|^2\,dx+\frac{1}{2}\int|\nabla p|^2+|\Delta p|^2\,dx$$
$$\le C\left(\|\nabla\phi_1\|_{L^4}^4+\|p_2\|_{L^2}^2+\|u_1\|_{L^4}^4+1\right)\left(\|p\|_{L^2}^2+\|\nabla p\|_{L^2}^2+\|u\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)\tag{2.26}$$
$$+\frac{1}{18}\|\Delta\phi\|_{L^2}^2+\frac{1}{18}\|\nabla u\|_{L^2}^2.$$

Testing (2.22) by $-\phi$, using (1.5), we deduce that

$$\frac{1}{2}\frac{d}{dt}\int|\nabla\phi|^2\,dx+\int|\Delta\phi|^2\,dx$$
$$=-\int(n+p)\nabla\phi_1\nabla\phi+(n_2+p_2)(\nabla\phi)^2+u_1\Delta\phi\nabla\phi+u(n_2-p_2)\nabla\phi\,dx\tag{2.27}$$
$$=: J_1+J_2+J_3+J_4.$$

Using (2.10), (2.18), and (2.19), each term J_i $(i=1,2,3,4)$ can be bounded as follows:

$$J_1\le\|n+p\|_{L^2}\|\nabla\phi_1\|_{L^4}\|\nabla\phi\|_{L^4}$$
$$\le C(\|\Delta N\|_{L^2}+\|\Delta P\|_{L^2}+\|N\|_{L^2}+\|P\|_{L^2})\|\nabla\phi_1\|_{L^4}\|\nabla\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2}$$
$$\le\frac{1}{18}\|\Delta N\|_{L^2}^2+\frac{1}{18}\|\Delta P\|_{L^2}^2+C\|N\|_{L^2}^2+C\|P\|_{L^2}^2$$
$$+\frac{1}{18}\|\Delta\phi\|_{L^2}^2+C\|\nabla\phi_1\|_{L^4}^4\|\nabla\phi\|_{L^2}^2,$$
$$J_2\le 0,$$
$$J_3\le\|u_1\|_{L^4}\|\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^4}$$

$$\begin{aligned}
&\le C\|u_1\|_{L^4}\|\Delta\phi\|_{L^2}\|\nabla\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2}\\
&\le \frac{1}{18}\|\Delta\phi\|_{L^2}^2 + C\|u_1\|_{L^4}^4\|\nabla\phi\|_{L^2}^2,\\
J_4 &\le \|u\|_{L^4}\|n_2 - p_2\|_{L^2}\|\nabla\phi\|_{L^4}\\
&\le \|n_2 + p_2\|_{L^2}\|u\|_{L^4}^2 + \|n_2 + p_2\|_{L^2}\|\nabla\phi\|_{L^4}^2\\
&\le C\|n_2 + p_2\|_{L^2}\|u\|_{L^2}\|\nabla u\|_{L^2} + C\|n_2 + p_2\|_{L^2}\|\nabla\phi\|_{L^2}\|\Delta\phi\|_{L^2}\\
&\le \frac{1}{18}\|\nabla u\|_{L^2}^2 + \frac{1}{18}\|\Delta\phi\|_{L^2}^2 + C\|n_2 + p_2\|_{L^2}^2\left(\|u\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2\right).
\end{aligned} \tag{2.28}$$

Substituting these estimates into (2.27), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla\phi|^2\,dx + \frac{1}{2}\int|\Delta\phi|^2\,dx\\
&\quad\le C\left(\|\nabla\phi_1\|_{L^4}^4 + \|u_1\|_{L^4}^4 + \|n_2 + p_2\|_{L^2}^2 + 1\right)\left(\|u\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2 + \|N\|_{L^2}^2 + \|P\|_{L^2}^2\right)\\
&\qquad + \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\Delta P\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2.
\end{aligned} \tag{2.29}$$

It is easy to find that u satisfies

$$\partial_t u + u_2\cdot\nabla u + u\cdot\nabla u_1 + \nabla\pi - \Delta u = \Delta\phi\nabla\phi_1 + \Delta\phi_2\nabla\phi. \tag{2.30}$$

Testing this equation by u, using (1.2), we have

$$\frac{1}{2}\frac{d}{dt}\int u^2\,dx + \int|\nabla u|^2\,dx = \int \Delta\phi\nabla\phi_1 u + \Delta\phi_2\nabla\phi\cdot u - u\cdot\nabla u_1\cdot u\,dx =: \ell_1 + \ell_2 + \ell_3. \tag{2.31}$$

Using (2.10), each term ℓ_i $(i = 1, 2, 3)$ can be bounded as follows:

$$\begin{aligned}
\ell_1 &\le \|\Delta\phi\|_{L^2}\|\nabla\phi_1\|_{L^4}\|u\|_{L^4}\\
&\le C\|\Delta\phi\|_{L^2}\|\nabla\phi_1\|_{L^4}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\\
&\le \frac{1}{18}\|\Delta\phi\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2 + C\|\nabla\phi_1\|_{L^4}^4\|u\|_{L^2}^2,\\
\ell_2 &\le \|\Delta\phi_2\|_{L^2}\|\nabla\phi\|_{L^4}\|u\|_{L^4}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\Delta\phi_2\|_{L^2}\|\nabla\phi\|_{L^2}^{1/2}\|\Delta\phi\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\\
&\le \frac{1}{18}\|\Delta\phi\|_{L^2}^2+\frac{1}{18}\|\nabla u\|_{L^2}^2+C\|\Delta\phi_2\|_{L^2}^2\left(\|u\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right),\\
\ell_3 &\le \int u\cdot\nabla u\cdot u_1\,dx\\
&\le \|u\|_{L^4}\|\nabla u\|_{L^2}\|u_1\|_{L^4}\\
&\le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{3/2}\|u_1\|_{L^4}\\
&\le \frac{1}{18}\|\nabla u\|_{L^2}^2+C\|u_1\|_{L^4}^4\|u\|_{L^2}^2.
\end{aligned}
\tag{2.32}
$$

Substituting these estimates into (2.31), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int u^2\,dx+\int|\nabla u|^2\,dx\\
&\quad\le \frac{1}{9}\|\Delta\phi\|_{L^2}^2+C\left(\|\nabla\phi_1\|_{L^4}^4+\|\Delta\phi_2\|_{L^2}^2+\|u_1\|_{L^4}^4\right)\left(\|u\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right).
\end{aligned}
\tag{2.33}
$$

Combining (2.25), (2.26), (2.29), and (2.33), using (2.8), (2.11), (2.12), (2.13), and the Gronwall inequality, we conclude that

$$N=P=0,\qquad u=0,\qquad \nabla\phi=0, \tag{2.34}$$

and thus

$$n=p=0. \tag{2.35}$$

This completes the proof.

3. Proof of Theorem 1.3

By the same calculations as that in [11], we can prove (1.13) and thus we omit the details here.

Now we are in a position to prove the uniqueness. We still use the same notations as that in Section 2, and similarly we get (2.23). But each term I_i $(i=1,2,3,4)$ can be bounded as follows:

$$
\begin{aligned}
I_1 &\le \|\Delta N\|_{L^2}\|\nabla N\|_{L^{30/13}}\|\nabla\phi_1\|_{L^{15}}+\|N\|_{L^6}\|\nabla\phi_1\|_{L^3}\|\nabla N\|_{L^2}\\
&\le C\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}^{4/5}\|\Delta N\|_{L^2}^{1/5}\|\nabla\phi_1\|_{L^{15}}+C\|\nabla N\|_{L^2}^2\\
&\le \frac{1}{18}\|\Delta N\|_{L^2}^2+C\|\nabla\phi_1\|_{L^{15}}^{5/2}\|\nabla N\|_{L^2}^2+C\|\nabla N\|_{L^2}^2,
\end{aligned}
\tag{3.1}
$$

by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
\|\nabla N\|_{L^{30/13}} &\le C\|\nabla N\|_{L^2}^{4/5}\|\Delta N\|_{L^2}^{1/5}, \\
I_2 &\le \|n_2\|_{L^2}\left\|\nabla\phi\right\|_{L^6}\|\nabla N\|_{L^3} \\
&\le C\|n_2\|_{L^2}\left\|\Delta\phi\right\|_{L^2}\|\nabla N\|_{L^2}^{1/2}\|\Delta N\|_{L^2}^{1/2} \\
&\le C\|n_2\|_{L^2}\|-\Delta N + N + \Delta P - P\|_{L^2}\|\nabla N\|_{L^2}^{1/2}\|\Delta N\|_{L^2}^{1/2} \\
&\le \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\Delta P\|_{L^2}^2 + C\|N\|_{L^2}^2 + C\|P\|_{L^2}^2 + C\|n_2\|_{L^2}^4\|\nabla N\|_{L^2}^2
\end{aligned} \tag{3.2}$$

by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
\|\nabla N\|_{L^3}^2 &\le C\|\nabla N\|_{L^2}\|\Delta N\|_{L^2}, \\
I_3 &= \int u_1 n\nabla N\, dx = -\int u_1\Delta N\nabla N\, dx.
\end{aligned} \tag{3.3}$$

Now we decompose u_1 into three parts in the phase variable:

$$\begin{aligned}
u_1 &= \sum_{j<-M}\eta_j * u_1 + \sum_{j=-M}^{M}\eta_j * u_1 + \sum_{j>M}\eta_j * u_1 \\
&=: u_1^{\ell} + u_1^{m} + u_1^{h}.
\end{aligned} \tag{3.4}$$

Thus

$$\begin{aligned}
I_3 &= -\int u_1^{\ell}\Delta N\nabla N\, dx + \sum_i \int \partial_i u_1^{m}\cdot\partial_i N\nabla N\, dx - \int u_1^{h}\Delta N\nabla N\, dx \\
&=: I_{31} + I_{32} + I_{33}.
\end{aligned} \tag{3.5}$$

Recalling the Bernstein inequality,

$$\left\|\eta_j * u\right\|_{L^q} \le C2^{3j(1/p-1/q)}\left\|\eta_j * u\right\|_{L^p}, \quad 1 \le p \le q \le \infty, \tag{3.6}$$

the low-frequency part is estimated as

$$
\begin{aligned}
I_{31} &\leq \|\nabla N\|_{L^6}\|\Delta N\|_{L^2}\left\|u_1^{\ell}\right\|_{L^3} \\
&\leq C\|\Delta N\|_{L^2}^2 \sum_{j<-M} 2^{j/2}\left\|\eta_j * u_1\right\|_{L^2} \\
&\leq C\|\Delta N\|_{L^2}^2 \left(\sum_{j<-M} 2^j\right)^{1/2}\left(\sum_{j=-\infty}^{\infty}\left\|\eta_j * u_1\right\|_{L^2}^2\right)^{1/2} \\
&\leq C2^{-M/2}\|\Delta N\|_{L^2}^2\|u_1\|_{L^2} \\
&\leq C2^{-M/2}\|\Delta N\|_{L^2}^2.
\end{aligned}
\tag{3.7}
$$

The second term can be bounded as follows:

$$
\begin{aligned}
I_{32} &\leq \sum_i \|\partial_i N\|_{L^2}\|\nabla N\|_{L^2}\left\|\partial_i u_1^m\right\|_{L^\infty} \\
&\leq C\|\nabla N\|_{L^2}^2\left\|\nabla u_1^m\right\|_{L^\infty} \\
&\leq C\|\nabla N\|_{L^2}^2 \sum_{j=-M}^{M}\left\|\eta_j * \nabla u_1\right\|_{L^\infty} \\
&\leq CM\|\nabla N\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}.
\end{aligned}
\tag{3.8}
$$

On the other hand, the last term is simply bounded by the Hausdorff-Young inequality as

$$
\begin{aligned}
I_{33} &\leq \|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\left\|u_1^h\right\|_{L^\infty} \\
&\leq \|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\sum_{j>M}\left\|\left\{(-\Delta)^{-1/2}(\eta_{j-1}+\eta_j+\eta_{j+1})\right\} * \eta_j * (-\Delta)^{1/2}u_1\right\|_{L^\infty} \\
&\leq C\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\sum_{j>M}2^{-j}\left\|\eta_j * (-\Delta)^{1/2}u_1\right\|_{L^\infty} \\
&\leq C\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\sum_{j>M}2^{-j}\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}} \\
&\leq C2^{-M}\|\Delta N\|_{L^2}\|\nabla N\|_{L^2}\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}} \\
&\leq C2^{-M}\|\nabla N\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}^2 + C2^{-M}\|\Delta N\|_{L^2}^2.
\end{aligned}
\tag{3.9}
$$

Choosing M properly large so that $C2^{-M/2} \leq 1/36$ and $C2^{-M}\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}} \leq 1$, we reach

$$
\begin{aligned}
I_3 &\leq \frac{1}{8}\|\Delta N\|_{L^2}^2 + C\|\nabla N\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right),\\
I_4 &\leq \|u\|_{L^6}\|n_2\|_{L^2}\|\nabla N\|_{L^3}\\
&\leq C\|\nabla u\|_{L^2}\|n_2\|_{L^2}\|\nabla N\|_{L^2}^{1/2}\|\Delta N\|_{L^2}^{1/2}\\
&\leq \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2 + C\|n\|_{L^2}^4\|\nabla N\|_{L^2}^2.
\end{aligned}
\tag{3.10}
$$

Substituting the above estimates into (2.23), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int N^2 + |\nabla N|^2\,dx + \frac{1}{2}\int |\nabla N|^2 + |\Delta N|^2 dx\\
&\quad\leq C\left(\left\|\nabla\phi_1\right\|_{L^{15}}^{5/2} + 1 + \|n_2\|_{L^2}^4\right)\left(\|N\|_{L^2}^2 + \|P\|_{L^2}^2 + \|\nabla N\|_{L^2}^2\right)\\
&\qquad + C\|\nabla N\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right)\\
&\qquad + \frac{1}{18}\|\Delta P\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{3.11}
$$

Similarly for the p-equation, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int P^2 + |\nabla P|^2\,dx + \int |\nabla P|^2 + |\Delta P|^2\,dx\\
&\quad\leq C\left(\left\|\nabla\phi_1\right\|_{L^{15}}^{5/2} + 1 + \left\|p_2\right\|_{L^2}^4\right)\left(\|N\|_{L^2}^2 + \|P\|_{L^2}^2 + \|\nabla P\|_{L^2}^2\right)\\
&\qquad + C\|\nabla P\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right)\\
&\qquad + \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{3.12}
$$

As in Section 2, we still have (2.31). But each term ℓ_i $(i = 1,2,3)$ can be bounded as follows:

$$
\begin{aligned}
\ell_1 &\leq \left\|\nabla\phi_1\right\|_{L^{15}}\left\|\Delta\phi\right\|_{L^2}\|u\|_{L^{30/13}}\\
&\leq C\left\|\nabla\phi_1\right\|_{L^{15}}(\|\Delta N\|_{L^2} + \|\Delta P\|_{L^2} + \|N\|_{L^2} + \|P\|_{L^2})\|u\|_{L^2}^{4/5}\|\nabla u\|_{L^2}^{1/5}\\
&\leq \frac{1}{18}\|\Delta N\|_{L^2}^2 + \frac{1}{18}\|\Delta P\|_{L^2}^2 + C\|N\|_{L^2}^2 + C\|P\|_{L^2}^2\\
&\quad + \frac{1}{18}\|\nabla u\|_{L^2}^2 + C\left\|\nabla\phi_1\right\|_{L^{15}}^{5/2}\|u\|_{L^2}^2,
\end{aligned}
\tag{3.13}
$$

by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
\|u\|_{L^{30/13}} &\leq C\|u\|_{L^2}^{4/5}\|\nabla u\|_{L^2}^{1/5}, \\
\ell_2 &\leq \left\|\Delta\phi_2\right\|_{L^2}\left\|\nabla\phi\right\|_{L^6}\|u\|_{L^3} \\
&\leq C\left\|\Delta\phi_2\right\|_{L^2}\left\|\Delta\phi\right\|_{L^2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2} \\
&\leq C\left\|\Delta\phi_2\right\|_{L^2}(\|\Delta N\|_{L^2}+\|\Delta P\|_{L^2}+\|N\|_{L^2}+\|P\|_{L^2})\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2} \\
&\leq \frac{1}{18}\|\Delta N\|_{L^2}^2+\frac{1}{18}\|\Delta P\|_{L^2}^2+C\|N\|_{L^2}^2+C\|P\|_{L^2}^2 \\
&\quad +\frac{1}{18}\|\nabla u\|_{L^2}^2+C\left\|\Delta\phi_2\right\|_{L^2}^4\|u\|_{L^2}^2,
\end{aligned} \tag{3.14}$$

by the Gagliardo-Nirenberg inequality

$$\|u\|_{L^3}^2 \leq C\|u\|_{L^2}\|\nabla u\|_{L^2}. \tag{3.15}$$

By the similar calculations as that of I_3, ℓ_3 can be bounded as follows:

$$\ell_3 \leq \frac{1}{18}\|\nabla u\|_{L^2}^2+C\|u\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right). \tag{3.16}$$

Substituting the above estimates into (2.31), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int u^2\,dx+\frac{1}{2}\int|\nabla u|^2\,dx \\
&\quad\leq \frac{1}{9}\|\Delta N\|_{L^2}^2+\frac{1}{9}\|\Delta P\|_{L^2}^2+C\|N\|_{L^2}^2+C\|P\|_{L^2}^2 \\
&\qquad+\left(\left\|\nabla\phi_1\right\|_{L^{15}}^{5/2}+\left\|\Delta\phi_2\right\|_{L^2}^4\right)\|u\|_{L^2}^2 \\
&\qquad+C\|u\|_{L^2}^2\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\left(1+\log\left(e+\|\nabla u_1\|_{\dot{B}^0_{\infty,\infty}}\right)\right).
\end{aligned} \tag{3.17}$$

Combining (3.11), (3.12), and (3.17), using (1.13) and the Gronwall inequality, we arrive at

$$N=P=0, \qquad u=0, \tag{3.18}$$

as thus

$$n=p=0, \qquad \nabla\phi=0. \tag{3.19}$$

This completes the proof.

Acknowledgments

The authors would like to thank the referee for careful reading and helpful suggestions. This work is partially supported by Zhejiang Innovation Project (Grant no. T200905), ZJNSF (Grant no. R6090109), and NSFC (Grant no. 11171154).

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