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Research Article

Multivariate Extension Principle and Algebraic Operations of Intuitionistic Fuzzy Sets

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This paper mainly focuses on multivariate extension of the extension principle of IFSs. Based on the Cartesian product over IFSs, the multivariate extension principle of IFSs is established. Furthermore, three kinds of representation of this principle are provided. Finally, a general framework of the algebraic operation between IFSs is given by using the multivariate extension principle.

1. Introduction

The concept of intuitionistic fuzzy sets was first proposed by Atanassov and Stoeva in 1983 [1]. However, this concept had not been widely concerned by many scholars, because it was only presented in the symposium proceedings with regard to interval and fuzzy mathematics in Poland. Until 1986, the notion of intuitionistic fuzzy sets was formally introduced by Atanassov [2]. Immediately, some new operators on intuitionistic fuzzy sets are defined and the corresponding properties are studied in 1989 [3].

The intuitionistic fuzzy sets can be viewed as an extension for fuzzy sets, which is more objective and comprehensive to describe the uncertainty of the problem. In 1986, Atanassov [2] established several different ways to change an intuitionistic fuzzy set into a fuzzy set and defined an operator called *Atanassov's operator*. Furthermore, the study of the properties on this operator is carried out in [2, 4]. Later, Burillo and Bustince [5] presented the *Atanassov's point operator* and studied the construction of intuitionistic fuzzy sets by using this type of operator. Meantime, they pointed out that it is possible to recover a fuzzy set from an intuitionistic fuzzy set constructed by means of different operators. Similarly, the intuitionistic fuzzy relations can also be seen as an extension for fuzzy relations, which were proposed by Bustince and Burillo in [6]. Afterwards, Bustince studied the construction of intuitionistic fuzzy relations with predetermined properties, which can allow us to build reflexive, symmetric, antisymmetric, perfect antisymmetric, and transitive intuitionistic fuzzy relations from fuzzy relations with the same properties on the basis of the Atanassov' operator in [7]. As mentioned above, one can see that it is an important and interesting research direction to extend some conclusions of fuzzy sets to intuitionistic fuzzy environment. For this issue, in recent years, further studies have been completed by different authors [6, 8–16].

In the theory of fuzzy sets, representation theorem, decomposition theorem, and extension principle are regarded as three important basic theorems, which provide an important theoretical basic for dealing with the fuzzy problems by the methods of classical mathematics. Especially, the multivariate extension principle, which can be viewed as a generalization of the extension principle of fuzzy sets, will provide a theoretical basic for the operations between fuzzy sets. In addition, it should be pointed out that the Cartesian product over fuzzy sets is an important tool to establish this principle. In 2007, Atanassova [17] constructed the extension principle of intuitionistic fuzzy sets. Meantime, several types of Cartesian products over intuitionistic fuzzy sets were also introduced in [18]. In 2008, Andonov [19] also introduced a Cartesian product over intuitionistic fuzzy sets and explored some of properties. Based on these existing results, in this paper, the multivariate extension principle of intuitionistic fuzzy sets will be considered. Meanwhile, some important operations between intuitionistic fuzzy sets can be obtained by using this principle.

The rest of the paper is organized into five parts. In Section 2, some related concepts and important conclusions on intuitionistic fuzzy sets are recalled. In Section 3, the Cartesian product over intuitionistic fuzzy sets is reviewed and some related properties are discussed. These results will establish a basis for the analysis and proof of the multivariate extension principle of intuitionistic fuzzy sets. Section 4 establishes a multivariate extension principle of intuitionistic fuzzy sets and provides three types of forms of this principle. In Section 5, a general framework of the algebraic operation between intuitionistic fuzzy sets is proposed by using the multivariate extension principle. Finally, a summarized conclusion is given in Section 6.

2. Preliminaries

For completeness and clarity, some basic notions and necessary conclusions on intuitionistic fuzzy sets are reviewed in this section.

2.1. Intuitionistic Fuzzy Sets

Let X be the universe of discourse, an intuitionistic fuzzy set on X is an expression E given by

$$E = \{\langle x, \mu_E(x), \nu_E(x)\rangle : x \in X\} \tag{2.1}$$

with

$$\mu_E : X \longrightarrow [0,1], \qquad \nu_E : X \longrightarrow [0,1] \tag{2.2}$$

such that $0 \le \mu_E(x) + \nu_E(x) \le 1$ for all $x \in X$.

Generally, $\mu_E(x)$ and $\nu_E(x)$ are called the degree of membership and the degree of nonmembership of the element x in the set E, respectively. The *complementary* of E is denoted by $E_c = \{\langle x, \nu_E(x), \mu_E(x)\rangle : x \in X\}$. The symbol $IFSs(X)$ denotes the set of all intuitionistic fuzzy sets on X. Especially, if $\nu_E(x) = 1 - \mu_E(x)$, the set E reduces to a fuzzy set. Meantime, FSs(X) denotes the set of all fuzzy sets on X. In addition, intuitionistic fuzzy sets are abbreviated as *IFSs*.

2.2. Cut Sets of IFSs and Its Properties

In this part, some concepts and conclusions associated with the cut sets of *IFSs* are summarized below.

Definition 2.1 (see [18]). Let $E \in IFSs(X)$, the parameters α, β satisfy the condition $\alpha + \beta \leq 1$ for all $\alpha, \beta \in [0,1]$, the following four sets:

$$\begin{aligned}
E_{\langle \alpha,\beta\rangle} &= \{x \in X : \mu_E(x) \geq \alpha, \nu_E(x) \leq \beta\},\\
E_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle} &= \{x \in X : \mu_E(x) > \alpha, \nu_E(x) < \beta\},\\
E_{\langle \underset{\cdot}{\alpha},\beta\rangle} &= \{x \in X : \mu_E(x) > \alpha, \nu_E(x) \leq \beta\},\\
E_{\langle \alpha,\underset{\cdot}{\beta}\rangle} &= \{x \in X : \mu_E(x) \geq \alpha, \nu_E(x) < \beta\},
\end{aligned} \tag{2.3}$$

are called the $\langle \alpha,\beta\rangle$-cut set, strong $\langle \alpha,\beta\rangle$-cut set, $\langle \underset{\cdot}{\alpha},\beta\rangle$-cut set, and $\langle \alpha,\underset{\cdot}{\beta}\rangle$-cut set, respectively.

For convenience, the symbol I^2 is given to denote the set $I^2 = \{\langle \alpha,\beta\rangle : \alpha + \beta \leq 1, \alpha, \beta \in [0,1]\}$. For all $\langle \alpha_1,\beta_1\rangle, \langle \alpha_2,\beta_2\rangle \in I^2$, the relations between them are defined as

(a) $\langle \alpha_1,\beta_1\rangle = \langle \alpha_2,\beta_2\rangle \Leftrightarrow \alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$;

(b) $\langle \alpha_1,\beta_1\rangle \leq \langle \alpha_2,\beta_2\rangle \Leftrightarrow \alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$;

(c) $\langle \alpha_1,\beta_1\rangle < \langle \alpha_2,\beta_2\rangle \Leftrightarrow \langle \alpha_1,\beta_1\rangle \leq \langle \alpha_2,\beta_2\rangle$ and $\langle \alpha_1,\beta_1\rangle \neq \langle \alpha_2,\beta_2\rangle$.

Additionally, for the need of the following narrative, a particular operation between the set I^2 and the *IFSs*(X) is quoted.

Definition 2.2 (see [18]). Let $E \in IFSs(X)$, $\langle \alpha,\beta\rangle \in I^2$, the set $\langle \alpha,\beta\rangle E \in IFSs(X)$ is defined as follows

$$\langle \alpha,\beta\rangle E \overset{\text{def}}{=} \{\langle x, \alpha \wedge \mu_E(x), \beta \vee \nu_E(x)\rangle : x \in X\}. \tag{2.4}$$

Next, some properties of the cut set of *IFSs* are summarized.

Lemma 2.3 (see [18]). *Let* $E \in IFSs(X)$, $\langle \alpha,\beta\rangle \in I^2$, *then*

(i) $E_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle} \subseteq E_{\langle \underset{\cdot}{\alpha},\beta\rangle} \subseteq E_{\langle \alpha,\beta\rangle}$;

(ii) $E_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle} \subseteq E_{\langle \alpha,\underset{\cdot}{\beta}\rangle} \subseteq E_{\langle \alpha,\beta\rangle}$;

(iii) $E_{\langle 0,1\rangle} = X$, $E_{\langle \underset{\cdot}{1},0\rangle} = E_{\langle 1,\underset{\cdot}{0}\rangle} = E_{\langle \underset{\cdot}{1},\underset{\cdot}{0}\rangle} = \emptyset$.

Lemma 2.4 (see [18]). *Let $E \in IFSs(X)$, $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in I^2$, and $\langle \alpha_1, \beta_1 \rangle \leq \langle \alpha_2, \beta_2 \rangle$, then*

(i) $E_{\langle \alpha_2, \beta_2 \rangle} \subseteq E_{\langle \alpha_1, \beta_1 \rangle}$;

(ii) $E_{\langle \underset{\cdot}{\alpha_2}, \underset{\cdot}{\beta_2} \rangle} \subset E_{\langle \underset{\cdot}{\alpha_1}, \underset{\cdot}{\beta_1} \rangle}$;

(iii) $E_{\langle \underset{\cdot}{\alpha_2}, \beta_2 \rangle} \subseteq E_{\langle \underset{\cdot}{\alpha_1}, \beta_1 \rangle}$;

(iv) $E_{\langle \alpha_2, \underset{\cdot}{\beta_2} \rangle} \subseteq E_{\langle \alpha_1, \underset{\cdot}{\beta_1} \rangle}$.

Based on the cut sets of *IFSs* and their properties, the basic theorems of *IFSs* are developed by Atanassov [18] and Liu [14]. In the following, some related theorems are recalled for the needs of the proofs in Sections 3 and 4.

Lemma 2.5 (decomposition theorem [18]). *Let $E \in IFSs(X)$, then*

$$E = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle E_{\langle \alpha, \beta \rangle}. \tag{2.5}$$

Notice that the set E can also be decomposed by the rest of cut sets of E. This case is similar with Lemma 2.5.

The concept of the binary nested set is introduced to give the representation theorem of *IFSs*. The symbol $P(X)$ denotes the power set of X.

Definition 2.6. Let $f : I^2 \to P(X)$ be a mapping, that is, $\langle \alpha, \beta \rangle \mapsto H(\alpha, \beta)$, for all $\langle \alpha_1, \beta_1 \rangle, \langle \alpha_2, \beta_2 \rangle \in I^2$, if $\langle \alpha_1, \beta_1 \rangle < \langle \alpha_2, \beta_2 \rangle$, there always has $H(\alpha_2, \beta_2) \subset H(\alpha_1, \beta_1)$, then the set H is called the binary nested set on X.

Lemma 2.7 (representation theorem [14]). *Let f be a mapping from $BN(X)$ to $IFSs(X)$, that is, $f : BN(X) \to IFSs(X)$, and*

$$H \longmapsto f(H) \triangleq \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle H(\alpha, \beta), \tag{2.6}$$

then

(i) *f is a surjection from $(BN(X), \cup, \cap, {}^c)$ to $(IFSs(X), \cup, \cap, {}^c)$;*

(ii) $(f(H))_{\langle \underset{\cdot}{\alpha}, \underset{\cdot}{\beta} \rangle} \subseteq H(\alpha, \beta) \subseteq (f(H))_{\langle \alpha, \beta \rangle}$,

where the notation $BN(X)$ denotes the set of all the binary nested sets on X.

Lemma 2.7 shows that there exists a unique $f(H)$ such that $f(H) = \bigcup_{\langle \alpha, \beta \rangle \in I^2} \langle \alpha, \beta \rangle H(\alpha, \beta)$ for all $H \in BN(X)$.

Lemma 2.8 (extension principle [14]). *Suppose $f : X \to Y$ is a mapping from the ordinary set X to the ordinary set Y, that is, $x \mapsto f(x)$, then the mapping f can induce into two mappings $f : IFSs(X) \to IFSs(Y)$ and $f^{-1} : IFSs(Y) \to IFSs(X)$*

$$f{:}IFSs(X) \longrightarrow IFSs(Y), \qquad A \longmapsto f(A) = \bigcup_{\langle\alpha,\beta\rangle\in I^2} \langle\alpha,\beta\rangle f\left(A_{\langle\alpha,\beta\rangle}\right),$$
$$f^{-1}{:}IFSs(Y) \longrightarrow IFSs(X), \qquad B \longmapsto f^{-1}(B) = \bigcup_{\langle\alpha,\beta\rangle\in I^2} \langle\alpha,\beta\rangle f^{-1}\left(B_{\langle\alpha,\beta\rangle}\right). \tag{2.7}$$

The membership and nonmembership functions of f, f^{-1} are defined, respectively, as follows

$$\mu_{f(A)}(y) = \bigvee_{f(x)=y} \mu_A(x), \qquad \nu_{f(A)}(y) = \bigwedge_{f(x)=y} \nu_A(x),$$
$$\mu_{f^{-1}(B)}(x) = \mu_B\left(f(x)\right), \qquad \nu_{f^{-1}(B)}(x) = \nu_B\left(f(x)\right), \tag{2.8}$$

where $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFSs(X)$, $f(A) = \{\langle y = f(x), \mu_{f(A)}(y), \nu_{f(A)}(y)\rangle : x \in X, f(x) \subseteq Y\} \in IFSs(Y)$. B and $f^{-1}(B)$ can be given in a similar manner.

3. Cartesian Product over *IFSs*

The concept of Cartesian product over *IFSs* is introduced by Atanassov [18]. Here we will review this concept in detail and extend it to n arguments. It should be noted that the main purpose of this section is to make a preparation for developing the multivariate extension principle of *IFSs*.

As we all know, the ordinary Cartesian product is defined as

$$A_1 \times A_2 \times \cdots \times A_n \overset{\text{def}}{=} \{(x_1, x_2, \cdots, x_n) : x_i \in A_i,\ i = 1, 2, \ldots, n\}, \tag{3.1}$$

the characteristic function is

$$\chi_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n) = \bigwedge_{k=1}^{n} A_k(x_k). \tag{3.2}$$

The Cartesian product over FSs is obtained by extending the ordinary sets to fuzzy sets.

Let $A^{(k)} \in \text{FSs}(X_k)$ $(k = 1, 2, \ldots, n)$. Based on the decomposition and representation theorems of FSs, the Cartesian product can be obtained as follows:

$$A^{(1)} \times A^{(2)} \times \cdots \times A^{(n)} = \bigcup_{\lambda\in[0,1]} \lambda\left(A_\lambda^{(1)} \times A_\lambda^{(2)} \times \cdots \times A_\lambda^{(n)}\right). \tag{3.3}$$

In fact, it has been proved that the membership function of Cartesian product over FSs is

$$\mu_{A^{(1)}\times A^{(2)}\times\cdots\times A^{(n)}}(x_1,x_2,\ldots,x_n)=\bigwedge_{k=1}^{n}\mu_{A^{(k)}}(x_k). \tag{3.4}$$

Similarly, the Cartesian product over *IFSs* can also be obtained.

Definition 3.1. Let $E^{(k)}\in IFSs(X_k)$ $(k=1,2,\ldots,n)$, $\langle\alpha,\beta\rangle\in I^2$, the following expression

$$E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}\overset{\text{def}}{=}\bigcup_{\langle\alpha,\beta\rangle\in I^2}\langle\alpha,\beta\rangle\left(E^{(1)}_{\langle\alpha,\beta\rangle}\times E^{(2)}_{\langle\alpha,\beta\rangle}\times\cdots\times E^{(n)}_{\langle\alpha,\beta\rangle}\right) \tag{3.5}$$

be called the Cartesian product over *IFSs*.

According to Lemma 2.4, we have

$$\begin{aligned}\langle\alpha_1,\beta_1\rangle<\langle\alpha_2,\beta_2\rangle&\Longrightarrow E^{(k)}_{\langle\alpha_1,\beta_1\rangle}\supseteq E^{(k)}_{\langle\alpha_2,\beta_2\rangle}\quad(k=1,2,\ldots,n)\\&\Longrightarrow E^{(1)}_{\langle\alpha_1,\beta_1\rangle}\times E^{(2)}_{\langle\alpha_1,\beta_1\rangle}\times\cdots\times E^{(n)}_{\langle\alpha_1,\beta_1\rangle}\supseteq E^{(1)}_{\langle\alpha_2,\beta_2\rangle}\times E^{(2)}_{\langle\alpha_2,\beta_2\rangle}\times\cdots\times E^{(n)}_{\langle\alpha_2,\beta_2\rangle}.\end{aligned} \tag{3.6}$$

Hence, it is easy to see that the set

$$\left\{E^{(1)}_{\langle\alpha,\beta\rangle}\times E^{(2)}_{\langle\alpha,\beta\rangle}\times\cdots\times E^{(n)}_{\langle\alpha,\beta\rangle}:\langle\alpha,\beta\rangle\in I^2\right\} \tag{3.7}$$

is a binary nested set on $X=X_1\times X_2\times\cdots\times X_n$.

According to Lemma 2.7, we know that the above binary nested set can uniquely identify an intuitionistic fuzzy set contained in $IFSs(X_1\times X_2\times\cdots\times X_n)$. Therefore, we have

$$\bigcup_{\langle\alpha,\beta\rangle\in I^2}\langle\alpha,\beta\rangle\left(E^{(1)}_{\langle\alpha,\beta\rangle}\times E^{(2)}_{\langle\alpha,\beta\rangle}\times\cdots\times E^{(n)}_{\langle\alpha,\beta\rangle}\right)\in IFSs(X_1\times X_2\times\cdots\times X_n), \tag{3.8}$$

and it satisfies

$$\begin{aligned}\left(E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}\right)_{\langle\alpha,\beta\rangle}&\subseteq E^{(1)}_{\langle\alpha,\beta\rangle}\times E^{(2)}_{\langle\alpha,\beta\rangle}\times\ldots\times E^{(n)}_{\langle\alpha,\beta\rangle}\\&\subseteq\left(E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}\right)_{\langle\alpha,\beta\rangle}.\end{aligned} \tag{3.9}$$

For the membership function and nonmembership function of the Cartesian product over *IFSs*, we have the following theorem.

Theorem 3.2. *Let* $E^{(k)} \in IFSs(X_k)$ $(k = 1, 2, \ldots, n)$, *then*

$$\mu_{E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}}(x_1, x_2, \ldots, x_n) = \bigwedge_{k=1}^{n} \mu_{E^{(k)}}(x_k), \tag{3.10}$$

$$\nu_{E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}}(x_1, x_2, \ldots, x_n) = \bigvee_{k=1}^{n} \nu_{E^{(k)}}(x_k). \tag{3.11}$$

Proof. First of all, we prove the membership function of Cartesian product over *IFSs*. For all $(x_1, x_2, \ldots, x_n) \in X = X_1 \times X_2 \times \cdots \times X_n$, according to Definitions 2.2 and 3.1, we can obtain

$$\begin{aligned}
&\mu_{E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}}(x_1, x_2, \ldots, x_n) \\
&\quad = \bigvee_{\langle\alpha,\beta\rangle\in I^2} \left(\alpha \wedge \left(\mu_{E^{(1)}_{\langle\alpha,\beta\rangle}\times E^{(2)}_{\langle\alpha,\beta\rangle}\times\cdots\times E^{(n)}_{\langle\alpha,\beta\rangle}}\right)(x_1, x_2, \ldots, x_n)\right) \\
&\quad = \bigvee_{\langle\alpha,\beta\rangle\in I^2} \left(\alpha \wedge \left(\bigwedge_{k=1}^{n}\left(\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right).
\end{aligned} \tag{3.12}$$

Now, we will prove that

$$\bigvee_{\langle\alpha,\beta\rangle\in I^2} \left(\alpha \wedge \left(\bigwedge_{k=1}^{n}\left(\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right) = \bigwedge_{k=1}^{n}\left(\bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha \wedge \mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right). \tag{3.13}$$

Since

$$\begin{aligned}
&\bigvee_{\langle\alpha,\beta\rangle\in I^2} \left(\alpha \wedge \left(\bigwedge_{k=1}^{n}\left(\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right) \le \bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha \wedge \mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right), \\
&\Longrightarrow \bigvee_{\langle\alpha,\beta\rangle\in I^2} \left(\alpha \wedge \left(\bigwedge_{k=1}^{n}\left(\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right) \le \bigwedge_{k=1}^{n}\left(\bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha \wedge \mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right).
\end{aligned} \tag{3.14}$$

Assume that $\bigvee_{\langle\alpha,\beta\rangle\in I^2}(\alpha \wedge (\bigwedge_{k=1}^{n}(\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)))) < \bigwedge_{k=1}^{n}(\bigvee_{\langle\alpha,\beta\rangle\in I^2}(\alpha \wedge \mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)))$, then there exists a constant λ such that

$$\bigvee_{\langle\alpha,\beta\rangle\in I^2} \left(\alpha \wedge \left(\bigwedge_{k=1}^{n}\left(\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right) < \lambda < \bigwedge_{k=1}^{n}\left(\bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha \wedge \mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right). \tag{3.15}$$

Since $\bigwedge_{k=1}^{n}\mu_{E^{(k)}_{\langle\lambda,\beta\rangle}}(x_k)=\{0,1\}$, if $\bigwedge_{k=1}^{n}\mu_{E^{(k)}_{\langle\lambda,\beta\rangle}}(x_k)=1$, then

$$\lambda > \bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha\wedge\left(\bigwedge_{k=1}^{n}\left(\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right)\geq\lambda\wedge\left(\bigwedge_{k=1}^{n}\left(\mu_{E^{(k)}_{\langle\lambda,\beta\rangle}}(x_k)\right)\right)=\lambda. \tag{3.16}$$

In addition, if $\bigwedge_{k=1}^{n}\mu_{E^{(k)}_{\langle\lambda,\beta\rangle}}(x_k)=0$, then there exists a constant k_0 $(1\leq k_0\leq n)$ such that $\mu_{E^{(k_0)}_{\langle\lambda,\beta\rangle}}(x_{k_0})=0$. By Definition 2.1, we know that $\mu_{E^{(k_0)}}(x_{k_0})<\lambda$ or $\nu_{E^{(k_0)}}(x_{k_0})>\beta$, and then for all $\alpha>\lambda$, we have $\mu_{E^{(k_0)}_{\langle\alpha,\beta\rangle}}(x_{k_0})=0$.

Hence,

$$\lambda<\bigwedge_{k=1}^{n}\left(\bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha\wedge\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\leq\bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha\wedge\mu_{E^{(k_0)}_{\langle\alpha,\beta\rangle}}(x_{k_0})\right)\leq\lambda. \tag{3.17}$$

Obviously, the expressions (3.16) and (3.17) are contradictory. Consequently, the previous hypothesis does not hold and the expression (3.13) holds.

By the expression (3.13), we can obtain

$$\begin{aligned}
&\mu_{E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}}(x_1,x_2,\ldots,x_n)\\
&\quad=\bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha\wedge\left(\bigwedge_{k=1}^{n}\left(\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right)\\
&\quad=\bigwedge_{k=1}^{n}\left(\bigvee_{\langle\alpha,\beta\rangle\in I^2}\left(\alpha\wedge\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\\
&\quad=\bigwedge_{k=1}^{n}\mu_{E^{(k)}}(x_k).
\end{aligned} \tag{3.18}$$

Now we start to prove the expression (3.11). Analogously, we have

$$\begin{aligned}
&\nu_{E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}}(x_1,x_2,\ldots,x_n)\\
&\quad=\bigwedge_{\langle\alpha,\beta\rangle\in I^2}\left(\beta\vee\left(\nu_{E^{(1)}_{\langle\alpha,\beta\rangle}\times E^{(2)}_{\langle\alpha,\beta\rangle}\times\cdots\times E^{(n)}_{\langle\alpha,\beta\rangle}}\right)(x_1,x_2,\ldots,x_n)\right)\\
&\quad=\bigwedge_{\langle\alpha,\beta\rangle\in I^2}\left(\beta\vee\left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right).
\end{aligned} \tag{3.19}$$

Next, we will prove that

$$\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right)=\bigvee_{k=1}^{n}\left(\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right). \tag{3.20}$$

Since

$$\begin{aligned}&\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right)\geq\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\\ \Longrightarrow&\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right)\geq\bigvee_{k=1}^{n}\left(\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right).\end{aligned} \tag{3.21}$$

Similarly, we assume that

$$\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right)>\bigvee_{k=1}^{n}\left(\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right), \tag{3.22}$$

then there exists a constant γ such that

$$\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right)>\gamma>\bigvee_{k=1}^{n}\left(\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right). \tag{3.23}$$

Since $\bigvee_{k=1}^{n}\nu_{E^{(k)}_{\langle\alpha,\gamma\rangle}}(x_k)=\{0,1\}$, if $\bigvee_{k=1}^{n}\nu_{E^{(k)}_{\langle\alpha,\gamma\rangle}}(x_k)=0$, then

$$\gamma<\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right)\leq\gamma\vee\left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)=\gamma. \tag{3.24}$$

On the other hand, if $\bigvee_{k=1}^{n}\nu_{E^{(k)}_{\langle\alpha,\gamma\rangle}}(x_k)=1$, then there exists a constant k_1 $(1\leq k_1\leq n)$ such that $\nu_{E^{(k_1)}_{\langle\alpha,\gamma\rangle}}(x_{k_1})=1$, that is, $\mu_{E^{(k_1)}}(x_{k_1})>\alpha$ and $\nu_{E^{(k_1)}}(x_{k_1})<\gamma$. Therefore, for all $\beta>\gamma$, we have $\nu_{E^{(k_1)}_{\langle\alpha,\beta\rangle}}(x_{k_1})=1$.

So we can obtain

$$\gamma>\bigvee_{k=1}^{n}\left(\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\geq\bigwedge_{\langle \alpha,\beta\rangle\in I^2}\left(\beta\vee\nu_{E^{(k_1)}_{\langle\alpha,\beta\rangle}}(x_{k_1})\right)\geq\gamma. \tag{3.25}$$

Notice that the expressions (3.24) and (3.25) are also contradictory. So we can see that the expression (3.20) holds, and then we can obtain

$$
\begin{aligned}
&\nu_{E^{(1)}\times E^{(2)}\times\cdots\times E^{(n)}}(x_1, x_2, \ldots, x_n) \\
&= \bigwedge_{\langle\alpha,\beta\rangle\in I^2}\left(\beta \vee \left(\bigvee_{k=1}^{n}\left(\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right)\right) \\
&= \bigvee_{k=1}^{n}\left(\bigwedge_{\langle\alpha,\beta\rangle\in I^2}\left(\beta \vee \nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)\right)\right) \\
&= \bigvee_{k=1}^{n}\nu_{E^{(k)}}(x_k).
\end{aligned}
\tag{3.26}
$$

□

Remark 3.3. According to the poof, one can see that the notations $\bigvee_{\langle\alpha,\beta\rangle}$ and $\bigwedge_{k=1}^{n}$, $\bigwedge_{\langle\alpha,\beta\rangle}$ and $\bigvee_{k=1}^{n}$ are commutative, respectively. The reason is that $\bigwedge_{k=1}^{n}\mu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)$ and $\bigvee_{k=1}^{n}\nu_{E^{(k)}_{\langle\alpha,\beta\rangle}}(x_k)$ are equal to $\{0,1\}$. Notice that these relations are not necessarily satisfied under general conditions.

Based on Theorem 3.2, we can obtain some properties of cut sets of the Cartesian product over *IFSs*.

Theorem 3.4. *Let $E^{(k)} \in IFSs(X_k)$ $(k = 1, 2, \ldots, n)$, $\langle\alpha,\beta\rangle \in I^2$, then*

(i) $(E^{(1)} \times E^{(2)} \times \cdots \times E^{(n)})_{\langle\alpha,\beta\rangle} = E^{(1)}_{\langle\alpha,\beta\rangle} \times E^{(2)}_{\langle\alpha,\beta\rangle} \times \cdots \times E^{(n)}_{\langle\alpha,\beta\rangle}$;

(ii) $(E^{(1)} \times E^{(2)} \times \cdots \times E^{(n)})_{\langle\underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle} = E^{(1)}_{\langle\underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle} \times E^{(2)}_{\langle\underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle} \times \cdots \times E^{(n)}_{\langle\underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle}$;

(iii) $(E^{(1)} \times E^{(2)} \times \cdots \times E^{(n)})_{\langle\underset{\cdot}{\alpha},\beta\rangle} = E^{(1)}_{\langle\underset{\cdot}{\alpha},\beta\rangle} \times E^{(2)}_{\langle\underset{\cdot}{\alpha},\beta\rangle} \times \cdots \times E^{(n)}_{\langle\underset{\cdot}{\alpha},\beta\rangle}$;

(iv) $(E^{(1)} \times E^{(2)} \times \cdots \times E^{(n)})_{\langle\alpha,\underset{\cdot}{\beta}\rangle} = E^{(1)}_{\langle\alpha,\underset{\cdot}{\beta}\rangle} \times E^{(2)}_{\langle\alpha,\underset{\cdot}{\beta}\rangle} \times \cdots \times E^{(n)}_{\langle\alpha,\underset{\cdot}{\beta}\rangle}$.

Proof. We only prove the first equality, the remaining equalities can be proved in a similar way.

According to Definition 2.1 and Theorem 3.2, for all

$$
\begin{aligned}
&(x_1, x_2, \ldots, x_n) \in \left(E^{(1)} \times E^{(2)} \times \cdots \times E^{(n)}\right)_{\langle\alpha,\beta\rangle} \\
&\Longleftrightarrow \bigwedge_{k=1}^{n}\mu_{E^{(k)}}(x_k) \geq \alpha, \qquad \bigvee_{k=1}^{n}\nu_{E^{(k)}}(x_k) \leq \beta \\
&\Longleftrightarrow \mu_{E^{(k)}}(x_k) \geq \alpha, \quad \nu_{E^{(k)}}(x_k) \leq \beta \quad (k = 1, 2, \ldots, n) \\
&\Longleftrightarrow x_k \in E^{(k)}_{\langle\alpha,\beta\rangle} \quad (k = 1, 2, \ldots, n) \\
&(x_1, x_2, \ldots, x_n) \in E^{(1)}_{\langle\alpha,\beta\rangle} \times E^{(2)}_{\langle\alpha,\beta\rangle} \times \cdots \times E^{(n)}_{\langle\alpha,\beta\rangle}.
\end{aligned}
\tag{3.27}
$$

Hence, the equality holds, namely,

$$\left(E^{(1)} \times E^{(2)} \times \cdots \times E^{(n)}\right)_{\langle \alpha,\beta \rangle} = E^{(1)}_{\langle \alpha,\beta \rangle} \times E^{(2)}_{\langle \alpha,\beta \rangle} \times \cdots \times E^{(n)}_{\langle \alpha,\beta \rangle}. \quad (3.28)$$

□

4. Multivariate Extension Principle of *IFSs*

Based on the Cartesian product over *IFSs*, we will discuss the multivariate extension principle of *IFSs*.

Let $E^{(k)} \in IFSs(X_k)$ $(k = 1, 2, \ldots, n)$, then the operation of Cartesian product may be viewed as a mapping, which is defined as follows

$$\begin{gathered} \times : IFSs(X_1) \times IFSs(X_2) \times \cdots \times IFSs(X_n) \longrightarrow IFSs(X_1 \times X_2 \times \cdots \times X_n), \\ \left(E^{(1)}, E^{(2)}, \ldots, E^{(n)}\right) \longmapsto E^{(1)} \times E^{(2)} \times \cdots \times E^{(n)}. \end{gathered} \quad (4.1)$$

Now we will define the multivariate extension principle. It is assumed that f is a mapping from X to Y, namely,

$$\begin{gathered} f : X = X_1 \times X_2 \times \cdots \times X_n \longrightarrow Y = Y_1 \times Y_2 \times \cdots \times Y_m, \\ (x_1, x_2, \ldots, x_n) \longmapsto (f(x_1, x_2, \ldots, x_n)) = y = (y_1, y_2, \ldots, y_m). \end{gathered} \quad (4.2)$$

By the extension principle of *IFSs* (Lemma 2.8), the mapping f can be induced to the following two mappings.

$$\begin{gathered} f{:}IFSs(X) \longrightarrow IFSs(Y), \qquad A \longmapsto f(A) = \bigcup_{\langle \alpha,\beta \rangle \in I^2} \langle \alpha, \beta \rangle f\left(A_{\langle \alpha,\beta \rangle}\right), \\ f^{-1}{:}IFSs(Y) \longrightarrow IFSs(X), \qquad B \longmapsto f^{-1}(B) = \bigcup_{\langle \alpha,\beta \rangle \in I^2} \langle \alpha, \beta \rangle f^{-1}\left(B_{\langle \alpha,\beta \rangle}\right). \end{gathered} \quad (4.3)$$

Next, we use the two mappings f, f^{-1} to make compound operations with the following two Cartesian products of *IFSs*, respectively.

$$\begin{gathered} \times_1{:}IFSs(X_1) \times IFSs(X_2) \times \cdots \times IFSs(X_n) \longrightarrow IFSs(X), \\ \left(A^{(1)}, A^{(2)}, \ldots, A^{(n)}\right) \longmapsto A^{(1)} \times A^{(2)} \times \cdots \times A^{(n)}, \\ \times_2{:}IFSs(Y_1) \times IFSs(Y_2) \times \cdots \times IFSs(Y_m) \longrightarrow IFSs(Y), \\ \left(B^{(1)}, B^{(2)}, \ldots, B^{(m)}\right) \longmapsto B^{(1)} \times B^{(2)} \times \cdots \times B^{(m)}. \end{gathered} \quad (4.4)$$

Based on the above compound operations, we will get the following definition about multivariate extension principle of *IFSs*.

Definition 4.1. Let $A^{(k)} \in IFSs(X_k)$ $(k = 1, 2, \ldots, n)$, $B^{(l)} \in IFSs(Y_l)$ $(l = 1, 2, \ldots, m)$, and f be a mapping from X to Y, that is,

$$\begin{aligned} f\colon X = X_1 \times X_2 \times \cdots \times X_n &\longrightarrow Y = Y_1 \times Y_2 \times \cdots \times Y_m, \\ (x_1, x_2, \ldots, x_n) &\longmapsto f(x_1, x_2, \ldots, x_n) = y = (y_1, y_2, \ldots, y_m). \end{aligned} \tag{4.5}$$

The two induced mappings of f can be defined as follows:

$$\begin{gathered} f\colon IFSs(X_1) \times IFSs(X_2) \times \cdots \times IFSs(X_n) \longrightarrow IFSs(Y), \\ \left(A^{(1)}, A^{(2)}, \ldots, A^{(n)}\right) \longmapsto f\left(A^{(1)}, A^{(2)}, \ldots, A^{(n)}\right) \overset{\text{def}}{=} f\left(A^{(1)} \times A^{(2)} \times \cdots \times A^{(n)}\right), \\ f^{-1}\colon IFSs(Y_1) \times IFSs(Y_2) \times \cdots \times IFSs(Y_m) \longrightarrow IFSs(X), \\ \left(B^{(1)}, B^{(2)}, \ldots, B^{(m)}\right) \longmapsto f^{-1}\left(B^{(1)}, B^{(2)}, \ldots, B^{(m)}\right) \\ \overset{\text{def}}{=} f^{-1}\left(B^{(1)} \times B^{(2)} \times \cdots \times B^{(m)}\right). \end{gathered} \tag{4.6}$$

It is obvious that the above result is an extension of Lemma 2.8 of the Cartesian product over *IFSs*. Hence, it is said to be the multivariate extension principle *of IFSs*.

According to Lemma 2.8 and Theorem 3.2, we can obtain the membership function and nonmembership function of f and f^{-1}. The corresponding conclusions are given by the following theorem.

Theorem 4.2. *Let f and f^{-1} be two induced mappings, which are given by Definition 4.1, then the membership function and nonmembership function of f and f^{-1} are given by*

$$\begin{aligned} \mu_{f(A^{(1)}, A^{(2)}, \ldots, A^{(n)})}(y) &= \bigvee_{f(x_1, x_2, \ldots, x_n) = y} \left(\bigwedge_{k=1}^{n} \mu_{A^{(k)}}(x_k) \right), \\ \nu_{f(A^{(1)}, A^{(2)}, \ldots, A^{(n)})}(y) &= \bigwedge_{f(x_1, x_2, \ldots, x_n) = y} \left(\bigvee_{k=1}^{n} \nu_{A^{(k)}}(x_k) \right), \\ \mu_{f^{-1}(B^{(1)}, B^{(2)}, \ldots, B^{(m)})}(x) &= \bigwedge_{l=1}^{m} \mu_{B^{(l)}}(y_l), \\ \nu_{f^{-1}(B^{(1)}, B^{(2)}, \ldots, B^{(m)})}(x) &= \bigvee_{l=1}^{m} \nu_{B^{(l)}}(y_l), \end{aligned} \tag{4.7}$$

where $(y_1, y_2, \ldots, y_m) = f(x)$.

Below we will discuss some other forms of the multivariate extension principle of *IFSs*.

Theorem 4.3 (multivariate extension principle I).

$$f\left(A^{(1)},A^{(2)},\dots,A^{(n)}\right)=\bigcup_{\langle\alpha,\beta\rangle\in I^2}\langle\alpha,\beta\rangle f\left(A^{(1)}_{\langle\alpha,\beta\rangle},A^{(2)}_{\langle\alpha,\beta\rangle},\dots,A^{(n)}_{\langle\alpha,\beta\rangle}\right),$$
$$f^{-1}\left(B^{(1)},B^{(2)},\dots,B^{(m)}\right)=\bigcup_{\langle\alpha,\beta\rangle\in I^2}\langle\alpha,\beta\rangle f^{-1}\left(B^{(1)}_{\langle\alpha,\beta\rangle},B^{(2)}_{\langle\alpha,\beta\rangle},\dots,B^{(m)}_{\langle\alpha,\beta\rangle}\right). \quad (4.8)$$

Proof. By Lemma 2.8 and Definition 4.1, we have

$$\begin{aligned}f\left(A^{(1)},A^{(2)},\dots,A^{(n)}\right)&=f\left(A^{(1)}\times A^{(2)}\times\cdots\times A^{(n)}\right)\\&=\bigcup_{\langle\alpha,\beta\rangle\in I^2}\langle\alpha,\beta\rangle f\left(\left(A^{(1)}\times A^{(2)}\times\cdots\times A^{(n)}\right)_{\langle\alpha,\beta\rangle}\right)\\&=\bigcup_{\langle\alpha,\beta\rangle\in I^2}\langle\alpha,\beta\rangle f\left(A^{(1)}_{\langle\alpha,\beta\rangle}\times A^{(2)}_{\langle\alpha,\beta\rangle}\times\cdots\times A^{(n)}_{\langle\alpha,\beta\rangle}\right).\end{aligned} \quad (4.9)$$

Since

$$\begin{aligned}&f\left(A^{(1)}_{\langle\alpha,\beta\rangle},A^{(2)}_{\langle\alpha,\beta\rangle},\dots,A^{(n)}_{\langle\alpha,\beta\rangle}\right)\\&\quad=\left\{y:\ \exists x_k\in A^{(k)}_{\langle\alpha,\beta\rangle}\quad(k=1,2,\dots,n),f(x_1,x_2,\dots,x_n)=y\right\}\\&\quad=\left\{y:\ \exists(x_1,x_2,\dots,x_n)\in A^{(1)}_{\langle\alpha,\beta\rangle}\times A^{(2)}_{\langle\alpha,\beta\rangle}\times\cdots\times A^{(n)}_{\langle\alpha,\beta\rangle},f(x_1,x_2,\dots,x_n)=y\right\}\\&\quad=f\left(A^{(1)}_{\langle\alpha,\beta\rangle}\times A^{(2)}_{\langle\alpha,\beta\rangle}\times\cdots\times A^{(n)}_{\langle\alpha,\beta\rangle}\right),\end{aligned} \quad (4.10)$$

we know that

$$f\left(A^{(1)},A^{(2)},\dots,A^{(n)}\right)=\bigcup_{\langle\alpha,\beta\rangle\in I^2}\langle\alpha,\beta\rangle f\left(A^{(1)}_{\langle\alpha,\beta\rangle},A^{(2)}_{\langle\alpha,\beta\rangle},\dots,A^{(n)}_{\langle\alpha,\beta\rangle}\right). \quad (4.11)$$

Similarly, we can prove the second equality. □

Theorem 4.4 (multivariate extension principle II).

$$
\begin{aligned}
f\left(A^{(1)}, A^{(2)}, \ldots, A^{(n)}\right) &= \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle f\left(A^{(1)}_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle}, A^{(2)}_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle}, \ldots, A^{(n)}_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle}\right) \\
&= \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle f\left(A^{(1)}_{\langle \underset{\cdot}{\alpha},\beta\rangle}, A^{(2)}_{\langle \underset{\cdot}{\alpha},\beta\rangle}, \ldots, A^{(n)}_{\langle \underset{\cdot}{\alpha},\beta\rangle}\right) \\
&= \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle f\left(A^{(1)}_{\langle \alpha,\underset{\cdot}{\beta}\rangle}, A^{(2)}_{\langle \alpha,\underset{\cdot}{\beta}\rangle}, \ldots, A^{(n)}_{\langle \alpha,\underset{\cdot}{\beta}\rangle}\right), \\
f^{-1}\left(B^{(1)}, B^{(2)}, \ldots, B^{(m)}\right) &= \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle f^{-1}\left(B^{(1)}_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle}, B^{(2)}_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle}, \ldots, B^{(m)}_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle}\right) \\
&= \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle f^{-1}\left(B^{(1)}_{\langle \underset{\cdot}{\alpha},\beta\rangle}, B^{(2)}_{\langle \underset{\cdot}{\alpha},\beta\rangle}, \ldots, B^{(m)}_{\langle \underset{\cdot}{\alpha},\beta\rangle}\right) \\
&= \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle f^{-1}\left(B^{(1)}_{\langle \alpha,\underset{\cdot}{\beta}\rangle}, B^{(2)}_{\langle \alpha,\underset{\cdot}{\beta}\rangle}, \ldots, B^{(m)}_{\langle \alpha,\underset{\cdot}{\beta}\rangle}\right).
\end{aligned}
\tag{4.12}
$$

Proof. The proof method is similar to that of Theorem 4.3. Thus, we omit it here. □

Theorem 4.5 (multivariate extension principle III).

$$
f\left(A^{(1)}, A^{(2)}, \ldots, A^{(n)}\right) = \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle f\left(H_A^{(1)}(\alpha,\beta), H_A^{(2)}(\alpha,\beta), \ldots, H_A^{(n)}(\alpha,\beta)\right), \tag{4.13}
$$

where $A^{(k)}_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle} \subseteq H_A^{(k)}(\alpha,\beta) \subseteq A^{(k)}_{\langle\alpha,\beta\rangle}$ $(k = 1, 2, \ldots, n)$.

$$
f^{-1}\left(B^{(1)}, B^{(2)}, \ldots, B^{(m)}\right) = \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle f^{-1}\left(H_B^{(1)}(\alpha,\beta), H_B^{(2)}(\alpha,\beta), \ldots, H_B^{(m)}(\alpha,\beta)\right), \tag{4.14}
$$

where $B^{(l)}_{\langle \underset{\cdot}{\alpha},\underset{\cdot}{\beta}\rangle} \subseteq H_B^{(l)}(\alpha,\beta) \subseteq B^{(l)}_{\langle\alpha,\beta\rangle}$ $(l = 1, 2, \ldots, m)$.

Proof. Since

$$
\begin{aligned}
& A^{(k)}_{\langle \alpha,\beta \rangle} \subseteq H^{(k)}_{A}(\alpha,\beta) \subseteq A^{(k)}_{\langle \alpha,\beta \rangle} \quad (k = 1,2,\ldots,n) \\
& \Longrightarrow A^{(1)}_{\langle \alpha,\beta \rangle} \times A^{(2)}_{\langle \alpha,\beta \rangle} \times \cdots \times A^{(n)}_{\langle \alpha,\beta \rangle} \subseteq H^{(1)}_{A}(\alpha,\beta) \times H^{(2)}_{A}(\alpha,\beta) \times \cdots \times H^{(n)}_{A}(\alpha,\beta) \\
& \qquad \subseteq A^{(1)}_{\langle \alpha,\beta \rangle} \times A^{(2)}_{\langle \alpha,\beta \rangle} \times \cdots \times A^{(n)}_{\langle \alpha,\beta \rangle} \\
& \Longrightarrow f\left(A^{(1)}_{\langle \alpha,\beta \rangle}, A^{(2)}_{\langle \alpha,\beta \rangle}, \ldots, A^{(n)}_{\langle \alpha,\beta \rangle}\right) \subseteq f\left(H^{(1)}_{A}(\alpha,\beta), H^{(2)}_{A}(\alpha,\beta), \ldots, H^{(n)}_{A}(\alpha,\beta)\right) \\
& \qquad \subseteq f\left(A^{(1)}_{\langle \alpha,\beta \rangle}, A^{(2)}_{\langle \alpha,\beta \rangle}, \ldots, A^{(n)}_{\langle \alpha,\beta \rangle}\right) \\
& \Longrightarrow f\left(A^{(1)}, A^{(2)}, \ldots, A^{(n)}\right) = \bigcup_{\langle \alpha,\beta \rangle \in I^2} \langle \alpha,\beta \rangle f\left(A^{(1)}_{\langle \alpha,\beta \rangle}, A^{(2)}_{\langle \alpha,\beta \rangle}, \ldots, A^{(n)}_{\langle \alpha,\beta \rangle}\right) \\
& \qquad \subseteq \bigcup_{\langle \alpha,\beta \rangle \in I^2} \langle \alpha,\beta \rangle f\left(H^{(1)}_{A}(\alpha,\beta), H^{(2)}_{A}(\alpha,\beta), \ldots, H^{(n)}_{A}(\alpha,\beta)\right) \\
& \qquad \subseteq \bigcup_{\langle \alpha,\beta \rangle \in I^2} \langle \alpha,\beta \rangle f\left(A^{(1)}_{\langle \alpha,\beta \rangle}, A^{(2)}_{\langle \alpha,\beta \rangle}, \ldots, A^{(n)}_{\langle \alpha,\beta \rangle}\right) \\
& \qquad = f\left(A^{(1)}, A^{(2)}, \ldots 0, A^{(n)}\right).
\end{aligned}
\tag{4.15}
$$

Similarly, the second equality can be proved by the above method. □

5. Algebraic Operations of the *IFSs*(R)

In this section, we will define several algebraic operations between *IFSs* on real number field R using the multivariate extension principle, such as the arithmetic operations $(+,-,\cdot,\div)$, conjunction and disjunction operations $(\wedge,\vee)$, and so forth. In general, we denote the set of all intuitionistic fuzzy sets on real number field R by the symbol $IFSs(R)$.

Let $*$ be an algebraic operation on R, that is,

$$
* : R \times R \longrightarrow R, \quad (x,y) \longmapsto z = x * y. \tag{5.1}
$$

According to the multivariate extension principle of *IFSs*, we can define several algebraic operations on $IFSs(R)$.

Definition 5.1. Let $*$ be an algebraic operation on R, then the corresponding algebraic operation on $IFSs(R)$ is defined as

$$\begin{gathered} *: IFSs(R) \times IFSs(R) \longrightarrow IFSs(R), \\ (A, B) \longmapsto A * B \overset{\text{def}}{=} \bigcup_{\langle\alpha,\beta\rangle \in I^2} \langle\alpha,\beta\rangle \left(A_{\langle\alpha,\beta\rangle} * B_{\langle\alpha,\beta\rangle}\right), \\ A_{\langle\alpha,\beta\rangle} * B_{\langle\alpha,\beta\rangle} = \left\{z :\ \exists(x,y) \in A_{\langle\alpha,\beta\rangle} \times B_{\langle\alpha,\beta\rangle}, x * y = z\right\}, \end{gathered} \tag{5.2}$$

the membership function and nonmembership function of the operational result are given, respectively, by

$$*: \begin{cases} \mu_{A*B}(z) = \bigvee_{x*y=z} \left(\mu_A(x) \wedge \mu_B(y)\right), \\ \nu_{A*B}(z) = \bigwedge_{x*y=z} \left(\nu_A(x) \vee \nu_B(y)\right). \end{cases} \tag{5.3}$$

By Definition 5.1, we can obtain the following some algebraic operations between *IFSs* on R.

(a) *Arithmetic operation*

$$+: \begin{cases} \mu_{A+B}(z) = \bigvee_{x+y=z} \left(\mu_A(x) \wedge \mu_B(y)\right) = \bigvee_{x\in R} \left(\mu_A(x) \wedge \mu_B(z-x)\right), \\ \nu_{A+B}(z) = \bigwedge_{x+y=z} \left(\nu_A(x) \vee \nu_B(y)\right) = \bigwedge_{x\in R} \left(\nu_A(x) \vee \nu_B(z-x)\right), \end{cases}$$

$$-: \begin{cases} \mu_{A-B}(z) = \bigvee_{x-y=z} \left(\mu_A(x) \wedge \mu_B(y)\right) = \bigvee_{x\in R} \left(\mu_A(x) \wedge \mu_B(x-z)\right), \\ \nu_{A-B}(z) = \bigwedge_{x-y=z} \left(\nu_A(x) \vee \nu_B(y)\right) = \bigwedge_{x\in R} \left(\nu_A(x) \vee \nu_B(x-z)\right), \end{cases}$$

$$\cdot: \begin{cases} \mu_{A\cdot B}(z) = \bigvee_{x\cdot y=z} \left(\mu_A(x) \wedge \mu_B(y)\right) \\ \qquad = \begin{cases} \bigvee_{x\in R} \left(\mu_A(x) \wedge \mu_B\left(\dfrac{z}{x}\right)\right) & (z \neq 0), \\ \left(\mu_A(0) \wedge \left(\bigvee_{y\in R} \mu_B(y)\right)\right) \vee \left(\mu_B(0) \wedge \left(\bigvee_{x\in R} \mu_A(x)\right)\right) & (z = 0), \end{cases} \\ \nu_{A\cdot B}(z) = \bigwedge_{x\cdot y=z} \left(\nu_A(x) \vee \nu_B(y)\right) \\ \qquad = \begin{cases} \bigwedge_{x\in R} \left(\nu_A(x) \vee \nu_B\left(\dfrac{z}{x}\right)\right) & (z \neq 0), \\ \left(\nu_A(0) \vee \left(\bigwedge_{y\in R} \nu_B(y)\right)\right) \wedge \left(\mu_B(0) \vee \left(\bigwedge_{x\in R} \nu_A(x)\right)\right) & (z = 0). \end{cases} \end{cases}$$

$$\div : \begin{cases} \mu_{A\div B}(z) = \bigvee_{x\div y=z} (\mu_A(x) \wedge \mu_B(y)) = \bigvee_{y\neq 0} (\mu_A(y\cdot z) \wedge \mu_B(y)), \\ \nu_{A\div B}(z) = \bigwedge_{x\div y=z} (\nu_A(x) \vee \nu_B(y)) = \bigwedge_{y\neq 0} (\nu_A(y\cdot z) \vee \nu_B(y)). \end{cases} \tag{5.4}$$

(b) *Conjunction and disjunction operations*

$$\begin{gathered} \wedge : \begin{cases} \mu_{A\wedge B}(z) = \bigvee_{x\wedge y=z} (\mu_A(x) \wedge \mu_B(y)), \\ \nu_{A\wedge B}(z) = \bigwedge_{x\wedge y=z} (\nu_A(x) \vee \nu_B(y)), \end{cases} \\ \vee : \begin{cases} \mu_{A\vee B}(z) = \bigvee_{x\vee y=z} (\mu_A(x) \wedge \mu_B(y)), \\ \nu_{A\vee B}(z) = \bigwedge_{x\vee y=z} (\nu_A(x) \vee \nu_B(y)). \end{cases} \end{gathered} \tag{5.5}$$

6. Conclusion

In this paper, we presented the multivariate extension principle of *IFSs*. First of all, we reviewed the notion of Cartesian product over *IFSs* proposed by Atanassov [18] and defined the membership function and nonmembership function of Cartesian product in intuitionistic fuzzy environment. In addition, some relevant properties were discussed. Afterwards, the multivariate extension principle of *IFSs* was presented. Finally, we provided a general framework of the algebraic operation between *IFSs* and gave several common operations. In short, this paper not only provides a theoretical foundation for algebraic operations between *IFSs*, but also enriches the theory of *IFSs*.

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