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*Research Article*

# The Existence of Solutions for a Fractional 2$m$-Point Boundary Value Problems

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By using the coincidence degree theory, we consider the following 2$m$-point boundary value problem for fractional differential equation $D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)) + e(t)$, $0 < t < 1$, $I_{0+}^{3-\alpha}u(t)|_{t=0} = 0$, $D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^{\alpha-2}u(\xi_i)$, $u(1) = \sum_{i=1}^{m-2} b_i u(\eta_i)$, where $2 < \alpha \le 3$, $D_{0+}^{\alpha}$ and $I_{0+}^{\alpha}$ are the standard Riemann-Liouville fractional derivative and fractional integral, respectively. A new result on the existence of solutions for above fractional boundary value problem is obtained.

## 1. Introduction

Fractional differential equations have been of great interest recently. This is because of the intensive development of the theory of fractional calculus itself as well as its applications. Apart from diverse areas of mathematics, fractional differential equations arise in a variety of different areas such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science (see [1–4] and references cited therein). The research of fractional differential equations on boundary value problems, as one of the focal topics has attained a great deal of attention from many researchers (see [5–13]).

However, there are few papers which consider the boundary value problem at resonance for nonlinear ordinary differential equations of fractional order. In [14], Hu and Liu studied the following BVP of fractional equation at resonance:

$$\begin{gathered} D_{0+}^{\alpha}x(t) = f\left(t, x(t), x'(t), x''(t)\right), \quad 0 \le t \le 1, \\ x(0) = x(1), \qquad x'(0) = x''(0) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \le 2$, $D_{0^+}^{\alpha}$ is the standard Caputo fractional derivative.

In [15], Zhang and Bai investigated the nonlinear nonlocal problem

$$\begin{gathered} D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad \beta u(\eta) = u(1), \end{gathered} \tag{1.2}$$

where $1 < \alpha \le 2$, they consider the case $\beta\eta^{\alpha-1} = 1$, that is, the resonance case.

In [16], Bai investigated the boundary value problem at resonance

$$\begin{gathered} D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t)\right) + e(t), \quad 0 < t < 1, \\ I_{0+}^{2-\alpha}u(t)|_{t=0} = 0, \qquad D_{0+}^{\alpha-1}u(1) = \sum_{i=0}^{m-2}\beta_i D_{0+}^{\alpha-1}u(\eta_i) \end{gathered} \tag{1.3}$$

is considered, where $1 < \alpha \le 2$ is a real number, $D_{0^+}^{\alpha}$ and $I_{0^+}^{\alpha}$ are the standard Riemann-Liouville fractional derivative and fractional integral, respectively, and $f : [0,1] \times R^2 \to R$ is continuous and $e(t) \in L^1[0,1]$, $m \ge 2$, $0 < \xi_i < 1$, $\beta_i \in R$, $i = 1, 2, \ldots, m-2$ are given constants such that $\sum_{i=1}^{m-2}\beta_i = 1$.

In this paper, we study the $2m$-point boundary value problem

$$D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)\right) + e(t), \quad 0 < t < 1, \tag{1.4}$$

$$I_{0+}^{3-\alpha}u(t)|_{t=0} = 0, \qquad D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m-2}a_i D_{0+}^{\alpha-2}u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2}b_i u(\eta_i), \tag{1.5}$$

where $2 < \alpha \le 3$, $m \ge 2$, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \eta_1 < \cdots < \eta_m < 1$, $a_i, b_i \in R$, $f : [0,1] \times R^3 \to R$, $f$ satisfies Carathéodory conditions, $D_{0^+}^{\alpha}$ and $I_{0^+}^{\alpha}$ are the standard Riemann-Liouville fractional derivative and fractional integral, respectively.

Setting

$$\begin{gathered} \Lambda_1 = \frac{1}{\alpha(\alpha+1)}\left(1 - \sum_{i=1}^{m-2}a_i\xi_i^{\alpha+1}\right), \qquad \Lambda_2 = \frac{1}{\alpha(\alpha-1)}\left(1 - \sum_{i=1}^{m-2}a_i\xi_i^{\alpha}\right), \\ \Lambda_3 = \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\left[1 - \sum_{i=1}^{m-2}b_i\eta_i^{2\alpha-1}\right], \qquad \Lambda_4 = \frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left[1 - \sum_{i=1}^{m-2}b_i\eta_i^{2\alpha-2}\right]. \end{gathered} \tag{1.6}$$

In this paper, we will always suppose that the following conditions hold:

(C1):

$$\sum_{i=1}^{m-2}a_i\xi_i = \sum_{i=1}^{m-2}a_i = 1, \qquad \sum_{i=1}^{m-2}b_i\eta_i^{\alpha-1} = \sum_{i=1}^{m-2}b_i\eta_i^{\alpha-2} = 1, \tag{1.7}$$

(C2):

$$\Lambda = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3 \neq 0. \tag{1.8}$$

We say that boundary value problem (1.4) and (1.5) is at resonance, if BVP

$$\begin{gathered} D_{0+}^{\alpha}u(t) = 0, \\ I_{0+}^{3-\alpha}u(t)|_{t=0} = 0, \qquad D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^{\alpha-2}u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\eta_i) \end{gathered} \tag{1.9}$$

has $u(t) = at^{\alpha-1} + bt^{\alpha-2}$, $a, b \in R$ as a nontrivial solution.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions, and lemmas. In Section 3, we establish a theorem on existence of solutions for BVP (1.4)-(1.5) under nonlinear growth restriction of $f$, basing on the coincidence degree theory due to Mawhin (see [17]).

Now, we will briefly recall some notation and an abstract existence result.

Let $Y, Z$ be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ a Fredholm map of index zero,s and $P : Y \to Y$, $Q : Z \to Z$ continuous projectors such that

$$Y = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Z = \operatorname{Im} L \oplus Q, \qquad \operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L. \tag{1.10}$$

It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of the map by $K_p$. If $\Omega$ is an open-bounded subset of $Y$ such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ will be called $L$-compact on $\Omega$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to Y$ is compact.

The lemma that we used is [17, Theorem 2.4].

**Lemma 1.1.** *Let $L$ be a Fredholm operator of index zero and let $N$ be $L$-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$, *for all* $(x, \lambda) \in [\operatorname{dom} L \setminus \operatorname{Ker} L \cap \partial\Omega] \times [0,1]$;

(ii) $Nx \notin \operatorname{Im} L$, *for all* $x \in \operatorname{Ker} L \cap \partial\Omega$;

(iii) $\deg(JQN|_{\operatorname{Ker} L},\ \operatorname{Ker} L \cap \Omega, 0) \neq 0$,

*where* $Q : Z \to Z$ *is a projection as above with* $\operatorname{Ker} Q = \operatorname{Im} L$ *and* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is any isomorphism. Then the equation* $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

## 2. Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory. These definitions can be found in the recent literature [1–16, 18].

*Definition 2.1.* The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \tag{2.1}$$

provided the right side is pointwise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

*Definition 2.2.* The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.2}$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

*Definition 2.3.* We say that the map $f : [0,1] \times R^n \to R$ satisfies Carathéodory conditions with respect to $L^1[0,1]$ if the following conditions are satisfied:

(i) for each $z \in R^n$, the mapping $t \to f(t,z)$ is Lebesgue measurable;

(ii) for almost every $t \in [0,1]$, the mapping $t \to f(t,z)$ is continuous on $R^n$;

(iii) for each $r > 0$, there exists $\rho_r \in L^1([0,1],R)$ such that for a.e. $t \in [0,1]$ and every $|z| \le r$, we have $f(t,z) \le \rho_r(t)$.

**Lemma 2.4** (see [15]). *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_1 t^{\alpha-2} + \cdots + c_N t^{\alpha-N} \tag{2.3}$$

*for some $c_i \in R$, $i = 1,2,\ldots,N$, where $N$ is the smallest integer greater than or equal to $\alpha$.*

We use the classical Banach space $C[0,1]$ with the norm

$$\|x\|_{\infty} = \max_{t\in[0,1]} |x(t)|, \tag{2.4}$$

$L[0,1]$ with the norm

$$\|x\|_1 = \int_0^1 |x(t)|\,dt. \tag{2.5}$$

*Definition 2.5.* For $n \in N$, we denote by $AC^n[0,1]$ the space of functions $u(t)$ which have continuous derivatives up to order $n-1$ on $[0,1]$ such that $u^{(n-1)}(t)$ is absolutely continuous: $AC^n[0,1] = \{u \mid [0,1] \to R$ and $(D^{(n-1)})u(t)$ is absolutely continuous in $[0,1]\}$.

**Lemma 2.6** (see [15]). *Given $\mu > 0$ and $N = [\mu] + 1$ we can define a linear space*

$$C^{\mu}[0,1] = \left\{ u(t) \mid u(t) = I_{0^+}^{\alpha} x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \cdots + c_N t^{\mu-(N-1)},\ t \in [0,1] \right\}, \tag{2.6}$$

*where $x \in [0,1]$, $c_i \in R$, $i = 1,2,\ldots,N-1$. By means of the linear functional analysis theory, we can prove that with the*

$$\|u\|_{C^\mu} = \left\|D_{0^+}^{\mu} u\right\|_\infty + \cdots + \left\|D_{0^+}^{\mu-(N-1)} u\right\|_\infty + \|u\|_\infty, \tag{2.7}$$

*$C^\mu[0,1]$ is a Banach space.*

*Remark 2.7.* If $\mu$ is a natural number, then $C^\mu[0,1]$ is in accordance with the classical Banach space $C^n[0,1]$.

**Lemma 2.8** (see [15]). *$f \subset C^\mu[0,1]$ is a sequentially compact set if and only if $f$ is uniformly bounded and equicontinuous. Here uniformly bounded means there exists $M > 0$, such that for every $u \in f$*

$$\|u\|_{C^\mu} = \left\|D_{0+}^{\mu} u\right\|_\infty + \cdots + \left\|D_{0+}^{\mu-(N-1)} u\right\|_\infty + \|u\|_\infty < M, \tag{2.8}$$

*and equicontinuous means that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that*

$$\begin{gathered} |u(t_1) - u(t_2)| < \varepsilon, \quad \left(\forall t_1,\ t_2 \in [0,1],\ |t_1 - t_2| < \delta,\ \forall u \in f\right), \\ \left|D_{0+}^{\alpha-i} u(t_1) - D_{0+}^{\alpha-i} u(t_2)\right| < \varepsilon, \quad \left(\forall t_1,\ t_2 \in [0,1],\ |t_1 - t_2| < \delta,\ \forall u \in f,\ \forall i = 1,2,\ldots,N-1\right). \end{gathered} \tag{2.9}$$

**Lemma 2.9** (see [1]). *Let $\alpha > 0$, $n = [\alpha] + 1$. Assume that $u \in L^1(0,1)$ with a fractional integration of order $n - \alpha$ that belongs to $AC^n[0,1]$. Then the equality*

$$\left(I_{0+}^{\alpha} D_{0+}^{\alpha} u\right)(t) = u(t) - \sum_{i=1}^{n} \frac{\left(\left(I_{0+}^{n-\alpha} u\right)(t)\right)^{n-i}\big|_{t=0}}{\Gamma(\alpha - i + 1)} t^{\alpha-i} \tag{2.10}$$

*holds almost everywhere on* $[0,1]$.

*Definition 2.10* (see [16]). Let $I_{0+}^{\alpha}(L^1(0,1))$, $\alpha > 0$ denote the space of functions $u(t)$, represented by fractional integral of order $\alpha$ of a summable function: $u = I_{0+}^{\alpha} v, v \in L^1(0,1)$.

Let $Z = L^1[0,1]$, with the norm $\|y\| = \int_0^1 |y(s)| ds$, $Y = C^{\alpha-1}[0,1]$ defined by Lemma 2.6, with the norm $\|u\|_{C^{\alpha-1}} = \|D_{0^+}^{\alpha-1} u\|_\infty + \|D_{0^+}^{\alpha-2} u\|_\infty + \|u\|_\infty$, where $Y$ is a Banach space.

Define $L$ to be the linear operator from $\operatorname{dom} L \subset Y$ to $Z$ with

$$\operatorname{dom} L = \left\{u \in C^{\alpha-1}[0,1] \mid D_{0+}^{\alpha} u \in L^1[0,1], u \text{ satisfies}(1.5)\right\}, \tag{2.11}$$

$$Lu = D_{0^+}^{\alpha} u, \quad u \in \operatorname{dom} L, \tag{2.12}$$

we define $N : Y \to Z$ by setting

$$Nu(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t)D_{0+}^{\alpha-2}u(t)\right) + e(t). \tag{2.13}$$

Then boundary value problem (1.4) and (1.5) can be written as $Lu = Nu$.

## 3. Main Results

**Lemma 3.1.** *Let L be defined by* (2.12), *then*

$$\begin{aligned}
\operatorname{Ker}L &= \left\{at^{\alpha-1} + bt^{\alpha-2} \mid a, b \in R\right\} \cong R^2, \\
\operatorname{Im}L &= \left\{y \in Z \mid \int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds = 0,\right. \\
&\qquad \left.\int_0^1 (1-s)^{\alpha-1}y(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1}y(s)ds = 0\right\}.
\end{aligned} \tag{3.1}$$

*Proof.* In the following lemma, we use the unified notation of both for fractional integrals and fractional derivatives assuming that $I_{0+}^{\alpha} = D_{0+}^{-\alpha}$ for $\alpha < 0$.

Let $Lu = D_{0+}^{\alpha}u$, by Lemma 2.9, $D_{0+}^{\alpha}u(t) = 0$ has solution

$$\begin{aligned}
u(t) &= \sum_{i=1}^{3} \frac{\left(\left(I_{0+}^{3-\alpha}u\right)(t)\right)^{3-i}|_{t=0}}{\Gamma(\alpha - i + 1)} t^{\alpha-i} \\
&= \frac{\left(\left(I_{0+}^{3-\alpha}u\right)(t)\right)''|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{\left(\left(I_{0+}^{3-\alpha}u\right)(t)\right)'|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{\left(\left(I_{0+}^{3-\alpha}u\right)(t)\right)|_{t=0}}{\Gamma(\alpha-2)} t^{\alpha-3} \\
&= \frac{D_{0+}^{\alpha-1}u(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + \frac{D_{0+}^{\alpha-2}u(t)|_{t=0}}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{\left(\left(I_{0+}^{3-\alpha}u\right)(t)\right)|_{t=0}}{\Gamma(\alpha-2)} t^{\alpha-3}.
\end{aligned} \tag{3.2}$$

Combine with (1.5), So,

$$\operatorname{Ker} L = \left\{at^{\alpha-1} + bt^{\alpha-2} \mid a, b \in R\right\} \cong R^2. \tag{3.3}$$

Let $y \in Z$ and let

$$u_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}. \tag{3.4}$$

Then $D_{0+}^{\alpha}u(t) = y(t)$ a.e. $t \in [0,1]$ and, if

$$\begin{aligned} &\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds = 0, \\ &\int_0^1 (1-s)^{\alpha-1}y(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1}y(s)ds = 0 \end{aligned} \tag{3.5}$$

hold, then $u(t)$ satisfies the boundary conditions (1.5). That is, $u \in \operatorname{dom} L$ and we have

$$\{y \in Z \mid y \text{ satisfies } (3.4)\} \subseteq \operatorname{Im} L. \tag{3.6}$$

Let $u \in \operatorname{dom} L$. Then for $D_{0+}^{\alpha}u \in \operatorname{Im} L$, we have

$$I_{0+}^{\alpha}y(t) = u(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - c_3 t^{\alpha-3}, \tag{3.7}$$

where

$$c_1 = \frac{D_{0+}^{\alpha-1}u(t)|_{t=0}}{\Gamma(\alpha)}, \qquad c_2 = \frac{D_{0+}^{\alpha-2}u(t)|_{t=0}}{\Gamma(\alpha-1)}, \qquad c_3 = \frac{I_{0+}^{3-\alpha}u(t)|_{t=0}}{\Gamma(\alpha-2)}, \tag{3.8}$$

which, due to the boundary value condition (1.5), implies that satisfies (3.5). In fact, from $I_{0+}^{3-\alpha}u(t)|_{t=0} = 0$ we have $c_3 = 0$, from $D_{0+}^{\alpha-2}u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^{\alpha-2}u(\xi_i)$, $u(1) = \sum_{i=1}^{m-2} b_i u(\eta_i)$, we have

$$\begin{aligned} &\int_0^1 (1-s)y(s)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds = 0, \\ &\int_0^1 (1-s)^{\alpha-1}y(s)ds - \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1}y(s)ds = 0. \end{aligned} \tag{3.9}$$

Hence,

$$\{y \in Z \mid y \text{ satisfies } (3.4)\} \supseteq \operatorname{Im} L. \tag{3.10}$$

Therefore,

$$\{y \in Z \mid y \text{ satisfies } (3.4)\} = \operatorname{Im} L. \tag{3.11}$$

The proof is complete. □

**Lemma 3.2.** *The mapping $L : \operatorname{dom} L \subset Y \to Z$ is a Fredholm operator of index zero, and*

$$Qy(t) = (T_1 y(t))t^{\alpha-1} + (T_2 y(t))t^{\alpha-2}, \tag{3.12}$$

*where*

$$T_1 y = \frac{1}{\Lambda}(\Lambda_4 Q_1 y - \Lambda_2 Q_2 y), \qquad T_2 y = \frac{1}{\Lambda}(\Lambda_3 Q_1 y - \Lambda_1 Q_2 y), \tag{3.13}$$

*define by* $K_p : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ *by*

$$K_p y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds = I_{0+}^{\alpha} y(t), \quad y \in \mathrm{Im}L, \tag{3.14}$$

*and for all* $y \in \mathrm{Im}L$, $\|K_p y\|_{C^{\alpha-1}} \leq ((1/\Gamma(\alpha)) + 2)\|y\|_1$.

*Proof.* Consider the continuous linear mapping $Q_1 : Z \to Z$ and $Q_2 : Z \to Z$ defined by

$$\begin{aligned} Q_1 y &= \int_0^1 (1-s) y(s)\,ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) y(s)\,ds, \\ Q_2 y &= \int_0^1 (1-s)^{\alpha-1} y(s)\,ds - \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s)\,ds. \end{aligned} \tag{3.15}$$

Using the above definitions, we construct the following auxiliary maps $T_1 : Z \to Z$ and $T_2 : Z \to Z$:

$$\begin{aligned} T_1 y &= \frac{1}{\Lambda}(\Lambda_4 Q_1 y - \Lambda_2 Q_2 y), \\ T_2 y &= \frac{1}{\Lambda}(\Lambda_3 Q_1 y - \Lambda_1 Q_2 y). \end{aligned} \tag{3.16}$$

Since the condition (C2) holds, the mapping defined by

$$Qy(t) = (T_1 y(t)) t^{\alpha-1} + (T_2 y(t)) t^{\alpha-2} \tag{3.17}$$

is well defined.

Recall (C2) and note that

$$\begin{aligned} T_1\left(T_1 y t^{\alpha-1}\right) &= \frac{1}{\Lambda}\left(\Lambda_4 Q_1\left(T_1 y t^{\alpha-1}\right) - \Lambda_2 Q_2\left(T_1 y t^{\alpha-1}\right)\right) \\ &= \frac{1}{\Lambda}\left[\Lambda_4\left(\frac{\Lambda_4\Lambda_1}{\Lambda} Q_1 y - \frac{\Lambda_1\Lambda_2}{\Lambda} Q_2 y\right) - \Lambda_2\left(\frac{\Lambda_4\Lambda_3}{\Lambda} Q_1 y - \frac{\Lambda_2\Lambda_3}{\Lambda} Q_2 y\right)\right] \\ &= T_1 y, \end{aligned} \tag{3.18}$$

and similarly we can derive that

$$T_1\left(T_2 y t^{\alpha-2}\right) = 0, \qquad T_2\left(T_1 y t^{\alpha-1}\right) = 0, \qquad T_2\left(T_2 y t^{\alpha-2}\right) = T_2 y. \tag{3.19}$$

So, for $y \in Z$, it follows from the four relations above that

$$\begin{aligned} Q^2 y &= Q\left((T_1 y)t^{\alpha-1} + (T_2 y)t^{\alpha-2}\right) \\ &= T_1\left((T_1 y)t^{\alpha-1} + (T_2 y)t^{\alpha-2}\right)t^{\alpha-1} + T_2\left((T_1 y)t^{\alpha-1} + (T_2 y)t^{\alpha-2}\right)t^{\alpha-2} \\ &= (T_1 y)t^{\alpha-1} + (T_2 y)t^{\alpha-2} \\ &= Qy, \end{aligned} \tag{3.20}$$

that is, the map $Q$ is idempotent. In fact $Q$ is a continuous linear projector.

Note that $y \in \operatorname{Im} L$ implies $Qy = 0$. Conversely, if $Qy = 0$, so

$$\begin{aligned} &\Lambda_4 Q_1 y - \Lambda_2 Q_2 y = 0, \\ &\Lambda_1 Q_2 y - \Lambda_3 Q_1 y = 0, \end{aligned} \tag{3.21}$$

but

$$\begin{vmatrix} \Lambda_4 & -\Lambda_2 \\ -\Lambda_3 & \Lambda_1 \end{vmatrix} = \Lambda_4\Lambda_1 - \Lambda_2\Lambda_3 \neq 0, \tag{3.22}$$

then we must have $Q_1 y = Q_2 y = 0$; since the condition (C2) holds, this can only be the case if $Q_1 y = Q_2 y = 0$, that is, $y \in \operatorname{Im} L$. In fact $\operatorname{Ker} Q = \operatorname{Im} L$, take $y \in Z$ in the form $y = (y - Qy) + Qy$ so that $y - Qy \in \operatorname{Ker} Q = \operatorname{Im} L, Qy \in \operatorname{Im} Q$, thus, $Z = \operatorname{Im} L + \operatorname{Im} Q$, Let $y \in \operatorname{Im} L \cap \operatorname{Im} Q$ and assume that $y = at^{\alpha-1} + bt^{\alpha-2}$ is not identically zero on $[0,1]$. Then, since $y \in \operatorname{Im} L$, from (3.5) and the condition (C2), we have

$$\begin{aligned} Q_1 y &= \int_0^1 (1-s)\left(as^{\alpha-1} + bs^{\alpha-2}\right)ds - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)\left(as^{\alpha-1} + bs^{\alpha-2}\right)ds = 0, \\ Q_2 y &= \int_0^1 (1-s)^{\alpha-1}\left(as^{\alpha-1} + bs^{\alpha-2}\right)ds - \sum_{i=1}^{m-2} b_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1}\left(as^{\alpha-1} + bs^{\alpha-2}\right)ds = 0. \end{aligned} \tag{3.23}$$

So

$$\begin{aligned} &a\Lambda_1 + b\Lambda_2 = 0, \\ &a\Lambda_3 + b\Lambda_4 = 0, \end{aligned} \tag{3.24}$$

but

$$\begin{vmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{vmatrix} = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3 \neq 0, \tag{3.25}$$

we derive $a = b = 0$, which is a contradiction. Hence, $\operatorname{Im} L \cap \operatorname{Im} Q = \{0\}$; thus $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$.

Now, $\dim\operatorname{Ker} L = 2 = \operatorname{codim}\operatorname{Im} L$ and so $L$ is a Fredholm operator of index zero.

Let $P : Y \to Y$ be defined by

$$Pu(t) = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} u(t)|_{t=0} t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2} u(t)|_{t=0} t^{\alpha-2}, \quad t \in [0,1]. \tag{3.26}$$

Note that $P$ is a continuous linear projector and

$$\operatorname{Ker} P = \left\{ u \in Y \mid D_{0+}^{\alpha-1} u(0) = D_{0+}^{\alpha-2} u(0) = 0 \right\}. \tag{3.27}$$

It is clear that $Y = \operatorname{Ker} P \oplus \operatorname{Ker} L$.

Note that the projectors $P$ and $Q$ are exact. Define by $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ by

$$K_p y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds = I_{0+}^{\alpha} y(t), \quad y \in \operatorname{Im} L. \tag{3.28}$$

Hence we have

$$D_{0+}^{\alpha-1} (K_p y) t = \int_0^t y(s) ds, \qquad D_{0+}^{\alpha-2} (K_p y) t = \int_0^t (t-s) y(s) ds, \tag{3.29}$$

then

$$\|K_p y\|_\infty \leq \frac{1}{\Gamma(\alpha)} \|y\|_1, \qquad \left\|D_{0+}^{\alpha-1}(K_p y)\right\|_\infty \leq \|y\|_1, \qquad \left\|D_{0+}^{\alpha-2}(K_p y)\right\|_\infty \leq \|y\|_1, \tag{3.30}$$

and thus

$$\|K_p y\|_{C^{\alpha-1}} \leq \left( \frac{1}{\Gamma(\alpha)} + 2 \right) \|y\|_1. \tag{3.31}$$

In fact, if $y \in \operatorname{Im} L$, then

$$(LK_p) y(t) = D_{0+}^{\alpha} I_{0+}^{\alpha} y(t) = y(t). \tag{3.32}$$

Also, if $u \in \operatorname{dom} L \cap \operatorname{Ker} P$, then

$$(K_p L) u(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \tag{3.33}$$

where

$$c_1 = \frac{D_{0+}^{\alpha-1}u(t)|_{t=0}}{\Gamma(\alpha)}, \qquad c_2 = \frac{D_{0+}^{\alpha-2}u(t)|_{t=0}}{\Gamma(\alpha-1)}, \qquad c_3 = \frac{I_{0+}^{3-\alpha}u(t)|_{t=0}}{\Gamma(\alpha-2)}, \tag{3.34}$$

and from the boundary value condition (1.5) and the fact that $u \in \operatorname{dom} L \cap \operatorname{Ker} P$, $Pu = 0$, $D_{0+}^{\alpha-1}u(t)|_{t=0} = D_{0+}^{\alpha-2}u(t)|_{t=0} = I_{0+}^{3-\alpha}u(t)\big|_{t=0} = 0$, we have $c_1 = c_2 = c_3 = 0$, thus

$$(K_pL)u(t) = u(t). \tag{3.35}$$

This shows that $K_p = [L|_{\operatorname{dom} L\cap \operatorname{Ker} P}]^{-1}$. The proof is complete. Using (3.16), we write

$$\begin{gathered} QNu(t) = (T_1Nu)t^{\alpha-1} + (T_2Nu)t^{\alpha-2}, \\ K_p(I-Q)Nu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[Nu(s) - QNu(s)]ds. \end{gathered} \tag{3.36}$$

□

By Lemma 2.8 and a standard method, we obtain the following lemma.

**Lemma 3.3** (see [16]). *For every given $e \in L^1[0,1]$, $K_p(I-Q)N : Y \to Y$ is completely continuous.*

Assume that the following conditions on the function $f(t,x,y,z)$ are satisfied.

(H1) There exist functions $a(t),\ b(t),\ c(t),\ d(t),\ r(t) \in L^1[0,1]$, and a constant $\theta \in [0,1)$ such that for all $(x,y,z) \in R^3$, $t \in [0,1]$, one of the following inequalities is satisfied:

$$\left|f(t,x,y,z)\right| \le a(t)|x| + b(t)\left|y\right| + c(t)|z| + d(t)|x|^\theta + r(t), \tag{3.37}$$

$$\left|f(t,x,y,z)\right| \le a(t)|x| + b(t)\left|y\right| + c(t)|z| + d(t)\left|y\right|^\theta + r(t), \tag{3.38}$$

$$\left|f(t,x,y,z)\right| \le a(t)|x| + b(t)\left|y\right| + c(t)|z| + d(t)|z|^\theta + r(t). \tag{3.39}$$

(H2) There exists a constant $A > 0$, such that for $x \in \operatorname{dom} L \setminus \operatorname{Ker} L$ satisfying $|D_{0+}^{\alpha-1}x(t)| + |D_{0+}^{\alpha-2}x(t)| > A$ for all $t \in [0,1]$, we have

$$Q_1Nx(t) \ne 0, \quad \text{or } Q_2Nx(t) \ne 0. \tag{3.40}$$

(H3) There exists a constant $B > 0$ such that for every $a, b \in R$ satisfying $a^2 + b^2 > B$ then either

$$aT_1N\left(at^{\alpha-1} + bt^{\alpha-2}\right) + bT_2N\left(at^{\alpha-1} + bt^{\alpha-2}\right) < 0, \tag{3.41}$$

or

$$aT_1N\left(at^{\alpha-1}+bt^{\alpha-2}\right)+bT_2N\left(at^{\alpha-1}+bt^{\alpha-2}\right)>0. \tag{3.42}$$

*Remark 3.4.* $T_1N(at^{\alpha-1}+bt^{\alpha-2})$ and $T_2N(at^{\alpha-1}+bt^{\alpha-2})$ from (H3) stand for the images of $u(t)=at^{\alpha-1}+bt^{\alpha-2}$ under the maps $T_1N$ and $T_2N$, respectively.

**Lemma 3.5.** *Suppose* (H1)-(H2) *hold, then the set*

$$\Omega_1=\{x\in \operatorname{dom}L\setminus \operatorname{Ker}L : Lx=\lambda Nx,\ \lambda\in[0,1]\} \tag{3.43}$$

*is bounded.*

*Proof.* Take

$$\Omega_1=\{x\in \operatorname{dom}L\setminus \operatorname{Ker}L : Lx=\lambda Nx,\ \lambda\in[0,1]\}. \tag{3.44}$$

Then for $x\in\Omega_1$, $Lx=\lambda Nx$ thus $\lambda\neq 0$, $Nx\in \operatorname{Im}L=\operatorname{Ker}Q$, and hence $QNx(t)$ for all $t\in[0,1]$. By the definition of $Q$, we have $Q_1Nx(t)=Q_2Nx(t)=0$. It follows from (H2) that there exists $t_0\in[0,1]$, such that $|D_{0+}^{\alpha-1}u(t_0)|+|D_{0+}^{\alpha-2}u(t_0)|\le A$.

Now

$$\begin{aligned} D_{0+}^{\alpha-1}x(t)&=D_{0+}^{\alpha-1}x(t_0)+\int_{t_0}^{t}D_{0+}^{\alpha}x(s)ds,\\ D_{0+}^{\alpha-2}x(t)&=D_{0+}^{\alpha-2}x(t_0)+\int_{t_0}^{t}D_{0+}^{\alpha-1}x(s)ds, \end{aligned} \tag{3.45}$$

and so

$$\begin{aligned} \left|D_{0+}^{\alpha-1}x(0)\right| &\le \left\|D_{0+}^{\alpha-1}x(t)\right\|_\infty\\ &\le \left|D_{0+}^{\alpha-1}x(t_0)\right|+\left\|D_{0+}^{\alpha-1}x\right\|_1\\ &\le A+\|Lx\|_1\le A+\|Nx\|_1,\\ \left|D_{0+}^{\alpha-2}x(0)\right| &\le \left\|D_{0+}^{\alpha-2}x(t)\right\|_\infty\\ &\le \left|D_{0+}^{\alpha-2}x(t_0)\right|+\left\|D_{0+}^{\alpha-1}x\right\|_\infty\\ &\le \left|D_{0+}^{\alpha-2}x(t_0)\right|+\left|D_{0+}^{\alpha-1}x(t_0)\right|+\left\|D_{0+}^{\alpha}x\right\|_1\\ &\le A+\|Lx\|_1\le A+\|Nx\|_1. \end{aligned} \tag{3.46}$$

Therefore, we have

$$
\begin{aligned}
\|Px\|_{C^{\alpha-1}} &= \left\| \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1}x(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2}x(0)t^{\alpha-2} \right\|_{C^{\alpha-1}} \\
&= \left\| \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1}x(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2}x(0)t^{\alpha-2} \right\|_{\infty} \\
&\quad + \left\| D_{0+}^{\alpha-1}x(0) \right\|_{\infty} + \left\| D_{0+}^{\alpha-1}x(0)t + D_{0+}^{\alpha-2}x(0) \right\|_{\infty} \\
&\leq \left(2 + \frac{1}{\Gamma(\alpha)}\right) \left| D_{0+}^{\alpha-1}x(0) \right| + \left(1 + \frac{1}{\Gamma(\alpha-1)}\right) \left| D_{0+}^{\alpha-2}x(0) \right| \\
&\leq \left(2 + \frac{1}{\Gamma(\alpha)}\right)(A + \|Nx\|_1) + \left(1 + \frac{1}{\Gamma(\alpha-1)}\right)(A + \|Nx\|_1).
\end{aligned}
\tag{3.47}
$$

Note that $(I-P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$ for all $x \in \Omega_1$. Then, by Lemma 3.2, we have

$$
\|(I-P)x\|_{C^{\alpha-1}} = \left\| K_p L(I-P)x \right\|_{C^{\alpha-1}} \leq \left(2 + \frac{1}{\Gamma(\alpha)}\right)\|Nx\|_1, \tag{3.48}
$$

so, we have

$$
\begin{aligned}
\|x\|_{C^{\alpha-1}} &\leq \|(I-P)x\|_{C^{\alpha-1}} + \|Px\|_{C^{\alpha-1}} \\
&\leq \left(2 + \frac{1}{\Gamma(\alpha)}\right)(A + \|Nx\|_1) + \left(1 + \frac{1}{\Gamma(\alpha-1)}\right)(A + \|Nx\|_1) \\
&\quad + \left(2 + \frac{1}{\Gamma(\alpha)}\right)\|Nx\|_1 \\
&= \left(\frac{2}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + 5\right)\|Nx\|_1 \\
&\quad + \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha-2)} + 3\right)A \\
&\leq m\|Nx\|_1 + nA,
\end{aligned}
\tag{3.49}
$$

where $m = ((2/\Gamma(\alpha)) + (1/\Gamma(\alpha-1)) + 5)$, $n = ((1/\Gamma(\alpha)) + (1/\Gamma(\alpha-1)) + (1/(\Gamma(\alpha-2)) + 3)$, $A$ is a constant. This is for all $x \in \Omega_1$. If the first condition of (H1) is satisfied, then, we have

$$
\begin{aligned}
\|x\|_{C^{\alpha-1}} &= \max\left\{ \|x\|_\infty, \left\| D_{0+}^{\alpha-1}x \right\|_\infty, \left\| D_{0+}^{\alpha-2}x \right\|_\infty \right\} \\
&\leq m\left[ \|a\|_1\|x\|_\infty + \|b\|_1 \left\| D_{0+}^{\alpha-1}x \right\|_\infty + \|c\|_1 \left\| D_{0+}^{\alpha-2}x \right\|_\infty \right. \\
&\qquad \left. + \|d\|_1 \left\| D_{0+}^{\alpha-2}x \right\|_\infty^\theta + D \right],
\end{aligned}
\tag{3.50}
$$

where $D = \|r\|_1 + \|e\|_1 + n/m$, and consequently, for

$$\|x\|_\infty \le \|x\|_{C^{\alpha-1}}, \qquad \left\|D_{0+}^{\alpha-1}x\right\|_\infty \le \|x\|_{C^{\alpha-1}}, \qquad \left\|D_{0+}^{\alpha-2}x\right\|_\infty \le \|x\|_{C^{\alpha-1}}, \tag{3.51}$$

so

$$\begin{gathered}
\|x\|_\infty \le \frac{m}{1-m\|a\|_1}\left[\|b\|_1\left\|D_{0+}^{\alpha-1}x\right\|_\infty + \|c\|_1\left\|D_{0+}^{\alpha-2}x\right\|_\infty + \|d\|_1\left\|D_{0+}^{\alpha-2}x\right\|_\infty^\theta + D\right], \\
\left\|D_{0+}^{\alpha-1}x\right\|_\infty \le \frac{m}{1-m\|a\|_1-m\|b\|_1}\left[\|c\|_1\left\|D_{0+}^{\alpha-2}x\right\|_\infty + \|d\|_1\left\|D_{0+}^{\alpha-2}x\right\|_\infty^\theta + D\right], \\
\left\|D_{0+}^{\alpha-2}x\right\|_\infty \le \frac{m}{1-m\|a\|_1-m\|b\|_1-m\|c\|_1}\left(\|d\|_1\left\|D_{0+}^{\alpha-2}x\right\|_\infty^\theta + D\right).
\end{gathered} \tag{3.52}$$

But $\theta \in [0,1)$ and $\|a\|_1 + \|b\|_1 + \|c\|_1 \le 1/m$, so there exists $A_1, A_2, A_3 > 0$ such that

$$\left\|D_{0+}^{\alpha-2}x\right\|_\infty \le A_1, \qquad \left\|D_{0+}^{\alpha-1}x\right\|_\infty \le A_2, \qquad \|x\|_\infty \le A_3. \tag{3.53}$$

Therefore, for all $x \in \Omega_1$,

$$\|x\|_{C^{\alpha-1}} = \max\left\{\|x\|_\infty, \left\|D_{0+}^{\alpha-1}x\right\|_\infty, \left\|D_{0+}^{\alpha-2}x\right\|_\infty\right\} \le \max\{A_1, A_2, A_3\}, \tag{3.54}$$

we can prove that $\Omega_1$ is also bounded.

If (3.38) or (3.39) holds, similar to the above argument, we can prove that $\Omega_1$ is bounded too. □

**Lemma 3.6.** *Suppose* (H3) *holds, then the set*

$$\Omega_2 = \{x \in \mathrm{Ker}L : Nx \in \mathrm{Im}L\} \tag{3.55}$$

*is bounded.*

*Proof.* Let

$$\Omega_2 = \{x \in \mathrm{Ker}\, L : Nx \in \mathrm{Im}\, L\}, \tag{3.56}$$

for $x \in \Omega_2$, $x \in \mathrm{Ker}\, L = \{x \in \mathrm{dom}\, L : x = at^{\alpha-1} + bt^{\alpha-2}, a, b \in R, t \in [0,1]\}$ and $QNx(t) = 0$; thus $T_1N(at^{\alpha-1}+bt^{\alpha-2}) = T_2N(at^{\alpha-1}+bt^{\alpha-2}) = 0$. By (H3), $a^2+b^2 \le B$, that is, $\Omega_2$ is bounded. □

**Lemma 3.7.** *Suppose* (H3) *holds, then the set*

$$\Omega_3 = \{x \in \mathrm{Ker}L : -\lambda Jx + (1-\lambda)QNx = 0, \quad \lambda \in [0,1]\} \tag{3.57}$$

*is bounded.*

*Proof.* We define the isomorphism $J : \operatorname{Ker} L \to \operatorname{Im} Q$ by

$$J\left(at^{\alpha-1} + bt^{\alpha-2}\right) = at^{\alpha-1} + bt^{\alpha-2}. \tag{3.58}$$

If the first part of (H3) is satisfied, let

$$\Omega_3 = \{x \in \operatorname{Ker} L : -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}. \tag{3.59}$$

For every $x = at^{\alpha-1} + bt^{\alpha-2} \in \Omega_3$,

$$\lambda\left(at^{\alpha-1} + bt^{\alpha-2}\right) = (1-\lambda)\left[T_1 N\left(at^{\alpha-1} + bt^{\alpha-2}\right)t^{\alpha-1} + T_2 N\left(at^{\alpha-1} + bt^{\alpha-2}\right)t^{\alpha-1}\right]. \tag{3.60}$$

If $\lambda = 1$, then $a = b = 0$, and if $a^2 + b^2 > B$, then by (H3)

$$\lambda\ \left(a^2 + b^2\right) = (1-\lambda)\left[aT_1 N\left(at^{\alpha-1} + bt^{\alpha-2}\right) + bT_2 N\left(at^{\alpha-1} + bt^{\alpha-2}\right)\right] < 0, \tag{3.61}$$

which, in either case, is a contradiction. If the other part of (H3) is satisfied, then we take

$$\Omega_3 = \{x \in \operatorname{Ker} L : \lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}, \tag{3.62}$$

and, again, obtain a contradiction. Thus, in either case

$$\begin{aligned}
\|x\|_{C^{\alpha-1}} &= \left\|at^{\alpha-1} + bt^{\alpha-2}\right\|_{C^{\alpha-1}} \\
&= \left\|at^{\alpha-1} + bt^{\alpha-2}\right\|_{\infty} + \|a\Gamma(\alpha)\|_{\infty} + \|a\Gamma(\alpha)t + b\Gamma(\alpha-1)\|_{\infty} \\
&\le (1 + 2\Gamma(\alpha))|a| + (1 + \Gamma(\alpha-1))|b| \\
&\le (2 + 2\Gamma(\alpha) + \Gamma(\alpha-1))B,
\end{aligned} \tag{3.63}$$

for all $x \in \Omega_3$, that is, $\Omega_3$ is bounded. □

*Remark 3.8.* Suppose the second part of (H3) holds, then the set

$$\Omega_3' = \{x \in \operatorname{Ker} L : \lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\} \tag{3.64}$$

is bounded.

**Theorem 3.9.** *If* (C1)-(C2) *and* (H1)–(H3) *hold, then the boundary value problem* (1.4)-(1.5) *has at least one solution.*

*Proof.* Set $\Omega$ to be a bounded open set of $Y$ such that $\cup_{i=1}^{3}\overline{\Omega} \subset \Omega$. It follows from Lemmas 3.2 and 3.3 that $L$ is a Fredholm operator of index zero, and the operator $K_p(I-Q)N : \overline{\Omega} \to Y$ is compact $N$, thus, is L-compact on $\overline{\Omega}$. By Lemmas 3.5 and 3.6, we get that the following two conditions are satisfied:

(i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [\operatorname{dom} L \setminus \operatorname{Ker} L \cap \partial\Omega] \times [0,1]$;

(ii) $Nx \notin \operatorname{Im} L$, for every $x \in \operatorname{Ker} L \cap \partial\Omega$.

Finally, we will prove that (iii) of Lemma 1.1 is satisfied. Let $H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx$, where $I$ is the identity operator in the Banach space $Y$. According to Lemma 3.7 (or Remark 3.8), we know that $H(x,\lambda) \neq 0$, for all $x \in \partial\Omega \cap \operatorname{Ker} L$, and thus, by the homotopy property of degree,

$$\begin{aligned}
\deg(QN|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) &= \deg(H(\cdot,0), \operatorname{Ker} L \cap \Omega, 0) \\
&= \deg(H(\cdot,1), \operatorname{Ker} L \cap \Omega, 0) \\
&= \deg(\pm I, \operatorname{Ker} L \cap \Omega, 0) \\
&= \operatorname{sgn}\left[\pm \begin{vmatrix} \dfrac{\Lambda_4}{\Lambda} & \dfrac{-\Lambda_2}{\Lambda} \\ \dfrac{-\Lambda_3}{\Lambda} & \dfrac{\Lambda_1}{\Lambda} \end{vmatrix}\right] = \operatorname{sgn}\left(\pm \frac{\Lambda_1\Lambda_4 - \Lambda_2\Lambda_3}{\Lambda}\right) \\
&= \pm 1 \neq 0.
\end{aligned} \tag{3.65}$$

Then by Lemma 1.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, so the boundary value problem (1.4) and (1.5) has at least one solution in the space $C^{\alpha-1}[0,1]$. The proof is finished. □

## 4. An Example

Let us consider the following boundary value problem:

$$\begin{gathered}
D_{0+}^{5/2}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)\right) + e(t), \quad 0 < t < 1, \\
I_{0+}^{3-\alpha}u(t)|_{t=0} = 0, \qquad D_{0+}^{1/2}u(1) = 2D_{0+}^{1/2}u\left(\frac{2}{3}\right) - D_{0+}^{1/2}u\left(\frac{1}{3}\right), \qquad u(1) = \frac{64}{5}u\left(\frac{1}{4}\right) - \frac{81}{5}u\left(\frac{1}{9}\right),
\end{gathered} \tag{4.1}$$

where

$$\begin{aligned}
f\left(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t)\right) = \frac{1}{40}\sin(u(t)) + \frac{1}{20}D_{0+}^{3/2}u(t) + \frac{1}{20}D_{0+}^{1/2}u(t) \\
+ 5\cos\left(D_{0+}^{1/2}u(t)\right)^{1/5}.
\end{aligned} \tag{4.2}$$

Corresponding to the problem (1.4)-(1.5), we have that $e(t) = 1 + 3\sin^2 t$, $\alpha = 5/2$, $a_1 = -1$, $a_2 = 2$, $\xi_1 = 1/3$, $\xi_2 = 2/3$, $b_1 = 64/5$, $b_2 = -81/5$, $\eta_1 = 1/4$, $\eta_2 = 1/9$ and

$$f(t, x, y, z) = \frac{1}{40}\sin x + \frac{1}{20}y + \frac{1}{20}z + 5\cos(z)^{1/5}, \tag{4.3}$$

then there is

$$\begin{gathered}
a_1 + a_2 = 1, \qquad a_1\xi_1 + a_2\xi_2 = 1, \\
b_1\eta_1^{3/2} + b_2\eta_2^{3/2} = 1, \qquad b_1\eta_1^{1/2} + b_2\eta_2^{1/2} = 1, \\
\Lambda_1 = \frac{4}{35}\left(1 - \sum_{i=1}^{2} a_i\xi_i^{7/2}\right), \qquad \Lambda_2 = \frac{4}{15}\left(1 - \sum_{i=1}^{2} a_i\xi_i^{5/2}\right), \\
\Lambda_3 = \frac{(\Gamma(5/2))^2}{24}\left[1 - \sum_{i=1}^{2} b_i\eta_i^4\right], \qquad \Lambda_4 = \frac{1}{6}\frac{\Gamma(5/2)\Gamma(3/2)}{24}\left[1 - \sum_{i=1}^{2} b_i\eta_i^3\right], \\
\Lambda = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3 \neq 0, \\
|f(t, x, y, z)| \leq \frac{1}{40}|x| + \frac{1}{20}|y| + \frac{1}{20}|z| + 5|z|^{1/5}.
\end{gathered} \tag{4.4}$$

Again, taking $a = 1/40$, $b = c = 1/20$, then

$$\begin{gathered}
\|a\|_1 + \|b\|_1 + \|c\|_1 = 1/8, \\
\frac{1}{m} = \frac{1}{(2/\Gamma(\alpha)) + (1/\Gamma(\alpha - 1)) + 5} \approx 0.131,
\end{gathered} \tag{4.5}$$

therefore

$$\|a\|_1 + \|b\|_1 + \|c\|_1 < \frac{1}{m}. \tag{4.6}$$

Take $A = 181$, $B = 81$. By simple calculation, we can get that (C1)-(C2) and (H1)–(H3) hold. By Lemma 1.1, we obtain that (4.1) has at least one solution.

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## References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B. V., Amsterdam, The Netherland, 2006.

[2] J. Sabatier, O.P. Agrawal, and J. A. T. Machado, Eds., *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherland, 2007.

[3] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, Academic Publishers, Oxford, UK, 2009.

[4] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press Inc., San Diego, Calif, USA, 1999.

[5] V. Lakshmikantham and A. S. Vatsala, "Theory of fractional differential inequalities and applications," *Communications in Applied Analysis*, vol. 11, no. 3-4, pp. 395–402, 2007.

[6] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 2, pp. 916–924, 2010.

[7] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[8] Z. Bai and T. Qiu, "Existence of positive solution for singular fractional differential equation," *Applied Mathematics and Computation*, vol. 215, no. 7, pp. 2761–2767, 2009.

[9] X. Su, "Boundary value problem for a coupled system of nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 64–69, 2009.

[10] N. Kosmatov, "A boundary value problem of fractional order at resonance," *Electronic Journal of Differential Equations*, vol. 135, pp. 1–10, 2010.

[11] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 8, pp. 2677–2682, 2008.

[12] C. Bai, "Positive solutions for nonlinear fractional differential equations with coefficient that changes sign," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 64, no. 4, pp. 677–685, 2006.

[13] M. Benchohra and F. Berhoun, "Impulsive fractional differential equations with variable times," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1245–1252, 2010.

[14] Z. Hu and W. Liu, "Solvability for fractional order boundary value problem at resonance," *Boundary Value Problem*, vol. 20, pp. 1–10, 2011.

[15] Y. Zhang and Z. Bai, "Existence of positive solutions for s nonlinear fractional three-point boundary value problen at resonance," *Journal of Applied Mathematics and Computing*, vol. 36, no. 1-2, pp. 417–440, 2010.

[16] Z. Bai, "On solutions of some fractional m-point boundary value problems at resonance," *Electronic Journal of Qualitative Theory of Differential Equations*, p. No. 37, 15, 2010.

[17] J. Mawhin, "Topological degree and boundary value problems for nonlinear differential equations," in *Topological Methods for Ordinary Differential Equations*, P. M. Fitzpatrick, M. Martelli, J. Mawhin, and R. Nuss-baum, Eds., vol. 1537 of *Lecture Notes in Mathematics*, pp. 74–142, Springer, Berlin, Germany, 1991.

[18] A. A. Kilbas and J. J. Trujillo, "Differential equations of fractional order: methods, results and problems. I," *Applicable Analysis*, vol. 78, no. 1-2, pp. 153–192, 2001.