Research Article

Spatially Nonhomogeneous Periodic Solutions in a Delayed Predator-Prey Model with Diffusion Effects

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This paper is concerned with a delayed predator-prey diffusion model with Neumann boundary conditions. We study the asymptotic stability of the positive constant steady state and the conditions for the existence of Hopf bifurcation. In particular, we show that large diffusivity has no effect on the Hopf bifurcation, while small diffusivity can lead to the fact that spatially nonhomogeneous periodic solutions bifurcate from the positive constant steady-state solution when the system parameters are all spatially homogeneous. Meanwhile, we study the properties of the spatially nonhomogeneous periodic solutions applying normal form theory of partial functional differential equations (PFDEs).

1. Introduction

Functional differential equations have merited a great deal of attention due to its theoretical and practical significance; they are often used in population dynamics, epidemiology, and other important areas of science; see [1–6]. In particular, Lu and Liu [7] proposed the following modified Holling-Tanner delayed predator-prey model:

$$\frac{du(t)}{dt} = ru(t)\left(1 - \frac{u(t)}{K}\right) - \frac{\alpha u(t)v(t)}{a + bu(t) + cv(t)},$$
$$\frac{dv(t)}{dt} = v(t)\left[s\left(1 - h\frac{v(t-\tau)}{u(t-\tau)}\right)\right],$$

(1.1)

where $u(t)$ and $v(t)$ denote the densities of prey species and predator species, respectively. The first equation states that the prey grows logistically with carrying capacity K and

intrinsic growth rate r in absence of predation. The second equation shows that predators grow logistically with intrinsic growth rate s and carrying capacity proportional to the prey populations size $u(t)$. The parameter h is the number of prey required to support one predator at equilibrium, when $v(t)$ equals $u(t)/h$. The term $hv(t)/u(t)$ of this equation is called the Leslie-Gower term. This interesting formulation for the predator dynamics has been discussed by Leslie and Gower in [8, 9]. τ is incorporated in the negative feedback of the predator density. $\alpha uv/(a+bu+cv)$ is Beddington-DeAngelis functional response. It is known that the Beddington-DeAngelis form of functional response has desirable qualitative features of ratio-dependent form but takes care of their controversial behaviors at low densities [10]. For more details on the background of this functional response, we refer to [10–12].

For convenience, a nondimensional form of system (1.1) will be useful. By defining $\tilde{t} = rt$, $\tilde{u} = u(t)/K$, $\tilde{v} = \alpha v(t)/rK$, and dropping the tildes for the sake of simplicity, model (1.1) becomes the following model:

$$
\begin{aligned}
\frac{du(t)}{dt} &= u(t)(1 - u(t)) - \frac{u(t)v(t)}{a_1 + bu(t) + c_1 v(t)}, \\
\frac{dv(t)}{dt} &= v(t)\left[\delta - \beta\frac{v(t-\tau)}{u(t-\tau)}\right],
\end{aligned}
\tag{1.2}
$$

where $\delta = s/r$, $\beta = sh/\alpha$, $a_1 = a/K$, $c_1 = cr/\alpha$, $\tilde{\tau} = r\tau$. Lu and Liu [7] proved the system (1.2) is permanent under some appropriate conditions and investigated the local and global stability of the equilibria.

In the earlier literature, most population models are often formulated by ordinary differential equations with or without time delays [1, 2, 13–18]. It is well known that the distribution of species is generally heterogeneous spatially, and therefore the species will migrate towards regions of lower population density to add the possibility of survival. Thus, partial differential equations with delay became the subject of a considerable interest in recent years. For a detailed theory and applications of delay equations with diffusion arising in biological and ecological problems, we refer to [19–23]. Therefore, time delays and spatial diffusion should be considered simultaneously in modeling biological interactions. Thus, the growth dynamics of two species corresponding to system (1.2) should be described by the following diffusion system with delay:

$$
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= d_1\Delta u(t,x) + u(t,x)\left[1 - u(t,x) - \frac{v(t,x)}{a_1 + bu(t,x) + c_1 v(t,x)}\right], \quad t > 0,\ x \in (0,\pi), \\
\frac{\partial v(t,x)}{\partial t} &= d_2\Delta v(t,x) + v(t,x)\left[\delta - \beta\frac{v(t-\tau,x)}{u(t-\tau,x)}\right], \quad t > 0,\ x \in (0,\pi), \\
\frac{\partial u(t,x)}{\partial x} &= \frac{\partial v(t,x)}{\partial x} = 0, \quad t \geq 0,\ x = 0,\pi, \\
u(t,x) &= \phi(t,x) \geq 0, \quad v(t,x) = \psi(t,x) \geq 0, \quad (t,x) \in [-\tau,0] \times (0,\pi),
\end{aligned}
\tag{1.3}
$$

where $u(t,x)$ and $v(t,x)$ can be interpreted as the densities of prey and predator populations at time t and space x, respectively; $d_1 > 0$, $d_2 > 0$ denote the diffusion coefficients of prey and predator two species, respectively; Δ is the Laplacian operator; Neumann boundary

conditions in (1.3) imply that two species have zero flux across the domain boundary. $(\phi, \psi) \in \mathbb{C} = C([-\tau, 0], X)$, and X defined by

$$X = \left\{ (u, v) : u, v \in W^{2,2}(0, \pi) : \frac{du}{dx} = \frac{dv}{dx} = 0, \ x = 0, \pi \right\} \tag{1.4}$$

with the inner product $< \cdot, \cdot >$.

In the remaining part of this paper, we focus on system (1.3). The main purpose of this paper is to consider the effects of the delay and diffusion on the dynamics of system (1.3).

The organization of this paper is as follows. In Section 2, we consider the stability of the positive constant steady-state solutions and the existences of Hopf bifurcations of surrounding the positive constant steady-state solutions. In particular, we show the existence of spatially nonhomogeneous periodic solutions while the system parameters are all spatially homogeneous. In Section 3, we present that the emergence of these spatially nonhomogeneous periodic solutions is clearly due to the effect of the small diffusivity. Finally, we study the properties of the spatially nonhomogeneous periodic solutions applying normal form theory of PFDEs.

2. Stability and Hopf Bifurcations

In this section, we investigate the stability of the positive constant steady state of (1.3) and obtain the conditions under which (1.3) undergoes a Hopf bifurcation.

It is easy to see that the solutions of system (1.2) have a unique boundary equilibrium $E_1(1, 0)$ and a unique positive equilibrium $E^*(u^*, v^*)$, where

$$u^* = \frac{-(a_1 - b - c_1\delta/\beta + \delta/\beta) + \sqrt{\Lambda}}{2(b + c_1\delta/\beta)}, \qquad v^* = \frac{\delta}{\beta}x^*,$$

$$\Lambda = (a_1 - b - c_1\delta/\beta + \delta/\beta)^2 + 4a_1(b + c_1\delta/\beta). \tag{2.1}$$

Obviously, $E_1(1, 0)$ and $E^*(u^*, v^*)$ are also the spatially homogeneous steady-state solutions of system (1.3). From the point of view of biology, we should consider system (1.3) in the closed first quadrant in the (u, v) plane, that is, the positive constant steady-state solutions $E^*(u^*, v^*)$ of system (1.3).

Let $\overline{u}(t, x) = u(t, x) - u^*; \overline{v}(t, x) = v(t, x) - v^*$, for convenience, we use $u(t, x)$ and $v(t, x)$ to replace $\overline{u}(t, x)$ and $\overline{v}(t, x)$, respectively; then system (1.3) can be transformed into

$$\frac{\partial u(t, x)}{\partial t} = d_1 \Delta u(t, x) + \alpha_{11} u(t, x) + \alpha_{12} v(t, x) + \sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} u^i(t, x) v^j(t, x),$$

$$\frac{\partial v(t, x)}{\partial t} = d_2 \Delta v(t, x) + \alpha_{21} u(t - \tau, x) + \alpha_{22} v(t - \tau, x) \tag{2.2}$$

$$+ \sum_{i+j+l\geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} u^i(t - \tau, x) v^j(t - \tau, x) v^l(t, x),$$

where

$$\alpha_{11} := f_{10}^{(1)} = -u^* + \frac{bu^*v^*}{(a_1 + bu^* + c_1 v^*)^2}, \qquad \alpha_{12} := f_{01}^{(1)} = -\frac{u^*(a_1 + bu^*)}{(a_1 + bu^* + c_1 v^*)^2} < 0,$$

$$\alpha_{21} := f_{100}^{(2)} = \frac{\delta^2}{\beta} > 0, \qquad \alpha_{22} := f_{010}^{(2)} = -\delta < 0,$$

$$f_{ij}^{(1)} = \left. \frac{\partial^{i+j} f^{(1)}}{\partial u^i \partial v^j} \right|_{(u^*,v^*)}, \qquad f_{ijl}^{(2)} = \left. \frac{\partial^{i+j+l} f^{(2)}}{\partial u^i \partial v^j \partial v_1^l} \right|_{(u^*,v^*,v^*)}, \qquad i, j, l \geq 0,$$

$$f^{(1)} = u(1-u) - \frac{uv}{a_1 + bu + c_1 v}, \qquad f^{(2)} = v_1 \left(\delta - \frac{\beta v}{u} \right).$$

(2.3)

Therefore, the positive constant stationary solution $E^*(u^*, v^*)$ of system (1.3) can be transformed into the origin of system (2.2).

Let $u_1(t) = u(t, \cdot)$, $u_2(t) = v(t, \cdot)$, $U(t) = (u_1(t), u_2(t))^T$; therefore, system (2.2) can be rewritten as an abstract form in the phase space $\mathbb{C} = C([-\tau, 0], X)$:

$$\dot{U}(t) = d\Delta U(t) + L(U_t) + f(U_t),$$

(2.4)

where $d = (d_1, d_2)^T$, $\Delta = \begin{pmatrix} \partial/\partial x^2 & 0 \\ 0 & \partial/\partial x^2 \end{pmatrix}$, $U_t(\theta) = U(t+\theta)$, $-\tau \leq \theta \leq 0$, $L : \mathbb{C} \to X$ and $f : \mathbb{C} \to X$ are given, respectively, by

$$L(\varphi) = \begin{pmatrix} \alpha_{11}\varphi(0) + \alpha_{12}\varphi(0) \\ \alpha_{21}\varphi_1(-\tau) + \alpha_{22}\varphi_2(-\tau) \end{pmatrix},$$

$$f(\varphi) = \begin{pmatrix} \displaystyle\sum_{i+j \geq 2} \frac{1}{ij} f_{ij}^{(1)} \varphi_1^i \varphi_2^j \\ \displaystyle\sum_{i+j+l \geq 2} \frac{1}{ijl} f_{ijl}^{(2)} \varphi_1^i(-\tau) \varphi_2^j(-\tau) \varphi_2^l(0) \end{pmatrix},$$

(2.5)

for $\varphi = (\varphi_1, \varphi_2)^T \in \mathbb{C}$, $\varphi(\theta) = U_t(\theta)$, $-\tau \leq \theta \leq 0$.

Linearizing (2.4) at $(0,0)$ gives the linear equation

$$\dot{U}(t) = d\Delta U(t) + L(U_t),$$

(2.6)

whose characteristic equation is

$$\lambda y - d\Delta y - L\left(e^{\lambda \cdot} y\right) = 0,$$

(2.7)

where $y \in \mathrm{dom}(\Delta) \setminus \{0\}$ and $\mathrm{dom}(\Delta) \subset X$.

It is well known that the linear operator Δ on $(0, \pi)$ with homogeneous Neumann boundary conditions has the eigenvalues $-k^2 (k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\})$, and the corresponding eigenfunctions are

$$\beta_k^1 = \begin{pmatrix} \gamma_k \\ 0 \end{pmatrix}, \quad \beta_k^2 = \begin{pmatrix} 0 \\ \gamma_k \end{pmatrix}, \quad \gamma_k = \frac{\cos(kx)}{\|\cos(kx)\|_{2,2}}, \quad k \in \mathbb{N}_0. \tag{2.8}$$

Notice that $(\beta_k^1, \beta_k^2)_{k=0}^{\infty}$ construct an orthogonal basis of the Banach space X. Therefore $L(\beta_k^1, \beta_k^2) \subset \text{span}\{\beta_k^1, \beta_k^2\}$, and thus any element y in X can be expanded a Fourier series in the form

$$y = \sum_{k=0}^{\infty} Y_k^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \tag{2.9}$$

$$Y_k^T = \begin{pmatrix} \langle y, \beta_k^1 \rangle \\ \langle y, \beta_k^2 \rangle \end{pmatrix}. \tag{2.10}$$

In addition, some easy computations can show that

$$L\left(\varphi^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \right) = [L(\varphi)]^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \tag{2.11}$$

for $\varphi = (\varphi_1, \varphi_2)^T \in \mathbb{C}$.

From (2.9) and (2.11), (2.7) is equivalent to

$$\sum_{k=0}^{\infty} Y_k^T \left[\begin{pmatrix} \lambda + d_1 k^2 & 0 \\ 0 & \lambda + d_2 k^2 \end{pmatrix} - \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} e^{-\lambda \tau} & \alpha_{22} e^{-\lambda \tau} \end{pmatrix} \right] \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} = 0. \tag{2.12}$$

Thus λ is a characteristic root of (2.7) if and only if for $k \in \mathbb{N}_0$, λ satisfies

$$\lambda^2 + A_k \lambda + B_k + (-\alpha_{22} \lambda + C_k) e^{-\lambda \tau} = 0, \tag{2.13}$$

where

$$A_k = d_1 k^2 + d_2 k^2 - \alpha_{11},$$

$$B_k = d_1 d_2 k^4 - \alpha_{11} d_2 k^2,$$

$$C_k = \alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} - d_1 k^2 \alpha_{22} > 0, \tag{2.14}$$

$$\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} = \delta u^* + \frac{a_1 \delta^2 u^*}{\beta(a_1 + bu^* + c_1 v^*)^2} > 0.$$

When $\tau = 0$, (2.13) reduces to the following quadratic equation with respect to λ:

$$\lambda^2 + (A_k - \alpha_{22})\lambda + B_k + C_k = 0. \tag{2.15}$$

If $\alpha_{11} < 0$, then $A_k > 0, B_k \geq 0$, and

$$
\begin{aligned}
A_k - \alpha_{22} &> 0, \\
B_k + C_k &> 0,
\end{aligned}
\tag{2.16}
$$

since $C_k > 0$.

Therefore, it is obvious that all roots of equations (2.15) have negative real parts, and we can conclude that the positive constant steady state $E^*(u^*, v^*)$ of system (2.2) is locally asymptotically stable in the absence of delay when $\alpha_{11} < 0$. Thus, we can have the following conclusions.

Theorem 2.1. *Suppose that the condition $\alpha_{11} < 0$ is satisfied. Then*

(i) *all roots of each equation in* (2.15) *have negative real parts for any wave number k,*

(ii) *for any wave number k, the positive constant steady-state solution $E^*(u^*, v^*)$ of system* (1.3) *is locally asymptotically stable in the absence of delay.*

In the following, we discuss the effects of delay τ on the stability of the trivial solution of (2.2). Notice that $i\omega(\omega > 0)$ is a root of (2.13) if and only if for a certain $k \in \mathbb{N}_0$, ω satisfies the following equation:

$$-\omega^2 + A_k\omega i + B_k + (-\alpha_{22}\omega i + C_k)(\cos \omega\tau - i\sin \omega\tau) = 0. \tag{2.17}$$

Thus

$$\omega^4 + \left(A_k^2 - 2B_k - \alpha_{22}^2\right)\omega^2 + B_k^2 - C_k^2 = 0, \quad k \in \mathbb{N}_0. \tag{2.18}$$

Letting $\omega^2 = z$, then (2.18) can be written as

$$z^2 + \left(A_k^2 - 2B_k - \alpha_{22}^2\right)z + B_k^2 - C_k^2 = 0, \quad k \in \mathbb{N}_0. \tag{2.19}$$

Equation (2.19) with $k = 0$ has only one positive real root:

$$z_0 = \frac{\sqrt{\left(A_0^2 - \alpha_{22}^2\right)^2 + 4C_0^2} - \left(A_0^2 - \alpha_{22}^2\right)}{2} > 0, \tag{2.20}$$

where $C_0 = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \delta u^* + (a_1\delta^2 u^*/\beta(a_1 + bu^* + c_1v^*)^2) > 0$.

In addition, from (2.17) and (2.20), we have

$$\cos(\omega_0\tau) = \frac{A_0\alpha_{22}(\omega_0)^2 + C_0(\omega_0)^2}{\alpha_{22}^2(\omega_0)^2 + C_0^2}, \tag{2.21}$$

where $\omega_0 = \sqrt{z_0}$. Thus

$$\tau_j^0 = \frac{1}{\omega_0}\left\{\arccos\frac{A_0\alpha_{22}(\omega_0)^2 + C_0(\omega_0)^2}{\alpha_{22}^2(\omega_0)^2 + C_0^2} + 2j\pi\right\}, \quad j \in \mathbb{N}_0. \tag{2.22}$$

Denote

(H) $(d_1^2 + d_2^2) - 2d_1\alpha_{11} > \alpha_{22}^2 - \alpha_{11}^2$ and $d_1d_2 + (d_1\alpha_{22} - d_2\alpha_{11}) > \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$.

Theorem 2.2. *Assume that the conditions* (H) *and* $\alpha_{11} < 0$ *hold. For* $\tau = \tau_j^0$, (2.13) *with* $k = 0$ *has a pair of purely imaginary eigenvalues* $\pm\omega_0 i$ *and there are no other roots of* (2.13) *with zero real parts.*

Proof. Assuming $\lambda = i\omega_*$, $\omega_* > 0$ is a solution of (2.13) with $k \geq 1$. From (2.17), (2.18), and (2.19), we get

$$\omega_*^4 + \left(A_k^2 - 2B_k - \alpha_{22}^2\right)\omega_*^2 + B_k^2 - C_k^2 = 0, \quad k \in \mathbb{N} = \{1, 2, \ldots\}, \tag{2.23}$$

so we have

$$\omega_*^2 = \frac{-\left(A_k^2 - 2B_k - \alpha_{22}^2\right) \pm \sqrt{\left(A_k^2 - 2B_k - \alpha_{22}^2\right)^2 - 4\left(B_k^2 - C_k^2\right)}}{2}, \quad k \in \mathbb{N}. \tag{2.24}$$

Clearly, if $A_k^2 - 2B_k - \alpha_{22}^2 > 0$ and $B_k^2 - C_k^2 > 0$, there are no ω_* such that (2.13) with $k \geq 1$ has purely imaginary roots $\pm\omega_* i$.

By computing, we have

$$A_k^2 - 2B_k - \alpha_{22}^2 = \left(d_1^2 + d_2^2\right)k^4 - 2d_1\alpha_{11}k^2 + \alpha_{11}^2 - \alpha_{22}^2,$$
$$B_k = d_1d_2k^4 - \alpha_{11}d_2k^2, \qquad C_k = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} - d_1k^2\alpha_{22} > 0. \tag{2.25}$$

In addition, according to $\alpha_{11} < 0$, we have $B_k \geq 0$. It is clear that $(d_1^2 + d_2^2)k^4 - 2d_1\alpha_{11}k^2 + \alpha_{11}^2 - \alpha_{22}^2 \geq (d_1^2 + d_2^2) - 2d_1\alpha_{11} + \alpha_{11}^2 - \alpha_{22}^2$ when $k \geq 1(\alpha_{11} < 0)$. Furthermore, if $d_1d_2 + (d_1\alpha_{22} - d_2\alpha_{11}) > \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$, we can get $B_k^2 - C_k^2 > 0$ when $k \geq 1$. Therefore, (2.13) with $k \geq 1$ has no purely imaginary roots when the conditions (H) and $\alpha_{11} < 0$ hold. Thus the proof of Theorem 2.2 is accomplished.

$\square$

Let

$$\lambda(\tau) = \sigma(\tau) + i\omega(\tau) \tag{2.26}$$

be a root of (2.13) with $k = 0$ near $\tau = \tau_j^0$ satisfying $\sigma(\tau_j^0) = 0$, $\omega(\tau_j^0) = \omega_0$, $j \in N_0$.

Then the following result holds.

Lemma 2.3. *The following transversality conditions hold:*

$$\frac{d \operatorname{Re} \lambda\left(\tau_j^0\right)}{d\tau} > 0, \quad j \in N_0. \tag{2.27}$$

From the previous discussions, we have the following theorem on the stability of positive steady-state solution (u^*, v^*) of system (1.3) and the existence of Hopf bifurcation near (u^*, v^*).

Theorem 2.4. *Assume that the conditions (H) and $\alpha_{11} < 0$ hold. Then*

(i) *if $\tau \in [0, \tau_j^0)$, the positive constant steady state (u^*, v^*) of (1.3) is asymptotically stable;*

(ii) *if $\tau > \tau_j^0$, the positive constant steady state (u^*, v^*) of (1.3) is unstable;*

(iii) *$\tau = \tau_j^0$ are Hopf bifurcation values of system (1.3), and these Hopf bifurcations are all spatially homogeneous.*

3. Effect of Small Diffusivity

In the previous section, we have obtained the conditions under which spatially homogeneous Hopf bifurcations bifurcate from the positive steady-state solutions $E^* = (u^*, v^*)$ of system (1.3) when the parameter τ crosses through the critical value τ_j^0. In this sense, we say that the diffusion terms do not have effect on the Hopf bifurcations. In this section, we discuss the effect of small diffusivity on Hopf bifurcations for system (1.3) when the condition (H) is not satisfied. For the simplicity of discussion which follows, throughout this section, we always suppose that the condition (H_1): $d_1 d_2 + (d_1 \alpha_{22} - d_2 \alpha_{11}) > \alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}$ holds.

Assume $\lambda = i w_k (w_k > 0)$ is a solution of (2.13) with $k \geq 1$. From the discussion in Section 2, we have

$$w_k^4 + \left(A_k^2 - 2B_k - \alpha_{22}^2\right)w_k^2 + B_k^2 - C_k^2 = 0, \quad k \in \mathbb{N}. \tag{3.1}$$

If the condition (H) is not satisfied, and

(H_2) $(A_k^2 - 2B_k - \alpha_{22}^2)^2 - 4(B_k^2 - C_k^2) \geq 0$,

then (2.13) with $k \geq 1$ has roots $\pm i w_k$, where

$$w_k = \sqrt{\frac{-(A_k^2 - 2B_k - \alpha_{22}^2) \pm \sqrt{(A_k^2 - 2B_k - \alpha_{22}^2)^2 - 4(B_k^2 - C_k^2)}}{2}}. \tag{3.2}$$

From the discussion in Section 2, we know that there exists $k_0 > 0$, $k_0 \in \mathbb{N}$ such that (2.13) with $k \geq 1$ has only characteristic roots with negative real parts when $k > k_0$ [24].

In addition, from (2.17), we have

$$\cos(w_k \tau) = \frac{A_k \alpha_{22} (w_k)^2 - B_k C_k + C_k (w_k)^2}{\alpha_{22}^2 (w_k)^2 + C_k^2}, \quad k \in \mathbb{N}. \tag{3.3}$$

Thus

$$\tau_j^k = \frac{1}{\omega_k}\left(\arccos\frac{A_k\alpha_{22}(\omega_k)^2 - B_kC_k + C_k(\omega_k)^2}{\alpha_{22}^2(\omega_k)^2 + C_k^2} + 2j\pi\right), \quad j \in \mathbb{N}_0, \ k \in \mathbb{N}. \tag{3.4}$$

In particular, it is easy to know from (H_1) that $B_k > C_k$ when $\alpha_{11} < 0$. By the same way in Theorem 2.2, we can see if

$$\left(d_1^2 + d_2^2\right)16 - 8d_1\alpha_{11} > \alpha_{22}^2 - \alpha_{11}^2, \tag{3.5}$$

then (2.13) with $k \geq 2$ has no purely imaginary eigenvalues.

Suppose the condition (H) is not satisfied, that is, $(d_1^2 + d_2^2) - 2d_1\alpha_{11} < \alpha_{22}^2 - \alpha_{11}^2$, assume further (H_2) satisfy. Then (2.13) with $k = 1$ has a pair of purely imaginary eigenvalues $i\omega_1$, and all other zeros have negative real parts, where ω_1 is given by

$$\omega_1 = \sqrt{\frac{-\left(A_1^2 - 2B_1 - \alpha_{22}^2\right) \pm \sqrt{\left(A_1^2 - 2B_1 - \alpha_{22}^2\right)^2 - 4\left(B_1^2 - C_1^2\right)}}{2}}. \tag{3.6}$$

Therefore, we can obtain the following.

Lemma 3.1. *Suppose that $\alpha_{11} < 0$ and $(d_1^2 + d_2^2)16 - 8d_1\alpha_{11} > \alpha_{22}^2 - \alpha_{11}^2 > (d_1^2 + d_2^2) - 2d_1\alpha_{11}$. If the condition (H_2) holds, then (2.13) with $k = 1$ has a simple pair of purely imaginary roots $\pm i\omega_1$ and all other roots except $\pm i\omega_1$ have strictly negative real parts, where ω_1 is defined by (3.6).*

For system (1.3), by the similar discussion to that of Theorem 2.2, when τ crosses through the critical values τ_j^1, where

$$\tau_j^1 = \frac{1}{\omega_1}\left(\arccos\frac{A_1\alpha_{22}(\omega_1)^2 - B_1C_1 + C_1(\omega_1)^2}{\alpha_{22}^2(\omega_1)^2 + C_1^2} + 2j\pi\right), \quad j \in \mathbb{N}_0, \tag{3.7}$$

it can give rise to Hopf bifurcation at the positive constant steady state (u^*, v^*). By the results in [22], bifurcating periodic solutions of (1.3) at $\tau = \tau_j^1$ are spatially nonhomogeneous.

Therefore, we have the following conclusion.

Theorem 3.2. *If the conditions in Lemma 3.1 are satisfied, then $\tau = \tau_j^1$ are Hopf bifurcation values of system (1.3), and these Hopf bifurcations are all spatially nonhomogeneous, where τ_j^1 is defined by (3.7).*

In general, we have the following.

Theorem 3.3. *Suppose that $\alpha_{11} < 0$, if there exist $k_0 > 0$, $k_0 \in \mathbb{N}$ such that*

$$\left(d_1^2 + d_2^2\right)k_0^4 - d_1\alpha_{11}k_0^2 > \alpha_{22}^2 - \alpha_{11}^2 > \left(d_1^2 + d_2^2\right) - 2d_1\alpha_{11}, \tag{3.8}$$

and (H_2) *holds, then (2.13) with* $k = k_0$ *has purely imaginary roots* $i\omega_{k_0}$ *and system (1.3) has a family of spatially nonhomogeneous periodic solutions bifurcating from the spatially homogeneous steady state* (u^*, v^*), *when* τ *crosses through the critical values* $\tau_j^{k_0}$, *where* ω_{k_0} *and* $\tau_j^{k_0}$ *are defined by (3.2) and (3.4) with* $k = k_0$, $k_0 \in \mathbb{N}$, *respectively.*

From Theorems 2.2 and 3.3, we can know that large diffusivity has no effect on the Hopf bifurcation, while small diffusivity can lead to the fact that the system bifurcates spatially nonhomogeneous periodic solutions at the positive constant steady state under which the system parameters are all spatially homogeneous. These exhibit that the emergence of these spatially nonhomogeneous periodic solutions is clearly due to the effect of the small diffusivity.

4. Properties of Hopf Bifurcation

In Theorem 3.2, we have obtained the conditions under which a family of spatially nonhomogeneous periodic solutions bifurcates from the spatially homogeneous steady-state solutions $E^* = (u^*, v^*)$ of system (1.3) when the parameter τ crosses through the critical value τ_j^1. In this section, we redefine an inner product to study the properties of the spatially nonhomogeneous Hopf bifurcation applying normal form theory of PFDEs by developed [22, 25].

Normalizing the delay τ in system (2.2) by the time-scaling $t \rightarrow t/\tau$, (2.2) is transformed into

$$\frac{\partial u(t,x)}{\partial t} = \tau \left\{ d_1 \Delta u(t,x) + \alpha_{11} u(t,x) + \alpha_{12} v(t,x) + \sum_{i+j \geq 2} \frac{1}{i!j!} f_{ij}^{(1)} u^i(t,x) v^j(t,x) \right\},$$

$$\frac{\partial v(t,x)}{\partial t} = \tau \left\{ d_2 \Delta v(t,x) + \alpha_{21} u(t-1,x) + \alpha_{22} v(t-1,x) \right. \tag{4.1}$$

$$\left. + \sum_{i+j+l \geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} u^i(t-1,x) v^j(t-1,x) v^l(t,x) \right\},$$

where $f^{(1)}$, $f^{(2)}$ are defined by (2.2). Letting $\tau = \tau_j^1 + \alpha$, $j \in \mathbb{N}_0$, then, (4.1) can be written in abstract form in $\mathcal{C} = C([-1,0] : X)$ as

$$\frac{d}{dt} U(t) = \tau_j^1 d\Delta U(t) + L\left(\tau_j^1\right)(U_t) + F(U_t, \alpha), \tag{4.2}$$

where $d = (d_1, d_2)^T$, $L(\tau_j^1)(\cdot) : C \to X$, $F(\cdot, \alpha) : C \times \mathbb{R}^+ \to X$ are given by

$$L\left(\tau_j^1\right)(\varphi) = \tau_j^1 \begin{pmatrix} \alpha_{11}\varphi_1(0) + \alpha_{12}\varphi_2(0) \\ \alpha_{21}\varphi_1(-1) + \alpha_{22}\varphi_2(-1) \end{pmatrix},$$

$$F(\varphi, \alpha) = \alpha d \Delta \varphi(0) + L(\alpha)\varphi + f(\varphi, \alpha),$$

$$f\left(\varphi, \tau_j^1\right) = \left(\tau_j^1 + \alpha\right) \begin{pmatrix} \displaystyle\sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} \varphi_1^i(0)\varphi_2^j(0) \\ \displaystyle\sum_{i+j+l\geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} \varphi_1^i(-1)\varphi_2^j(-1)\varphi_2^l(0) \end{pmatrix},$$

(4.3)

for $\varphi = (\varphi_1, \varphi_2)^T \in C$.

Linearizing (4.2) at $(0,0)$ leads to the following linear equation:

$$\frac{d}{dt}U(t) = \tau_j^1 d\Delta U(t) + L\left(\tau_j^1\right)(U_t).$$

(4.4)

Let $\Lambda_1 = \{-i\omega_1, i\omega_1\}$; consider the following FDE on $C([-1,0], X)$:

$$\dot{z}(t) = \tau_j^1 d\Delta z(t) + L\left(\tau_j^1\right)(z_t),$$

(4.5)

that is,

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \tau_j^1 \left\{ \begin{pmatrix} \alpha_{11} - d_1 & \alpha_{12} \\ 0 & -d_2 \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} z_1(t-1) \\ z_2(t-1) \end{pmatrix} \right\}.$$

(4.6)

Obviously, $L(\tau_j^1)$ is a continuous linear function mapping $C([-1,0], X)$ into X. According to the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \tau)$. $-1 \leq \theta \leq 0$, whose elements are of bounded variation such that

$$L\left(\tau_j^1\right)(\phi) = \int_{-1}^0 d\eta\left(\theta, \tau_j^1\right)\phi(\theta) \quad \text{for } \phi \in C.$$

(4.7)

Thus, we can choose

$$\eta\left(\theta, \tau_j^1\right) = \begin{cases} \tau_j^1 \begin{pmatrix} \alpha_{11} - d_1 & \alpha_{12} \\ 0 & -d_2 \end{pmatrix}, & \theta \in (-1, 0], \\ \tau_j^1 \begin{pmatrix} 0 & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, & \theta = -1, \end{cases}$$

(4.8)

then (4.7) is satisfied.

Letting $A(\tau_j^1)$ denote the infinitesimal generator of strongly continuous semigroup, according to [2], then,

$$
A\left(\tau_j^1\right)\phi(\theta) = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0), \\[3mm] L\left(\tau_j^1\right)(\phi) \overset{\text{def}}{=} \displaystyle\int_{-1}^{0} d\eta\left(t,\tau_j^1\right)\phi(t), & \theta = 0, \end{cases} \tag{4.9}
$$

where $\phi \in C^1([-1,0], X)$.

For $\psi \in C^1([0,1], (X)^*)$, define

$$
A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\[3mm] \displaystyle\int_{-1}^{0} \psi(-t)d\eta\left(t,\tau_j^1\right), & s = 0 \end{cases} \tag{4.10}
$$

and a bilinear inner product of the Sobolev space $W^{2.2}(0,\pi)$:

$$
\begin{aligned}
(\psi(s), \phi(\theta)) &= \psi(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi \\[2mm]
&= \psi(0)\phi(0) - \tau_j^1 \int_{-1}^{0} \psi(\theta+1)\begin{pmatrix} 0 & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\phi(\theta)d\theta,
\end{aligned} \tag{4.11}
$$

where $\eta(\theta) = \eta(\theta, \tau_j^1)$ and A^* are the formal adjoint of $A(\tau_j^1)$.

It is easy to see from Section 2 that $A(\tau_j^1)$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_1$ and they are also eigenvalues of A^* since $A(\tau_j^1)$ and A^* are adjoint operators. Let $\mathcal{P}$ and $\mathcal{P}^*$ be the center spaces, that is, the generalized eigenspaces, of $A(\tau_j^1)$ and A^* associated with Λ_1, respectively. Then $\mathcal{P}^*$ is the adjoint space of $\mathcal{P}$ and $\dim \mathcal{P} = \dim \mathcal{P}^* = 2$.

In addition, according to [22, 25], by a few simple calculations, we can choose Φ and Ψ be the bases for $\mathcal{P}$ and $\mathcal{P}^*$, respectively. It is known that $\dot{\Phi} = \Phi B$, where B is the $I \times I$ diagonal matrix $B = \begin{pmatrix} i\tau_j^1 & 0 \\ 0 & i\tau_j^1 \end{pmatrix}$.

Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)^T$, where

$$
\Phi_1(\theta) = (1,\xi)^T e^{i\omega_1 \tau_j^1 \theta}, \quad \Phi_2(\theta) = \overline{\Phi_1(\theta)}, \quad -1 \le \theta \le 0,
$$

$$
\Psi_1(s) = \frac{1}{\rho}(1,\zeta)e^{-i\omega_1 \tau_j^1 s}, \quad \Psi_2(s) = \overline{\Psi_1(s)}, \quad 0 \le s \le 1,
$$

$$
\xi = \frac{i\omega_1 - \alpha_{11} + d_1}{\alpha_{12}}, \qquad \zeta = -\frac{i\omega_1 + \alpha_{11} - d_1}{\alpha_{21}}e^{-i\omega_1 \tau_j^1}, \tag{4.12}
$$

$$
\overline{\rho} = (1 + \xi\zeta) - \tau_j^1(-d_1 + \alpha_{11} + \zeta\alpha_{21} + \xi\alpha_{12} - d_2\xi\zeta + \xi\zeta\alpha_{22})e^{-i\omega_1 \tau_j^1}.
$$

From the above expression, we can easily see that $(\Psi_1, \Phi_1) = 1$, $(\Psi_1, \overline{\Phi}_1) = 0$.

Let $f_1 = (\beta_1^1, \beta_1^2)$, $c \cdot f_1$ be defined by $c \cdot f_1 = c_1 \beta_1^1 + c_2 \beta_1^2$ for $c = (c_1, c_2)^T \in \mathbb{R}^2$ and $(\psi \cdot f_1)(\theta) = \psi(\theta) \cdot f_1$ for $\theta \in [-1, 0]$. Then the center space of linear equation (4.4) is given by $P_{\mathrm{CN}}\mathcal{C}$, where

$$P_{\mathrm{CN}}\varphi = \Phi(\Psi, < \varphi, f_1 >) \cdot f_1, \quad \varphi \in \mathcal{C} \tag{4.13}$$

and $\mathcal{C} = P_{\mathrm{CN}}\mathcal{C} \oplus P_Q\mathcal{C}$; here $P_Q\mathcal{C}$ denotes the complementary subspace of $P_{\mathrm{CN}}\mathcal{C}$ in $\mathcal{C}$.

Let $A_{\tau_j^1}$ be defined by

$$A_{\tau_j^1}\varphi(\theta) = \dot{\varphi}(\theta) + X_0(\theta)\left[\tau_j^1 \Delta\varphi(0) + L_*\left(\tau_j^1\right)(\varphi(0)) - \dot{\varphi}(0)\right], \quad \varphi \in \mathcal{C}, \tag{4.14}$$

where $X_0: [-1, 0] \rightarrow B(X, X)$ is given by

$$X_0 = \begin{cases} 0, & \theta \in [-1, 0), \\ I, & \theta = 0. \end{cases} \tag{4.15}$$

Then $A_{\tau_j^1}$ is the infinitesimal generator induced by the solution of (4.4) and (4.2) and can be rewritten as the following operator differential equation:

$$\dot{U}_t = A_{\tau_j^1} U_t + X_0 F(U_t, \alpha). \tag{4.16}$$

Using the decomposition $\mathcal{C} = P_{\mathrm{CN}}\mathcal{C} \oplus P_Q\mathcal{C}$ and (4.13), the solution of (4.16) can be written as

$$U_t = \Phi\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_1 + h(x_1, x_2, \alpha), \tag{4.17}$$

where $(x_1, x_2)^T = (\Psi, < U_t, f_1 >)$, and $h(x_1, x_2, \alpha) \in P_Q\mathcal{C}$ with $h(0, 0, 0) = Dh(0, 0, 0) = 0$.

Thus, we describe the flow on the center manifold for (4.2) as

$$U_t^* = \Phi\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_1 + h(x_1, x_2), \tag{4.18}$$

where $h(x_1, x_2) = h(x_1, x_2, 0)$.

Letting $z = x_1 - ix_2$ and $\Psi(0) = (\Psi_1(0), \Psi_2(0))^T$, when $\alpha = 0$, then z satisfies

$$\dot{z} = i\omega_1 \tau_j^1 z + g(z, \overline{z}), \tag{4.19}$$

where

$$g(z,\overline{z}) = (\Psi_1(0) - i\Psi_2(0))\langle F(U_t,0), f_1\rangle, \tag{4.20}$$

$$w(z,\overline{z}) = h\left(\frac{z+\overline{z}}{2}, \frac{(z-\overline{z})i}{2}, 0\right), \tag{4.21}$$

$$w(z,\overline{z}) = w_{20}\frac{z^2}{2} + w_{11}z\overline{z} + w_{02}\frac{\overline{z}^2}{2} + w_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{4.22}$$

Noticing that $p_1 = \Phi_1 + i\Phi_2$, therefore, solutions of (4.16) can be rewritten as

$$U_t^* = \frac{1}{2}\Phi\left(\begin{matrix}(z+\overline{z})\\i(z-\overline{z})\end{matrix}\right) \cdot f_1 + w(z,\overline{z}) = \frac{1}{2}(p_1z + p_2\overline{z}) \cdot f_1 + w(z,\overline{z}). \tag{4.23}$$

In addition, (4.19) can be rewritten as the following form:

$$\dot{z} = i\omega_1\tau_j^1 z + g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{4.24}$$

Let

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{4.25}$$

From (4.20), we have

$$\langle F(U_t,0), f_1\rangle$$

$$= \frac{\tau_j^1}{4}\left(\begin{matrix}\left(\xi f_{11}^{(1)} + \frac{1}{2}f_{20}^{(1)} + \frac{1}{2}\xi^2 f_{02}^{(1)}\right)z^2\\ e^{-2i\omega_1\tau_j^1}\left(\xi f_{110}^{(2)} + e^{i\omega_1\tau_j^1}\xi f_{101}^{(2)} + e^{i\omega_1\tau_j^1}\xi^2 f_{011}^{(2)} + \frac{1}{2}f_{200}^{(2)} + \frac{1}{2}\xi^2 f_{020}^{(2)}\right)z^2\end{matrix}\right)$$

$$+ \frac{\tau_j^1}{4}\left(\begin{matrix}\left[\left(\overline{\xi}+\xi\right)f_{11}^{(1)} + f_{20}^{(1)} + \xi\overline{\xi}f_{02}^{(1)}\right]z\overline{z}\\ \left[\left(\overline{\xi}+\xi\right)f_{110}^{(2)} + e^{-i\omega_1\tau_j^1}\overline{\xi}\left(f_{101}^{(2)} + \xi f_{011}^{(2)}\right) + e^{i\omega_1\tau_j^1}\xi\left(f_{101}^{(2)} + \overline{\xi}f_{011}^{(2)}\right) + f_{200}^{(2)} + \xi\overline{\xi}f_{020}^{(2)}\right]z\overline{z}\end{matrix}\right)$$

$$+ \frac{\tau_j^1}{4}\left(\begin{matrix}\left(\overline{\xi}f_{11}^{(1)} + \frac{1}{2}f_{20}^{(1)} + \frac{1}{2}\overline{\xi}^2 f_{02}^{(1)}\right)\overline{z}^2\\ e^{2i\omega_1\tau_j^1}\left(\overline{\xi}f_{110}^{(2)} + e^{-i\omega_1\tau_j^1}\overline{\xi}f_{101}^{(2)} + e^{-i\omega_1\tau_j^1}\overline{\xi}^2 f_{011}^{(2)} + \frac{1}{2}f_{200}^{(2)} + \frac{1}{2}\overline{\xi}^2 f_{020}^{(2)}\right)\overline{z}^2\end{matrix}\right)$$

$$+ \frac{\tau_j^1}{2} \begin{pmatrix} \left\langle f_{11}^{(1)} \left(w_{11}^2(0) + \frac{w_{20}^2(0)}{2} + w_{11}^1(0)\xi + \frac{w_{20}^1(0)}{2}\bar{\xi} \right) \right. \\ \left. + f_{20}^{(1)} \left(w_{11}^1(0) + \frac{w_{20}^1(0)}{2} \right) + f_{02}^{(1)} \left(w_{11}^1(0)\xi + \frac{w_{20}^1(0)}{2}\bar{\xi} \right), 1 \right\rangle \\ \left\langle f_{110}^{(2)} e^{-i\omega_1\tau_j^1} \left(w_{11}^2(-1) + e^{2i\omega_1\tau_j^1} \frac{w_{20}^2(-1)}{2} + w_{11}^1(-1)\xi + e^{2i\omega_1\tau_j^1} \frac{w_{20}^1(-1)}{2}\bar{\xi} \right) \right. \\ + f_{101}^{(2)} \left(e^{-i\omega_1\tau_j^1} w_{11}^2(0) + e^{i\omega_1\tau_j^1} \frac{w_{20}^2(0)}{2} + w_{11}^1(-1)\xi + \frac{w_{20}^1(-1)}{2}\bar{\xi} \right) \\ + f_{011}^{(2)} \left(e^{-i\omega_1\tau_j^1} w_{11}^2(0)\xi + e^{i\omega_1\tau_j^1} \frac{w_{20}^2(0)}{2}\bar{\xi} + w_{11}^2(-1)\xi + \frac{w_{20}^2(-1)}{2}\bar{\xi} \right) \\ + \frac{1}{2} f_{200}^{(2)} \left(2e^{-i\omega_1\tau_j^1} w_{11}^1(-1) + e^{i\omega_1\tau_j^1} w_{20}^1(-1) \right) \\ \left. + \frac{1}{2} f_{020}^{(2)} \left(2e^{-i\omega_1\tau_j^1} w_{11}^2(-1)\xi + e^{i\omega_1\tau_j^1} w_{20}^2(-1)\xi \right), 1 \right\rangle \end{pmatrix} z^2\bar{z}$$

$$+ \cdots .$$

$$(4.26)$$

Noting that $\Psi_1(0) - i\Psi_2(0) = (2(1 - i\omega_1)/(1 + \omega_1^2)(1 + \xi\zeta))(1, \zeta)$, therefore,

$$g_{20} = \frac{\tau_j^1(1 - i\omega_1)}{(1 + \omega_1^2)(1 + \xi\zeta)} \left[\left(\xi f_{11}^{(1)} + \frac{1}{2} f_{20}^{(1)} + \frac{1}{2} \xi^2 f_{02}^{(1)} \right) \right.$$

$$\left. + e^{-2i\omega_1\tau_j^1} \left(\xi f_{110}^{(2)} + e^{i\omega_1\tau_j^1} \xi f_{101}^{(2)} + e^{i\omega_1\tau_j^1} \xi^2 f_{011}^{(2)} + \frac{1}{2} f_{200}^{(2)} + \frac{1}{2} \xi^2 f_{020}^{(2)} \right) \zeta \right],$$

$$g_{11} = \frac{\tau_j^1(1 - i\omega_1)}{(1 + \omega_1^2)(1 + \xi\zeta)} \left\{ \left[\left(\bar{\xi} + \xi \right) f_{11}^{(1)} + f_{20}^{(1)} + \xi\bar{\xi} f_{02}^{(1)} \right] \right.$$

$$+ \left[\left(\bar{\xi} + \xi \right) f_{110}^{(2)} + e^{-i\omega_1\tau_j^1} \bar{\xi} \left(f_{101}^{(2)} + \xi f_{011}^{(2)} \right) + e^{i\omega_1\tau_j^1} \xi \left(f_{101}^{(2)} + \bar{\xi} f_{011}^{(2)} \right) \right.$$

$$\left. + f_{200}^{(2)} + \xi\bar{\xi} f_{020}^{(2)} \right] \zeta \right\},$$

$$g_{02} = \overline{g_{20}}$$

$$g_{21} = \frac{2\tau_j^1(1 - i\omega_1)}{(1 + \omega_1^2)(1 + \xi\zeta)}$$

$$\times \left[\left\langle f_{11}^{(1)} \left(w_{11}^2(0) + \frac{w_{20}^2(0)}{2} + w_{11}^1(0)\xi + \frac{w_{20}^1(0)}{2}\bar{\xi} \right) \right. \right.$$

$$\left. + f_{20}^{(1)} \left(w_{11}^1(0) + \frac{w_{20}^1(0)}{2} \right) + f_{02}^{(1)} \left(w_{11}^1(0)\xi + \frac{w_{20}^1(0)}{2}\bar{\xi} \right), 1 \right\rangle$$

$$
\begin{aligned}
+ \Bigg\langle & f_{110}^{(2)} e^{-i\omega_1 \tau_j^1} \left(w_{11}^2(-1) + e^{2i\omega_1 \tau_j^1} \frac{w_{20}^2(-1)}{2} + w_{11}^1(-1)\xi + e^{2i\omega_1 \tau_j^1} \frac{w_{20}^1(-1)}{2}\bar{\xi} \right) \\
& + f_{101}^{(2)} \left(e^{-i\omega_1 \tau_j^1} w_{11}^2(0) + e^{i\omega_1 \tau_j^1} \frac{w_{20}^2(0)}{2} + w_{11}^1(-1)\xi + \frac{w_{20}^1(-1)}{2}\bar{\xi} \right) \\
& + f_{011}^{(2)} \left(e^{-i\omega_1 \tau_j^1} w_{11}^2(0)\xi + e^{i\omega_1 \tau_j^1} \frac{w_{20}^2(0)}{2}\bar{\xi} + w_{11}^2(-1)\xi + \frac{w_{20}^2(-1)}{2}\bar{\xi} \right) \\
& + \frac{1}{2} f_{200}^{(2)} \left(2e^{-i\omega_1 \tau_j^1} w_{11}^1(-1) + e^{i\omega_1 \tau_j^1} w_{20}^1(-1) \right) \\
& + \frac{1}{2} f_{020}^{(2)} \left(2e^{-i\omega_1 \tau_j^1} w_{11}^2(-1)\xi + e^{i\omega_1 \tau_j^1} w_{20}^2(-1)\xi \right), 1 \Bigg\rangle \zeta \Bigg].
\end{aligned}
$$

$$(4.27)$$

Since $w_{20}(\theta)$ and $w_{11}(\theta)$ for $(\theta \in [-1,0])$ appear in g_{21}, we still need them.

It follows easily from (4.22) that

$$
\dot{w}(z,\bar{z}) = w_{20} z\dot{z} + w_{11}\left(\dot{z}\bar{z} + z\dot{\bar{z}}\right) + w_{02}\bar{z}\dot{\bar{z}} + \cdots, \tag{4.28}
$$

$$
A_{\tau_j^1} w = A_{\tau_j^1} w_{20}\frac{z^2}{2} + A_{\tau_j^1} w_{11} z\bar{z} + A_{\tau_j^1} w_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{4.29}
$$

According to [22] we can know,

$$
\dot{w} = A_{\tau_j^1} w + H(z,\bar{z}), \tag{4.30}
$$

where

$$
\begin{aligned}
H(z,\bar{z}) &= H_{20}\frac{z^2}{2} + H_{11} z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots \\
&= X_0 F(U_t^*, 0) - \Phi\big(\Psi, \langle X_0 F(U_t^*, 0), f_1 \rangle\big) \cdot f_1
\end{aligned} \tag{4.31}
$$

and $H_{ij} \in \mathcal{P}_Q \mathcal{C}$, $i + j = 2$.

Thus, by using the chain rule

$$
\dot{w} = \frac{\partial w(z,\bar{z})}{\partial z}\dot{z} + \frac{\partial w(z,\bar{z})}{\partial \bar{z}}\dot{\bar{z}}. \tag{4.32}
$$

From (4.23) and (4.30), we can obtain

$$\left(2i\omega_1\tau_j^1 - A_{\tau_j^1}\right)w_{20} = H_{20},$$

$$-A_{\tau_j^1}w_{11} = H_{11}, \tag{4.33}$$

$$\left(-2i\omega_1\tau_j^1 - A_{\tau_j^1}\right)w_{02} = H_{02}.$$

Noticing that $A_{\tau_j^1}$ has only two eigenvalues $\pm i\omega_1$, therefore, (4.33) has the unique solution $w_{ij}(i + j = 2)$ in $\dot{p}_Q C$ and

$$w_{20} = \left(2i\omega_1\tau_j^1 - A_{\tau_j^1}\right)^{-1}H_{20},$$

$$w_{11} = -A_{\tau_j^1}^{-1}H_{11}, \tag{4.34}$$

$$w_{02} = \left(-2i\omega_1\tau_j^1 - A_{\tau_j^1}\right)^{-1}H_{02}.$$

Note that for $-1 \le \theta < 0$,

$$
\begin{aligned}
H(z,\overline{z}) &= -\Phi(\theta)\Psi(0)\langle F(U_t,0), f_1\rangle \cdot f_1 \\
&= -\left(\frac{p_1(\theta) + p_2(\theta)}{2}, \frac{p_1(\theta) - p_2(\theta)}{2i}\right)\begin{pmatrix}\Psi_1(0)\\\Psi_2(0)\end{pmatrix}\langle F(U_t,0), f_1\rangle \cdot f_1 \\
&= -\frac{1}{2}\left[p_1(\theta)(\Psi_1(0) - i\Psi_2(0)) + p_2(\theta)(\Psi_1(0) + i\Psi_2(0))\right]\langle F(U_t,0), f_1\rangle \cdot f_1 \\
&= -\frac{1}{4}\left[g_{20}p_1(\theta) + \overline{g}_{02}p_2(\theta)\right]z^2 \cdot f_1 - \frac{1}{2}\left[g_{11}p_1(\theta) + \overline{g}_{11}p_2(\theta)\right]z\overline{z} \cdot f_1.
\end{aligned}
\tag{4.35}
$$

So, for $-1 \le \theta < 0$,

$$H_{20}(\theta) = -\frac{1}{2}\left[g_{20}p_1(\theta) + \overline{g}_{02}p_2(\theta)\right] \cdot f_1,$$

$$H_{11}(\theta) = -\frac{1}{2}\left[g_{11}p_1(\theta) + \overline{g}_{11}p_2(\theta)\right]z\overline{z} \cdot f_1,$$

$$H_{20}(0) = \frac{\tau_j^1}{2} \left(\begin{array}{c} \xi f_{11}^{(1)} + \dfrac{1}{2} f_{20}^{(1)} + \dfrac{1}{2}\xi^2 f_{02}^{(1)} \\ e^{-2i\omega_1\tau_j^1}\left(\xi f_{110}^{(2)} + e^{i\omega_1\tau_j^1}\xi f_{101}^{(2)} + e^{i\omega_1\tau_j^1}\xi^2 f_{011}^{(2)} + \dfrac{1}{2}f_{200}^{(2)} + \dfrac{1}{2}\xi^2 f_{020}^{(2)}\right) \end{array} \right)$$

$$- \frac{1}{2}\left[g_{20}p_1(0) + \overline{g}_{02}p_2(0)\right]\cdot f_1,$$

$$H_{11}(0) = \frac{\tau_j^1}{2} \left(\begin{array}{c} \left(\overline{\xi}+\xi\right)f_{11}^{(1)} + f_{20}^{(1)} + \xi\overline{\xi}f_{02}^{(1)} \\ \left(\overline{\xi}+\xi\right)f_{110}^{(2)} + e^{-i\omega_1\tau_j^1}\overline{\xi}\left(f_{101}^{(2)} + \xi f_{011}^{(2)}\right) + e^{i\omega_1\tau_j^1}\xi\left(f_{101}^{(2)} + \overline{\xi}f_{011}^{(2)}\right) + f_{200}^{(2)} + \xi\overline{\xi}f_{020}^{(2)} \end{array} \right)$$

$$- \frac{1}{2}\left[g_{11}p_1(0) + \overline{g}_{11}p_2(0)\right]\cdot f_1.$$

$$(4.36)$$

By the definition of $A_{\tau_j^1}$, for $-1 \le \theta < 0$, we have

$$\dot{w}_{20}(\theta) = 2i\omega_1 w_{20}(\theta) + \frac{1}{2}\left[g_{20}p_1(\theta) + \overline{g}_{02}p_2(\theta)\right]\cdot f_1, \quad -1 \le \theta < 0. \tag{4.37}$$

Note that $p_1(\theta) = p_1(0)e^{i\omega_1\theta}$, $-1 \le \theta \le 0$; hence

$$w_{20}(\theta) = \left[\frac{ig_{20}}{\omega_1\tau_j^1}p_1(\theta) + \frac{i\overline{g}_{02}}{3\omega_1\tau_j^1}p_2(\theta)\right]\cdot f_1 + e^{2i\omega_1\theta}E_1, \quad -1 \le \theta < 0,$$

$$w_{11}(\theta) = \left[g_{11}p_1(\theta) + \overline{g}_{11}p_2(\theta)\right]\cdot f_1 + E_2, \quad -1 \le \theta < 0,$$

$$E_1 = w_{20}(0) - \left[\frac{ig_{20}}{\omega_1\tau_j^1}p_1(0) + \frac{i\overline{g}_{02}}{3\omega_1\tau_j^1}p_2(0)\right]\cdot f_1, \tag{4.38}$$

$$E_2 = w_{11}(0) - \left[\frac{ig_{20}}{\omega_1\tau_j^1}p_1(0) + \frac{i\overline{g}_{02}}{\omega_1\tau_j^1}p_2(0)\right]\cdot f_1.$$

Using the definition of $A_{\tau_j^1}$ again and combining (4.29) and (4.33), we get

$$2i\omega_1\tau_j^1\left[\frac{ig_{20}}{\omega_1\tau_j^1}p_1(0)\cdot f_1 + \frac{i\overline{g}_{02}}{3\omega_1\tau_j^1}p_2(0)\cdot f_1 + E_1\right]$$

$$- \tau_j^1 A_{\tau_j^1}\left[\frac{ig_{20}}{\omega_1}p_1(0)\cdot f_1 + \frac{i\overline{g}_{02}}{3\omega_1}p_2(0)\cdot f_1 + E_1\right]$$

$$-L\left(\tau_j^1\right)\left[\frac{ig_{20}}{2\omega_1}p_1(\theta)\cdot f_1 + \frac{i\overline{g}_{02}}{6\omega_1}p_2(\theta)\cdot f_1 + E_1 e^{2i\omega_1\theta}\right]$$

$$= \frac{\tau_j^1}{2}\begin{pmatrix} \xi f_{11}^{(1)} + \frac{1}{2}f_{20}^{(1)} + \frac{1}{2}\xi^2 f_{02}^{(1)} \\ e^{-2i\omega_1\tau_j^1}\left(\xi f_{110}^{(2)} + e^{i\omega_1\tau_j^1}\xi f_{101}^{(2)} + e^{i\omega_1\tau_j^1}\xi^2 f_{011}^{(2)} + \frac{1}{2}f_{200}^{(2)} + \frac{1}{2}\xi^2 f_{020}^{(2)}\right) \end{pmatrix}$$

$$-\frac{1}{2}\left[g_{20}p_1(0) + \overline{g}_{02}p_2(0)\right]\cdot f_1.$$

$$(4.39)$$

As

$$\tau_j^1 A_{\tau_j^1}p_1(0)\cdot f_1 + L\left(\tau_j^1\right)\left(p_1(\theta)\cdot f_1\right) = i\omega_1\tau_j^1 p_1(0)\cdot f_1,$$

$$\tau_j^1 A_{\tau_j^1}p_2(0)\cdot f_1 + L\left(\tau_j^1\right)\left(p_2(\theta)\cdot f_1\right) = -i\omega_1\tau_j^1 p_2(0)\cdot f_1,$$

$$(4.40)$$

then

$$2i\omega_1\tau_j^1 E_1 - \tau_j^1 A_{\tau_j^1} - L\left(\tau_j^1\right)\left(E_1 e^{2i\omega_1\tau_j^1\theta}\right)$$

$$= \frac{\tau_j^1}{2}\begin{pmatrix} \xi f_{11}^{(1)} + \frac{1}{2}f_{20}^{(1)} + \frac{1}{2}\xi^2 f_{02}^{(1)} \\ e^{-2i\omega_1\tau_j^1}\left(\xi f_{110}^{(2)} + e^{i\omega_1\tau_j^1}\xi f_{101}^{(2)} + e^{i\omega_1\tau_j^1}\xi^2 f_{011}^{(2)} + \frac{1}{2}f_{200}^{(2)} + \frac{1}{2}\xi^2 f_{020}^{(2)}\right) \end{pmatrix}.$$

$$(4.41)$$

From the above expression, we can see easily that

$$E_1 = \frac{1}{2}\begin{pmatrix} -2i\omega_1 + \alpha_{11} + d_1 & \alpha_{12} \\ \alpha_{21} & -2i\omega_1 + \alpha_{22} + d_2 \end{pmatrix}^{-1}$$

$$\times\begin{pmatrix} \xi f_{11}^{(1)} + \frac{1}{2}f_{20}^{(1)} + \frac{1}{2}\xi^2 f_{02}^{(1)} \\ e^{-2i\omega_1\tau_j^1}\left(\xi f_{110}^{(2)} + e^{i\omega_1\tau_j^1}\xi f_{101}^{(2)} + e^{i\omega_1\tau_j^1}\xi^2 f_{011}^{(2)} + \frac{1}{2}f_{200}^{(2)} + \frac{1}{2}\xi^2 f_{020}^{(2)}\right) \end{pmatrix}.$$

$$(4.42)$$

Similarly

$$E_2 = \frac{1}{2}\begin{pmatrix} \alpha_{11} + d_1 & \alpha_{12} \\ \alpha_{21} & \alpha_{22} + d_2 \end{pmatrix}^{-1}$$

$$\times\begin{pmatrix} \left(\overline{\xi} + \xi\right)f_{11}^{(1)} + f_{20}^{(1)} + \xi\overline{\xi}f_{02}^{(1)} \\ \left(\overline{\xi} + \xi\right)f_{110}^{(2)} + e^{-i\omega_1\tau_j^1}\overline{\xi}\left(f_{101}^{(2)} + \xi f_{011}^{(2)}\right) + e^{i\omega_1\tau_j^1}\xi\left(f_{101}^{(2)} + \overline{\xi}f_{011}^{(2)}\right) + f_{200}^{(2)} + \xi\overline{\xi}f_{020}^{(2)} \end{pmatrix}.$$

$$(4.43)$$

Thus, g_{21} can be determined by the parameters and delay; we get

$$c_1(0) = \frac{i}{2\omega_1}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}^2|}{3}\right) + \frac{g_{21}}{2}. \tag{4.44}$$

Then, we can compute the following values:

$$\sigma_2 = \frac{\mathrm{Re}(c_1(0))}{\mathrm{Re}\left(\lambda'\left(\tau_j^1\right)\right)},$$

$$\epsilon_2 = 2\,\mathrm{Re}(c_1(0)), \tag{4.45}$$

$$T_2 = \frac{\mathrm{Im}(c_1(0)) + \sigma_2\,\mathrm{Im}\left(\lambda'\left(\tau_j^1\right)\right)}{w_1}.$$

Therefore, we have the following result.

Theorem 4.1. *(i) σ_2 determines the directions of the spatially nonhomogeneous Hopf bifurcation. If $\sigma_2 < 0\ (>0)$, then the spatially nonhomogeneous Hopf bifurcation is subcritical (supercritical).*

(ii) ϵ_2 determines the stability of bifurcated periodic solutions. If $\epsilon_2 < 0\ (>0)$, then the bifurcated periodic solutions are stable (unstable).

(iii) T_2 determines the period of the bifurcating periodic solutions; if $T_2 < 0\ (>0)$, the period decreases (increases).

5. Conclusions

In this paper, we considered a delayed predator-prey system with diffusion effects. By investigating the linearized system of the original system, the distribution of the roots of the characteristic equations at the positive constant steady-state solution was obtained and its stability was discussed. The obtained results indicate that the positive constant steady-state solution of the system is asymptotically stable when $\tau \in [0, \tau_j^0)$. As the delay τ crosses through each τ_j^0, there exist a sequence of critical values τ_j^0 $(j = 0, 1, 2, \ldots)$ of τ such that the system undergoes a Hopf bifurcation at the positive constant steady-state solution. Besides, we show that large diffusivity has no effect on the Hopf bifurcation, while small diffusivity can lead to the fact that the system can bifurcate a spatially nonhomogeneous periodic solutions at the positive constant steady-state solution. Furthermore, we study the properties of the spatially nonhomogeneous periodic solutions. The conclusions demonstrate that system (1.3) may have more complex and richer dynamics than system (1.1).

Acknowledgment

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