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Research Article

Multiple Solutions for Nonlinear Doubly Singular Three-Point Boundary Value Problems with Derivative Dependence

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We study the existence of multiple nonnegative solutions for the doubly singular three-point boundary value problem with derivative dependent data function $-(p(t)y'(t))' = q(t)f(t,y(t),p(t)y'(t)), 0 < t < 1, y(0) = 0, y(1) = \alpha_1 y(\eta)$. Here, $p \in C[0,1] \cap C^1(0,1]$ with $p(t) > 0$ on $(0,1]$ and $q(t)$ is allowed to be discontinuous at $t = 0$. The fixed point theory in a cone is applied to achieve new and more general results for existence of multiple nonnegative solutions of the problem. The results are illustrated through examples.

1. Introduction

In this paper, we consider the following three-point boundary value problem of Sturm-Liouville type:

$$-\left(p(t)y'(t)\right)' = q(t)f\left(t,y(t),p(t)y'(t)\right), \quad 0 < t < 1, \tag{1.1}$$

with boundary conditions

$$y(0) = 0, \qquad y(1) = \alpha_1 y(\eta), \tag{1.2}$$

where $0 < \alpha_1 < (h(1))/(h(\eta))$ and $h(t) = \int_0^t (1/p(x))dx$.

Throughout this paper, we assume the following conditions on the functions $p(t)$, $q(t)$, and $f(t,y,py')$:

(E1) $p \in C[0,1] \cap C^1(0,1]$ with $p(t) > 0$ on $(0,1]$ and $1/p \in L^1(0,1)$;

(E2) $q(t) > 0$, q is not identically zero on $[0,1]$ and $q \in L^1(0,1)$;

(E3) $f \in C([0,1] \times [0,\infty) \times \mathbb{R}, [0,\infty))$ and f is not identically zero.

Note that condition (E2) allows $q(t)$ be discontinuous at $t = 0$, and if $p(0) = 0$, then the differential equation (1.1) is called doubly singular [1].

Nonlocal boundary value problem have variety of applications in the area of applied mathematics and physical sciences. The design of a large size bridge with multipoint supports can be considered as an application of these types of boundary value problem [2]. Some more applications can be found in [3–5] and the references therein. Recently, motivated by the wide application of boundary value problems in physical and applied mathematics, the study of multipoint boundary value problems has received increasing interest (see [2, 6–12] and the references therein).

Nonsingular multipoint boundary value problems have been extensively studied in literature, see [13–16] for derivative dependent data function $f(t,y,z)$ and [8, 10, 12] for derivative independent data function $f(t,y,z) = f(t,y)$.

Some attention has been devoted to singular multipoint boundary value problems (see [17, 18] and the references therein). When $p(t) = 1$ and $q(t)f(t,y,z)$ may have singularity at $t = 0$, $t = 1$, $y = 0$ and $z = 0$, differential equation (1.1) with boundary conditions $y'(0) = 0$, $y(1) = \alpha_1 y(\eta)$ is considered by Chen et al. [17] and Agarwal et al. [18]. Chen et al. proved the existence of at least one positive solution while Agarwal et al. established that this problem may have at least two positive solutions and also may have no positive solutions under some conditions on $q(t)$ and $f(t,y,z)$.

Bai and Ge [19] have generalized the Leggett-Williams fixed point theory and applied to

$$\begin{aligned} y'' + q(t)f\left(t,y,y'\right) = 0, \quad t \in (0,1), \\ y(0) = 0, \qquad y(1) = 0, \end{aligned} \tag{1.3}$$

to achieve at least three positive solutions of the two-point boundary value problem.

In this work, we consider the problem (1.1)-(1.2) with unbounded coefficient of y' along with singularity in the data function $f(t,y,z)$.

Existence of nonnegative solution(s) of the problem (1.1)-(1.2) may be established either directly or by reducing the problem to

$$y'' + q(t)f\left(t,y,y'\right) = 0, \tag{1.4}$$

and applying the existing results. But direct consideration of the problem provides better results, especially as the order of singularity increases. This may be demonstrated by the following simple linear three-point boundary value problem:

$$\begin{aligned} -\left(t^r y'(t)\right)' = t^{r-1}\left[2(r+1)t - \frac{5}{4}\, r\right], \quad 0.5 \le r < 1,\ t \in (0,1), \\ y(0) = 0, \qquad y(1) = y\left(\frac{1}{4}\right). \end{aligned} \tag{1.5}$$

The problem (1.5) can be reduced to the following boundary value problem:

$$-v''(x) = \frac{1}{(1-r)^2} x^{(2r-1)/(1-r)} \left[2(r+1)x^{1/(1-r)} - \frac{5}{4}r\right], \quad x \in (0,1),$$
$$v(0) = 0, \qquad v(1) = v\left[\left(\frac{1}{4}\right)^{(1-r)}\right], \tag{1.6}$$

by change of variable $x = (1-r)\int_0^t \tau^{-r} d\tau = t^{1-r}$.

Now we apply the result (Theorem 4.2) of this work to the problem (1.5) and conclude that the problem has at least one nonnegative solution $y(t)$ with

$$\sup_{t\in[0,1]} |y(t)| \le \frac{3}{2}\left[\frac{3(1-r)}{4-4^r}\right]^{(1-r)/r}\left[\frac{3}{4}r + 2\right]. \tag{1.7}$$

Further, for $p(t) = 1$, Theorems 4.2 and 4.3 may be regarded as extension of Theorem 3.1 in [19] for three-point singular boundary value problem. Now applying Theorem 4.2 with $p(t) = 1$ to the reduced problem (1.6), we get that the problem (1.5) has at least one nonnegative solution $y(t)$ with

$$\sup_{t\in[0,1]} |y(t)| = \sup_{t\in[0,1]} |v(t)| \le \frac{3}{4-4^r}\left[\frac{3(1-r)}{4-4^r}\right]^{(1-r)/r}\left[\frac{3}{4}r + 2\right]. \tag{1.8}$$

Now as r approaches to one, that is, the order of singularity increases, the upper bound for $\sup_{t\in[0,1]}|y(t)|$ in (1.8) approaches to ∞ while in (1.7) approaches to 4.125, which can be seen from Figure 1. As smaller upper bound for $\sup_{t\in[0,1]}|y(t)|$ will enable to find nonnegative solution(s) faster and hence will be helpful in constructing efficient numerical algorithms to find multiple nonnegative solutions, thus it is justified to consider the singular problem directly. A detailed working is given in Example 5.1.

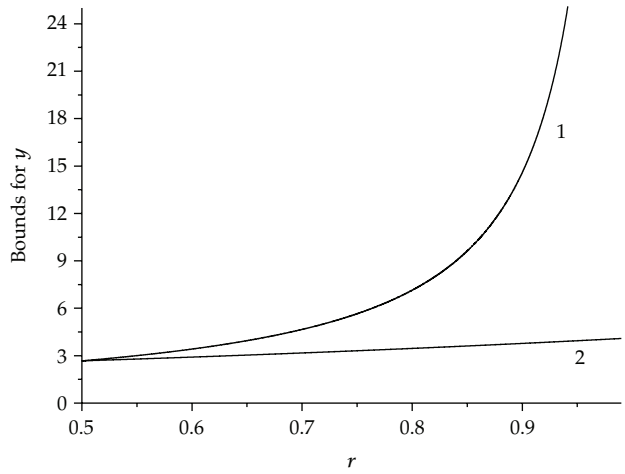
In this work, we are concerned with existence of multiple nonnegative solutions of the three-point doubly singular boundary value problem (1.1)-(1.2). To achieve this, we use generalized Leggett-Williams fixed point theorem established by Bai and Ge [19].

For this purpose, we first establish certain properties of Green's function of the corresponding homogeneous boundary value problem. Then fixed point theorem of functional type (generalized Leggett-Williams fixed point theorem) is applied to yield multiple nonnegative solutions for the boundary value problem (1.1)-(1.2).

We organize this work as follows. In Section 2, we present some definitions and basic results required for this work. Section 3 deals with nonnegativity of Green's function and some basic properties. Section 4 is devoted to existence of at least one and three or odd number of nonnegative solutions. In Section 5, we demonstrate the results through examples.

2. Background and Definitions

The proof of main results is based on fixed point theorem of functional type in a cone given by Bai and Ge [19], which deals with three fixed points of completely continuous nonlinear



(1) When singular problem is reduced to regular problem

(2) When singular problem is solved directly

Figure 1: Variation of bounds for y in both cases.

operators defined in a cone of an ordered Banach space. In this section, we provide some background material from the theory of cone in Banach spaces to make the paper self-contained.

Definition 2.1. A subset D of Banach space E is said to be retract of E if $\exists$ a continuous map $r : E \to D$ such that $r(x) = x$ for every $x \in D$.

Corollary 2.2. *Every close convex set of a Banach space is a retract of Banach space.*

Definition 2.3. Let E be a Banach space, $P \subset E$ is nonempty convex, closed set, P is said to be cone provided that

(1) $\lambda u \in P$ for all $\lambda \geq 0,\ u \in P$, and

(2) $u \in P,\ -u \in P$ implies $u = 0$.

Note. From Corollary 2.2, a cone P of Banach Space E is retract of E.

Definition 2.4. A subset R of Banach space X is called relatively compact if $\overline{R}$ (closure of R) is compact.

Definition 2.5. Consider two Banach spaces X and Y, a subset Ω of X, and a map $T : \Omega \to Y$. Then T is said to be completely continuous operator if

(1) T is continuous, and

(2) T maps bounded subset of Ω into relatively compact sets.

Definition 2.6. The map α is said to be a nonnegative continuous convex functional on P provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \le t\alpha(x) + (1-t)\alpha(y), \tag{2.1}$$

for all $x, y \in P$ and $0 \le t \le 1$. Similarly, the map γ is said to be a nonnegative continuous concave functional on P provided that $\gamma : P \to [0,\infty)$ is continuous and

$$\gamma\big(tx + (1-t)y\big) \ge t\gamma(x) + (1-t)\gamma(y), \tag{2.2}$$

for all $x, y \in P$, and $0 \le t \le 1$.

Definition 2.7. Suppose $\alpha,\ \beta : P \to [0,\infty)$ are two continuous convex functionals satisfying

$$\|x\| \le M \max\{\alpha(x), \beta(x)\}, \quad \text{for } x \in P, \tag{2.3}$$

where M is positive constant, and

$$\Omega = \{x \in P : \alpha(x) < r, \beta(x) < L\} \ne \phi \quad \text{for } r > 0,\ L > 0. \tag{2.4}$$

From (2.3) and (2.4), Ω is a bounded nonempty open subset of P.

Definition 2.8. Let $r > a > 0$, $L > 0$ be given constants, $\alpha,\ \beta : P \to [0,\infty)$ two nonnegative continuous convex functionals satisfying (2.3) and (2.4), and ψ a nonnegative continuous concave functional on the cone P. Then bounded convex sets are defined as

$$\begin{aligned} P\left(\alpha^r, \beta^L\right) &= \{x \in P : \alpha(x) < r,\ \beta(x) < L\}, \\ \overline{P}\left(\alpha^r, \beta^L\right) &= \{x \in P : \alpha(x) \le r,\ \beta(x) \le L\}, \\ P\left(\alpha^r, \beta^L, \psi_a\right) &= \{x \in P : \alpha(x) < r,\ \beta(x) < L,\ \psi(x) > a\}, \\ \overline{P}\left(\alpha^r, \beta^L, \psi_a\right) &= \{x \in P : \alpha(x) \le r,\ \beta(x) \le L,\ \psi(x) \ge a\}. \end{aligned} \tag{2.5}$$

Theorem 2.9 (see [20]). *Let X be retract of real Banach space E. Then for every bounded relatively open subset U of X and every completely continuous operator $T : \overline{U} \to X$ which has no fixed point on ∂U (relative to X), there exists an integer $i(T, U, X)$ such that if $i(T, U, X) \ne 0$, then T has at least one fixed point in U. Moreover, $i(T, U, X)$ is uniquely defined.*

Theorem 2.10 (see [20]). *Let E be Banach space, X retract of E, X_1 a bounded convex retract of X, and $U \subset X$ nonempty open subset, such that $U \subset X_1$. If $T : X_1 \to X$ is completely continuous, $T(X_1) \subset X_1$, such that there is no fixed point of T in $X_1 \setminus U$, then $i(T, U, X) = 1$.*

Theorem 2.11 ((see [19]) (fixed point theorem of functional type)). *Let E be Banach space, $P \subset E$ a cone, and $c \ge b > a > d > 0$, $L_2 \ge L_1 > 0$ given constants. Assume that α, β*

are nonnegative continuous convex functionals on P such that (2.3) *and* (2.4) *are satisfied. ψ is a nonnegative continuous concave functional on P such that $\psi(x) \le \alpha(x)$ for all $x \in \overline{P}(\alpha^c, \beta^{L_2})$ and let $T : \overline{P}(\alpha^c, \beta^{L_2}) \to \overline{P}(\alpha^c, \beta^{L_2})$ be a completely continuous operator. Suppose that*

(1) $\{x \in \overline{P}(\alpha^b, \beta^{L_2}, \psi_a) : \psi(x) > a\} \neq \Phi$ *and* $\psi(Tx) > a$ *for* $x \in \overline{P}(\alpha^b, \beta^{L_2}, \psi_a)$,

(2) $\alpha(Tx) < d$, $\beta(Tx) < L_1$ *for all* $x \in \overline{P}(\alpha^d, \beta^{L_1})$,

(3) $\psi(Tx) > a$ *for all* $x \in \overline{P}(\alpha^c, \beta^{L_2}, \psi_a)$ *with* $\alpha(Tx) > b$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P}(\alpha^c, \beta^{L_2})$ such that

$$x_1 \in P\left(\alpha^d, \beta^{L_1}\right), \qquad x_2 \in \left\{x \in \overline{P}\left(\alpha^c, \beta^{L_2}, \psi_a\right) : \psi(x) > a\right\},$$
$$x_3 \in \overline{P}\left(\alpha^c, \beta^{L_2}\right) \setminus \left(\overline{P}\left(\alpha^c, \beta^{L_2}, \psi_a\right) \cup \overline{P}\left(\alpha^d, \beta^{L_1}\right)\right). \tag{2.6}$$

3. Some Preliminary Results

In this section, we construct the Green's function and establish some properties, required to establish the main results in Section 4.

Lemma 3.1. *The Green's function for the following boundary value problem:*

$$B.C. := y(0) = 0, \quad y(1) = \alpha_1 y(\eta) \Big\}, \quad t \in (0,1) \tag{3.1}$$

is given by

$$G(t,s) = \begin{cases} G_1(t,s), & 0 \le s \le \min\{t, \eta\} < 1; \\ G_2(t,s), & 0 \le t \le s \le \eta; \\ G_3(t,s), & \eta \le s \le t \le 1; \\ G_4(t,s), & 0 < \max\{\eta, t\} \le s \le 1. \end{cases} \tag{3.2}$$

Here

$$G_1(t,s) = \frac{h(s)}{\delta}[\delta - h(t) + \alpha_1 h(t)],$$
$$G_2(t,s) = \frac{h(t)}{\delta}[\delta - h(s) + \alpha_1 h(s)],$$
$$G_3(t,s) = \frac{1}{\delta}\left[h(t)\{\alpha_1 h(\eta) - h(s)\} + \delta h(s)\right], \tag{3.3}$$
$$G_4(t,s) = \frac{h(t)}{\delta}[h(1) - h(s)],$$
$$\delta = h(1) - \alpha_1 h(\eta) > 0, \quad h(t) = \int_0^t \frac{1}{p(x)}\, dx, \quad 0 < \alpha_1 < \frac{h(1)}{h(\eta)}.$$

Proof. Consider the following linear differential equation:

$$-\left(p(t)y'(t)\right)' = q(t)F(t), \tag{3.4}$$

where $F \in C([0,1],[0,\infty))$. Integrating the above differential equation twice first from t to 1 and then from 0 to t, changing the order of integration, and applying the boundary conditions, we get

$$\begin{aligned} y(t) = {} & \int_0^t h(s)q(s)F(s)ds + h(t)\int_t^1 q(s)F(s)ds \\ & + \frac{h(t)}{\delta}\left[\alpha_1\int_0^\eta h(s)q(s)F(s)ds - \int_0^1 h(s)q(s)F(s)ds + \alpha_1 h(\eta)\int_\eta^1 q(s)F(s)ds\right]. \end{aligned} \tag{3.5}$$

For $t \in [0,\eta]$, $y(t)$ can be written as

$$\begin{aligned} y(t) = {} & \int_0^t \frac{h(s)}{\delta}[\delta - h(t) + \alpha_1 h(t)]q(s)F(s)ds \\ & + \int_t^\eta \frac{h(t)}{\delta}[\delta - h(s) + \alpha_1 h(s)]q(s)F(s)ds + \int_\eta^1 \frac{h(t)}{\delta}[h(1) - h(s)]q(s)F(s)ds, \end{aligned} \tag{3.6}$$

or

$$y(t) = \int_0^t G_1(t,s)q(s)F(s)ds + \int_t^\eta G_2(t,s)q(s)F(s)ds + \int_\eta^1 G_4(t,s)q(s)F(s)ds. \tag{3.7}$$

Similarly, for $t \in [\eta,1]$, $y(t)$ can be written as

$$\begin{aligned} y(t) = {} & \int_0^\eta \frac{h(s)}{\delta}[\delta - h(t) + \alpha_1 h(t)]q(s)F(s)ds \\ & + \int_\eta^t \frac{1}{\delta}\left[h(t)\left\{\alpha_1 h(\eta) - h(s)\right\} + \delta h(s)\right]q(s)F(s)ds + \int_t^1 \frac{h(t)}{\delta}[h(1) - h(s)]q(s)F(s)ds, \end{aligned} \tag{3.8}$$

or

$$y(t) = \int_0^\eta G_1(t,s)q(s)F(s)ds + \int_\eta^t G_3(t,s)q(s)F(s)ds + \int_t^1 G_4(t,s)q(s)F(s)ds. \tag{3.9}$$

From (3.7) and (3.9), we may write

$$y(t) = \int_0^1 G(t,s)q(s)F(s)ds, \tag{3.10}$$

where $G(t,s)$ is given in the lemma. It is easy to see that $G(t,s)$ satisfies all the properties of Green's function. Hence $G(t,s)$ is the Green's function for the boundary value problem (3.1). □

Lemma 3.2. *The Green's function $G(t,s)$ satisfies the following properties:*

(i) $\max_{t\in[0,1]} p(t)(\partial G(t,s)/\partial t) < \infty$,

(ii) $G(t,s) \geq 0$ *for all* $(t,s) \in \{[0,1]\times[0,1]\}$,

(iii) *there exist a constant* λ *in (0,1) such that* $\min_{t\in[\eta,1]} G(t,s) \geq \lambda \max_{t\in[0,1]} G(t,s)$ *for* $s \in [0,1]$, *where*

$$\lambda = \begin{cases} \min\left\{\dfrac{\alpha_1[h(1)-h(\eta)]}{h(1)-\alpha_1 h(\eta)}, \dfrac{\alpha_1 h(\eta)}{h(1)}\right\}, & 0 < \alpha_1 \leq 1, \\ \min\left\{\dfrac{h(1)-\alpha_1 h(\eta)}{\alpha_1[h(1)-h(\eta)]}, \dfrac{h(\eta)}{\alpha_1 h(1)}\right\}, & 1 \leq \alpha_1 < \dfrac{h(1)}{h(\eta)}. \end{cases} \tag{3.11}$$

Proof. (i)

$$p(t)\frac{\partial G(t,s)}{\partial t} = \begin{cases} \dfrac{h(s)}{\delta}[\alpha_1 - 1], & 0 \leq s \leq \min\{t,\eta\} < 1; \\ \dfrac{1}{\delta}[\delta - h(s) - \alpha_1 h(s)], & 0 \leq t \leq s \leq \eta; \\ \dfrac{1}{\delta}[\alpha_1 h(\eta) - h(s)], & \eta \leq s \leq t \leq 1; \\ \dfrac{1}{\delta}[h(1) - h(s)], & 0 < \max\{\eta,t\} \leq s \leq 1. \end{cases} \tag{3.12}$$

Since $p(t)(\partial G(t,s)/\partial t)$ is independent of t, therefore $\max_{t\in[0,1]} p(t)(\partial G(t,s)/\partial t) < \infty$.

(ii) For $t < \eta$, $\alpha_1[h(t) - h(\eta)] > h(1)/h(\eta)[h(t) - h(\eta)]$ and

$$G_1(t,s) \geq \frac{1}{\delta} h(s)h(t)\left[\frac{h(1)}{h(\eta)} - 1\right] \geq 0, \tag{3.13}$$

it is easy to show that $G_1(t,s) \geq 0$, for $t \geq \eta$.

Next we show $G_2(t,s) \geq 0$, $G_3(t,s) \geq 0$ and $G_4(t,s) \geq 0$ as follows:

$$\begin{aligned} G_2(t,s) &\geq \frac{h(t)}{\delta h(\eta)}\left[h(s)\left(h(1) - h(\eta)\right)\right] \geq 0; \\ G_3(t,s) &= \frac{1}{\delta}\left[h(s)\{h(1) - h(t)\} + \alpha_1 h(\eta)\{h(t) - h(s)\}\right] \geq 0, \\ G_4(t,s) &= \frac{h(t)}{\delta}[h(1) - h(s)] \geq 0. \end{aligned} \tag{3.14}$$

Thus $G(t,s) \geq 0$ for all $(t,s) \in \{[0,1]\times[0,1]\}$.

(iii) We prove the inequality for the following cases:
(a) $s \in [0,\eta]$ and (b) $s \in [\eta,1]$,
(a) for $s \in [0,\eta]$, we further divide this case in two parts as follows.
(1) When $0 < \alpha_1 \leq 1$, (2) when $1 \leq \alpha_1 < h(1)/h(\eta)$.
Case 1 (For $0 < \alpha_1 \leq 1$). It is easy to see that

$$G_1(t,s) = \frac{h(s)}{\delta}\left[\left(h(1) - \alpha_1 h(\eta)\right) + h(t)(\alpha_1 - 1)\right] \tag{3.15}$$

implies

$$\max_{t\in[0,1]} G_1(t,s) = \frac{h(s)}{\delta}\left[h(1) - \alpha_1 h(\eta)\right], \qquad \min_{t\in[\eta,1]} G_1(t,s) = \frac{\alpha_1 h(s)}{\delta}\left[h(1) - h(\eta)\right]. \tag{3.16}$$

Next,

$$\begin{aligned} G_2(t,s) &= \frac{h(t)}{\delta}\left[h(1) - \alpha_1 h(\eta) + h(s)(\alpha_1 - 1)\right], \\ &\leq \frac{h(s)}{\delta}\left[h(1) - \alpha_1 h(\eta)\right], \quad \text{as } h(t) \leq h(s). \end{aligned} \tag{3.17}$$

Thus for $0 < \alpha_1 \leq 1$,

$$\max_{t\in[0,1]} G(t,s) = \frac{h(s)}{\delta}\left[h(1) - \alpha_1 h(\eta)\right], \qquad \min_{t\in[\eta,1]} G(t,s) = \frac{\alpha_1 h(s)}{\delta}\left[h(1) - h(\eta)\right]. \tag{3.18}$$

Case 2 (For $1 \leq \alpha_1 < h(1)/h(\eta)$). It is easy to see that

$$\max_{t\in[0,1]} G_1(t,s) = \frac{\alpha_1 h(s)}{\delta}\left[h(1) - h(\eta)\right], \qquad \min_{t\in[\eta,1]} G_1(t,s) = \frac{h(s)}{\delta}\left[h(1) - h(\eta)\right]. \tag{3.19}$$

Next,

$$\begin{aligned} G_2(t,s) &= \frac{h(t)}{\delta}\left[h(1) - \alpha_1 h(\eta) + h(s)(\alpha_1 - 1)\right], \\ &\leq \frac{\alpha_1 h(s)}{\delta}\left[h(1) - h(\eta)\right], \quad \text{as } h(t) \leq h(s) \leq h(\eta). \end{aligned} \tag{3.20}$$

Thus for $1 \leq \alpha_1 < h(1)/h(\eta)$,

$$\max_{t\in[0,1]} G(t,s) = \frac{\alpha_1 h(s)}{\delta}\left[h(1) - h(\eta)\right], \qquad \min_{t\in[\eta,1]} G(t,s) = \frac{h(s)}{\delta}\left[h(1) - h(\eta)\right]. \tag{3.21}$$

Combining (3.18) and (3.21), we may write for $s \in [0,\eta]$,

$$\max_{t\in[0,1]} G(t,s) = \begin{cases} \dfrac{h(s)}{\delta}\left[h(1)-\alpha_1 h(\eta)\right], & 0<\alpha_1\le 1, \\ \dfrac{\alpha_1 h(s)}{\delta}\left[h(1)-h(\eta)\right], & 1\le\alpha_1<\dfrac{h(1)}{h(\eta)}, \end{cases}$$
$$\min_{t\in[\eta,1]} G(t,s) = \begin{cases} \dfrac{\alpha_1 h(s)}{\delta}\left[h(1)-h(\eta)\right], & 0<\alpha_1\le 1, \\ \dfrac{h(s)}{\delta}\left[h(1)-h(\eta)\right], & 1\le\alpha_1<\dfrac{h(1)}{h(\eta)}. \end{cases} \tag{3.22}$$

(b) For $s \in [\eta,1]$. For this case, $G_3(t,s)$ and $G_4(t,s)$ are considered. From (3.2), it can be easily seen that for $s \in [\eta,1]$,

$$\max_{t\in[0,1]} G(t,s) = \frac{h(1)}{\delta}[h(1)-h(s)], \qquad \min_{t\in[\eta,1]} G(t,s) = \frac{h(\eta)}{\delta}[h(1)-h(s)]. \tag{3.23}$$

Thus from (3.22) and (3.23), we get

$$\max_{t\in[0,1]} G(t,s) = \begin{cases} s\in[0,\eta]: \begin{cases} \dfrac{h(s)}{\delta}\left[h(1)-\alpha_1 h(\eta)\right], & 0<\alpha_1\le 1, \\ \dfrac{\alpha_1 h(s)}{\delta}\left[h(1)-h(\eta)\right], & 1\le\alpha_1<\dfrac{h(1)}{h(\eta)}, \end{cases} \\ s\in[\eta,1]: \left\{ \dfrac{h(1)}{\delta}[h(1)-h(s)], \quad 0<\alpha_1<\dfrac{h(1)}{h(\eta)}. \right. \end{cases}$$
$$\min_{t\in[\eta,1]} G(t,s) = \begin{cases} s\in[0,\eta]: \begin{cases} \dfrac{\alpha_1 h(s)}{\delta}\left[h(1)-h(\eta)\right], & 0<\alpha_1\le 1, \\ \dfrac{h(s)}{\delta}\left[h(1)-h(\eta)\right], & 1\le\alpha_1<\dfrac{h(1)}{h(\eta)}, \end{cases} \\ s\in[\eta,1]: \left\{ \dfrac{h(\eta)}{\delta}[h(1)-h(s)], \quad 0<\alpha_1<\dfrac{h(1)}{h(\eta)}. \right. \end{cases} \tag{3.24}$$

From (3.24),

$$\min_{t\in[\eta,1]} G(t,s) \ge \begin{cases} \min\left\{ \dfrac{\alpha_1\left[h(1)-h(\eta)\right]}{h(1)-\alpha_1 h(\eta)}, \dfrac{\alpha_1 h(\eta)}{h(1)} \right\} \max_{t\in[0,1]} G(t,s), & 0<\alpha_1\le 1, \\ \min\left\{ \dfrac{h(1)-\alpha_1 h(\eta)}{\alpha_1\left[h(1)-h(\eta)\right]}, \dfrac{h(\eta)}{\alpha_1 h(1)} \right\} \max_{t\in[0,1]} G(t,s), & 1\le\alpha_1<\dfrac{h(1)}{h(\eta)}. \end{cases} \tag{3.25}$$

Consequently, setting

$$\lambda = \begin{cases} \min\left\{ \dfrac{\alpha_1[h(1)-h(\eta)]}{h(1)-\alpha_1 h(\eta)}, \dfrac{\alpha_1 h(\eta)}{h(1)} \right\}, & 0 < \alpha_1 \le 1, \\ \min\left\{ \dfrac{h(1)-\alpha_1 h(\eta)}{\alpha_1[h(1)-h(\eta)]}, \dfrac{h(\eta)}{\alpha_1 h(1)} \right\}, & 1 \le \alpha_1 < \dfrac{h(1)}{h(\eta)}, \end{cases} \tag{3.26}$$

there holds

$$\min_{t\in[\eta,1]} G(t,s) \ge \lambda \max_{t\in[0,1]} G(t,s). \tag{3.27}$$

It can be easily seen that $0 < \lambda < 1$. This completes the proof. □

4. Existence of Multiple Nonnegative Solutions

Let $X = C[0,1] \bigcap C^2(0,1]$ be endowed with ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0,1]$ and $\|x\| = \max\{\|x\|_1, \|x'\|_1\}$, where

$$\begin{aligned} \|x\|_1 &= \sup_{t\in[0,1]} |x(t)|, \\ \left\|x'\right\|_1 &= \sup_{t\in(0,1]} \left|p(t)x'(t)\right|. \end{aligned} \tag{4.1}$$

Let $E = \{x : x \in X, \|x\| < \infty\}$ be bounded subset of X. E is Banach Space.

Now define a cone $P \subset E$ as

$$P = \left\{ x \in E : x(t) \ge 0, \min_{t\in[\eta,1]} x(t) \ge \lambda \max_{t\in[0,1]} x(t), \left(p(t)x'(t)\right)' \le 0 \right\}. \tag{4.2}$$

The boundary value problem (1.1)-(1.2) has a solution $y(t)$ if and only if $y(t)$ solves the following operator equation:

$$y(t) = Ty(t), \tag{4.3}$$

where the operator $T : P \to P$ is given by

$$(Ty)(t) = \int_0^1 G(t,s)q(s)f\left(s, y(s), p(s)y'(s)\right)ds, \quad 0 \le t \le 1. \tag{4.4}$$

Here $G(t,s)$ is the Green's function of the problem (3.1) defined in Lemma 3.1.

Lemma 4.1. *Let (E1)–(E3) hold, then the operator $T : P \to P$ is well defined and is completely continuous.*

Proof. First we show that the operator T is well defined. For this, we take $y \in P$. From (E2), (E3), and $G(t,s) \geq 0$, it follows that $Ty(t) \geq 0$.

Now applying Lemma 3.2, we get

$$\begin{aligned}
\min_{t\in[\eta,1]} Ty(t) &= \min_{t\in[\eta,1]} \int_0^1 G(t,s)q(s)f\big(s,y(s),p(s)y'(s)\big)ds,\\
\min_{t\in[\eta,1]} Ty(t) &\geq \lambda \int_0^1 \max_{t\in[0,1]} G(t,s)q(s)f\big(s,y(s),p(s)y'(s)\big)ds,\\
&= \lambda \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)f\big(s,y(s),p(s)y'(s)\big)ds,\\
&= \lambda \max_{t\in[0,1]} (Ty)(t).
\end{aligned} \tag{4.5}$$

It is easy to show that $(p(t)Ty'(t))' \leq 0$. Thus T is well defined.

We now show that T is completely continuous. Let $\{y_n\}$ be a sequence in P and $y_0 \in P$ with $\lim_{n\to\infty} y_n = y_0$. Then, there exists a constant $k_1 > 0$ such that $\|y_n\| < k_1$ for all $n \in \mathbb{N} \cup \{0\}$. Thus $\|y_n - y_0\| \to 0$ as $n \to \infty$ implies $\sup_{t\in[0,1]}|(y_n - y_0)(t)|$ and $\sup_{t\in(0,1]}|p(t)(y_n - y_0)'(t)| \to 0$ as $n \to \infty$. So $y_n(t) \to y_0(t)$ and $p(t)y_n'(t) \to p(t)y_0'(t)$ as $n \to \infty$.

Since f is continuous on $\{[0,1] \times [0,k_1] \times [-k_1,k_1]\}$, so

$$\begin{aligned}
\big|Ty_n(t) - Ty_0(t)\big| &= \left|\int_0^1 G(t,s)q(s)\big[f\big(s,y_n(s),p(s)y_n'(s)\big) - f\big(s,y_0(s),p(s)y_0'(s)\big)\big]ds\right|\\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\\
&\Longrightarrow \big\|Ty_n - Ty_0\big\|_1 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\big|p(t)\big(Ty_n'(t) - Ty_0'(t)\big)\big| &= \left|p(t)\int_0^1 \frac{\partial G(t,s)}{\partial t} q(s)\big[f\big(s,y_n(s),p(s)y_n'(s)\big)\right.\\
&\qquad \left. -f\big(s,y_0(s),p(s)y_0'(s)\big)\big]ds\right|\\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\\
&\Longrightarrow \Big\|\big(Ty_n - Ty_0\big)'\Big\|_1 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.7}$$

From (4.6) and (4.7),

$$\big\|Ty_n - Ty_0\big\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.8}$$

Hence $T : P \to P$ is a continuous operator.

Next we prove that T maps every bounded subset of P into relatively compact set. Let $B = \{x \in P : \|x\| \leq k_2, k_2 \text{ is a positive constant}\}$ be any bounded subset of P. For $y \in B$,

$$\begin{aligned} Ty(t) &= \int_0^1 G(t,s)q(s)f\big(s,y(s),p(s)y'(s)\big)ds \\ &\leq \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)f\big(s,y(s),p(s)y'(s)\big)ds \\ &\leq \max_{(s,u,v)\in[0,1]\times[0,k_2]\times[-k_2,k_2]} f(s,u,v) \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)ds \\ &\Longrightarrow Ty(t) < \infty. \end{aligned} \tag{4.9}$$

Therefore $T(B)$ is uniformly bounded. Further, equicontinuity of $T(B)$ follows from

$$\begin{aligned} \big|Ty(t_1) - Ty(t_2)\big| &= \left|\int_0^1 \big[G(t,s)|_{t=t_1} - G(t,s)|_{t=t_2}\big]q(s)f\big(s,y(s),p(s)y'(s)\big)ds\right| \\ &\longrightarrow 0 \quad \text{as } t_1 \longrightarrow t_2, \\ \big|p(t)\big(Ty'(t_1) - Ty'(t_2)\big)\big| &= \left|p(t)\int_0^1 \left[\frac{\partial G(t,s)}{\partial t}\bigg|_{t=t_1} - \frac{\partial G(t,s)}{\partial t}\bigg|_{t=t_2}\right] q(s)f\big(s,y(s),p(s)y'(s)\big)ds\right| \\ &\leq 2\max\left|\frac{\partial G(t,s)}{\partial t}\right| \cdot \sup_{t\in[0,1]} p(t) \int_{t_1}^{t_2} q(s)f\big(s,y(s),p(s)y'(s)\big)ds, \\ &\longrightarrow 0 \quad \text{as } t_1 \longrightarrow t_2. \end{aligned} \tag{4.10}$$

Thus from Arzela-Ascoli Theorem, $T(B)$ is relatively compact subset of P and also $T : P \to P$ is completely continuous. □

Next, define functionals $\alpha, \beta, \psi : P \to [0,\infty)$ such that

$$\begin{aligned} \alpha(y) &= \sup_{t\in[0,1]} \big|y(t)\big|, \\ \beta(y) &= \sup_{t\in(0,1]} \big|p(t)y'(t)\big|, \\ \psi(y) &= \min_{t\in[\eta,1]} \big|y(t)\big|. \end{aligned} \tag{4.11}$$

Clearly, α, β are nonnegative continuous convex functionals such that $\|y\| = \max\{\alpha(y), \beta(y)\}$ satisfying (2.3) and (2.4), and ψ is nonnegative concave functional with $\psi(y) \leq \alpha(y)$.

Let

$$C = \min_{t\in[\eta,1]} \int_{\eta}^{1} G(t,s)q(s)ds,$$

$$L = \sup_{t\in[0,1]} \int_{0}^{1} G(t,s)q(s)ds, \tag{4.12}$$

$$N = \sup_{t\in(0,1]} \left| \int_{0}^{1} p(t) \frac{\partial G(t,s)}{\partial t} q(s)ds \right|.$$

Now we state the main results of this work.

Theorem 4.2. *Suppose that (E1)–(E3) are satisfied and $f(t,y,z)$ satisfies the following condition.*

(H_1) *if there exist real constants $d > 0$ and $L_1 > 0$ such that $f(t,y,z) < \min\{d/L, L_1/N\}$ for $(t,y,z) \in \{[0,1] \times [0,d] \times [-L_1, L_1]\}$,*

then boundary value problem (1.1)-(1.2) *has at least one nonnegative solution y_1 such that $\sup_{t\in[0,1]}|y_1(t)| < d$ with $\sup_{t\in(0,1]}|p(t)y_1'(t)| < L_1$.*

Theorem 4.3. *Suppose that (E1)–(E3) are satisfied. There exist real constants $a, c, d, L_1,$ and L_2 with $c \geq a/\lambda = b > a > d > 0$, $L_2 \geq L_1 > 0$ such that $a/C \leq \min\{c/L, L_2/N\}$ and $f(t,y,z)$ satisfies following conditions.*

(H_1) $f(t,y,z) < \min\{d/L, L_1/N\}$ *for* $(t,y,z) \in \{[0,1] \times [0,d] \times [-L_1, L_1]\}$,

(H_2) $f(t,y,z) > a/C$ *for* $(t,y,z) \in \{[\eta,1] \times [a, a/\lambda] \times [-L_2, L_2]\}$,

(H_3) $f(t,y,z) \leq \min\{c/L, L_2/N\}$ *for* $(t,y,z) \in \{[0,1] \times [0,c] \times [-L_2, L_2]\}$.

Then boundary value problem (1.1)-(1.2) *has at least three nonnegative solutions y_1, y_2, and y_3 in $\overline{P}(\alpha^c, \beta^{L_2})$ such that*

$$\sup_{t\in[0,1]} |y_1(t)| < d, \qquad \sup_{t\in(0,1]} |p(t)y_1'(t)| < L_1,$$

$$a < \min_{t\in[\eta,1]} |y_2(t)| \leq \sup_{t\in[0,1]} |y_2(t)| \leq c, \tag{4.13}$$

$$\sup_{t\in(0,1]} |p(t)y_2'(t)| \leq L_2, \qquad \sup_{t\in[0,1]} |y_3(t)| \leq \frac{a}{\lambda} \text{ with } \sup_{t\in(0,1]} |p(t)y_3'(t)| \leq L_2.$$

Proof of Theorem 4.2. Let $U = P(\alpha^d, \beta^{L_1})$ be open subset of P. We now show that $T(\overline{U}) \subset \overline{U}$. For $y \in \overline{U}$,

$$\begin{aligned}
\alpha(Ty) &= \sup_{t\in[0,1]} \left|Ty(t)\right| \\
&= \sup_{t\in[0,1]} \int_0^1 G(t,s)q(s)f\left(s, y(s), p(s)y'(s)\right)ds, \\
&< \frac{d}{L} \sup_{t\in[0,1]} \int_0^1 G(t,s)q(s)ds < d, \text{ from condition } (H_1)
\end{aligned} \tag{4.14}$$

implies that $\alpha(Ty) < d$.

Consider that

$$\begin{aligned}
\beta(Ty) &= \sup_{t\in(0,1]} \left|p(t)Ty'(t)\right|, \\
&= \sup_{t\in(0,1]} \left|p(t) \int_0^1 \frac{\partial G(t,s)}{\partial t} q(s) f\left(s, y(s), p(s)y'(s)\right)\right|, \\
&< \frac{L_1}{N} \sup_{t\in(0,1]} \left|\int_0^1 p(t) \frac{\partial G(t,s)}{\partial t} q(s)ds\right|, \text{ from condition } (H_1),
\end{aligned} \tag{4.15}$$

implies that $\beta(Ty) < L_1$.

Thus $T(\overline{U}) \subset \overline{U}$. Next, we show that T has no fixed point on $\partial U (= \overline{U} \setminus U)$. On contrary, suppose there exists a fixed point $y(t)$ on ∂U such that $Ty(t) = y(t)$. Then from (4.14) and (4.15), $\alpha(y) < d$ and $\beta(y) < L_1$, which are not possible. So the operator T has no fixed point on ∂U and from Theorem 2.10 $i(T, U, P) = 1$. Thus the operator T has at least one fixed point in U and also the boundary value problem (1.1)-(1.2) has at least one nonnegative solution y_1 such that $\sup_{t\in[0,1]}|y_1(t)| < d$ with $\sup_{t\in(0,1]}|p(t)y_1'(t)| < L_1$. □

Proof of Theorem 4.3. It is easy to see that $\psi(y) \leq \alpha(y)$ for each $y \in \overline{P}(\alpha^c, \beta^{L_2})$. We now show that $T : \overline{P}(\alpha^c, \beta^{L_2}) \to \overline{P}(\alpha^c, \beta^{L_2})$ is well defined. For $y \in \overline{P}(\alpha^c, \beta^{L_2})$,

$$\begin{aligned}
\alpha(Ty) &= \sup_{t\in[0,1]} \left|Ty(t)\right|, \\
&= \sup_{t\in[0,1]} \int_0^1 G(t,s)q(s)f\left(s, y(s), p(s)y'(s)\right)ds, \\
&\leq \frac{c}{L} \sup_{t\in[0,1]} \int_0^1 G(t,s)q(s)ds, \text{ from assumption } (H_3), \\
&\leq c.
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\beta(Ty) &= \sup_{t\in(0,1]} \left|p(t)Ty'(t)\right|, \\
&= \sup_{t\in(0,1]} \left|p(t)\int_0^1 \frac{\partial G(t,s)}{\partial t} q(s) f\left(s, y(s), p(s)y'(s)\right)\right|, \\
&\leq \frac{L_2}{N} \sup_{t\in(0,1]} \left|\int_0^1 p(t)\frac{\partial G(t,s)}{\partial t} q(s)ds\right|, \text{ from assumption } (H_3), \\
&\leq L_2.
\end{aligned} \tag{4.17}$$

From (4.16) and (4.17),

$$T(y) \in \overline{P}\left(\alpha^c, \beta^{L_2}\right). \tag{4.18}$$

Thus, $T : \overline{P}(\alpha^c, \beta^{L_2}) \to \overline{P}(\alpha^c, \beta^{L_2})$ is well defined, and by Lemma 4.1, it is completely continuous. Now Condition (2) of Theorem 2.11 can be proved by similar manner. Choose $y(t) = a/\lambda \in P$, $0 \leq t \leq 1$, then $\alpha(y) = \sup_{t\in[0,1]}|y(t)| = a/\lambda$, $\beta(y) = \sup_{t\in(0,1]}|p(t)y'(t)| = 0 < L_2$, $\varphi(x) = \min_{t\in[\eta,1]}|y(t)| = a/\lambda > a$. Thus, $\{y \in \overline{P}(\alpha^{b=a/\lambda}, \beta^{L_2}, \Psi_a) : \Psi(y) > a\} \neq \Phi$. Further if $y \in \overline{P}(\alpha^{a/\lambda}, \beta^{L_2}, \Psi_a)$, then $a \leq y(t) \leq a/\lambda$ for $\eta \leq t \leq 1$. Then by definition of φ and assumption (H_2), we have

$$\begin{aligned}
\varphi(Ty) &= \min_{t\in[\eta,1]} \left|Ty(t)\right| \\
&= \min_{t\in[\eta,1]} \int_0^1 G(t,s)q(s)f\left(s, y(s), p(s)y'(s)\right)ds \\
&\geq \min_{t\in[\eta,1]} \int_\eta^1 G(t,s)q(s)f\left(s, y(s), p(s)y'(s)\right)ds \\
&> \frac{a}{C}\cdot C = a
\end{aligned} \tag{4.19}$$

Thus, Condition (1) of Theorem 2.11 is satisfied. We finally show that condition (3) of Theorem 2.11 holds, too. Suppose $y \in \overline{P}(\alpha^c, \beta^{L_2}, \Psi_a)$ with $\alpha(Ty) > b$. Then by definition of φ and $Ty \in P$, we have

$$\begin{aligned}
\Psi(Ty) &= \min_{t\in[\eta,1]} \left|Ty(t)\right| \\
&\geq \lambda \max_{t\in[0,1]} \left|Ty(t)\right| \\
&\geq \lambda\alpha(Ty) \\
&= a.
\end{aligned} \tag{4.20}$$

So, Condition (3) of Theorem 2.11 is also satisfied. Therefore, Theorem 2.11 yields that boundary value problem (1.1)-(1.2) has at least three nonnegative solutions y_1, y_2, and y_3 in $\overline{P}(\alpha^c, \beta^{L_2})$ such that

$$\begin{gathered}\sup_{t\in[0,1]}\left|y_1(t)\right| < d, \qquad \sup_{t\in(0,1]}\left|p(t)y_1'(t)\right| < L_1;\\ a < \min_{t\in[\eta,1]}\left|y_2(t)\right| \le \sup_{t\in[0,1]}\left|y_2(t)\right| \le c, \qquad \sup_{t\in(0,1]}\left|p(t)y_2'(t)\right| \le L_2;\\ \sup_{t\in[0,1]}\left|y_3(t)\right| \le \frac{a}{\lambda}, \qquad \sup_{t\in(0,1]}\left|p(t)y_3'(t)\right| \le L_2.\end{gathered} \tag{4.21}$$

□

Corollary 4.4. *Suppose that (E1)–(E3) are satisfied. If there exist constants* $0 < d_1 < a_1 < a_1/\lambda < d_2 < a_2 < a_2/\lambda < d_3 < \cdots < d_n$, $0 < L_1 \le L_2 \le L_3 \le \cdots \le L_n$ $n \in \mathbb{N}$, *with* $a_i/C < \min\{d_{i+1}/L, L_{i+1}/N\}, 1 < i \le n-1$ *such that* f *satisfies the following conditions:*

(M1) $f(t,y,z) < \min\{d_i/L, L_i/N\}$, *for* $(t,y,z) \in \{[0,1]\times[0,d_i]\times[-L_i,L_i]\}$, $1 \le i \le n$,

(M2) $f(t,y,z) > a_i/C$, *for* $(t,y,z) \in \{[\eta,1]\times[a_i,a_i/\lambda]\times[-L_{i+1},L_{i+1}]\}$, $1 \le i \le n-1$,

then boundary value problem (1.1)-(1.2) *has at least* $2n-1$ *nonnegative solutions.*

Proof. When $n = 1$, the result follows from Theorem 4.2. When $n = 2$, it is clear that all the conditions of Theorem 4.3 hold (with $c = d_2$, $d = d_1$, $a = a_1$). Thus the boundary value problem (1.1)-(1.2) has at least three positive solutions y_1, y_2, and y_3. Following this way, we complete the proof by induction method. □

Finally, we demonstrate these results through examples.

5. Example

In Example 5.1, we demonstrate the detailed working of the boundary value problem (1.5) mentioned in the introduction. Example 5.2 verifies our results.

Example 5.1. Consider the boundary value problem (1.5).

Here,

$$f(t,y,py') = 2(r+1)t - \frac{5}{4}r, \qquad q(t) = t^{r-1}, \qquad \max_{t\in[0,1]} f(t,y,py') = \frac{3}{4}r + 2. \tag{5.1}$$

Following the notations of this work, it is easy to see that

$$L = \frac{3}{2}\left[\frac{3(1-r)}{4-4^r}\right]^{(1-r)/r}, \qquad N = \frac{3(1-r)}{r(4-4^r)}. \tag{5.2}$$

Now for $d \ge (3/2)[3(1-r)/4-4^r]^{(1-r)/r}[(3/4)r+2]$ and $L_1 \ge 3(1-r)((3/4)r+1)/r(4-4^r)$, $f(t,y,py') \le \min\{d/L, L_1/N\}$.

Then from Theorem 4.2, the problem has at least one nonnegative solution $y(t)$ with

$$\sup_{t\in[0,1]}\left|y(t)\right| \le \frac{3}{2}\left[\frac{3(1-r)}{4-4^r}\right]^{(1-r)/r}\left[\frac{3}{4}r+2\right], \qquad \sup_{t\in(0,1]}\left|p(t)y'(t)\right| \le \frac{3(1-r)}{r(4-4^r)}\left[\frac{3}{4}r+2\right]. \tag{5.3}$$

Next we reduce the problem and then apply Theorem 4.2 for $p(t) = 1$. Using the transformation $x = (1-r)\int_0^t (1/x^r)dx = t^{1-r}$, the boundary value problem (1.5) can be reduced to regular boundary value problem as

$$\begin{aligned} -v''(x) &= \frac{1}{(1-r)^2}x^{(2r-1)/(1-r)}\left[2(r+1)x^{1/(1-r)} - \frac{5}{4}r\right], \quad x\in(0,1), \\ v(0) &= 0, \qquad v(1) = v\left[\left(\frac{1}{4}\right)^{(1-r)}\right]. \end{aligned} \tag{5.4}$$

Here,

$$\begin{aligned} F(x,v,v') &= \frac{1}{(1-r)^2}t^{(2r-1)/(1-r)}\left[2(r+1)t^{1/(1-r)} - \frac{5}{4}r\right], \\ \max_{x\in[0,1]}F(x,v,v') &= \frac{1}{(1-r)^2}\left[2(1+r) - \frac{5}{4}r\right]. \end{aligned} \tag{5.5}$$

Now following the notation of this work for $p(t) = 1$,

$$L = \frac{3(1-r)^2}{4-4^r}\left[\frac{3(1-r)}{r(4-4^r)}\right]^{(1-r)/r}, \qquad N = \frac{3(1-r)^2}{r(4-4^r)}. \tag{5.6}$$

Now for $d \ge 3/(4-4^r)[3(1-r)/(4-4^r)]^{(1-r)/r}[(3/4)r+2]$ and $L_1 \ge (3/r(4-4^r))[(3/4)r+2]$, $f(t,v,v') \le \min\{d/L, L_1/N\}$. So the problem has at least one nonnegative solution $v(t)$ with

$$\sup_{t\in[0,1]}|v(t)| \le \frac{3}{4-4^r}\left[\frac{3(1-r)}{4-4^r}\right]^{(1-r)/r}\left[\frac{3}{4}r+2\right], \qquad \sup_{t\in(0,1]}\left|v'(t)\right| \le \frac{3}{r(4-4^r)}\left[\frac{3}{4}r+2\right]. \tag{5.7}$$

Hence the boundary value problem (1.5) has at least one nonnegative solution $y(t)$ with

$$\sup_{t\in[0,1]}\left|y(t)\right| \le \frac{3}{4-4^r}\left[\frac{3(1-r)}{4-4^r}\right]^{(1-r)/r}\left[\frac{3}{4}r+2\right], \qquad \sup_{t\in(0,1]}\left|y'(t)\right| \le \frac{3}{r(4-4^r)}\left[\frac{3}{4}r+2\right]. \tag{5.8}$$

Now in this case it is easy to show that if r approaches one, that is, the order of singularity increases, upper bound for $\sup_{t\in[0,1]}|y(t)|$ approaches ∞ while in case of direct solving, upper bound for $\sup_{t\in[0,1]}|y(t)|$ approaches 4.125. As smaller upper bound for $\sup_{t\in[0,1]}|y(t)|$ will enable to find nonnegative solution(s) faster and hence will be helpful in constructing efficient numerical algorithms to find multiple nonnegative solutions.

Example 5.2. Consider the following boundary value problem:

$$-\left(t^{1/2}y'(t)\right)' = t^{-1/2}f\left(t, y(t), t^{1/2}y'(t)\right), \quad 0 < t < 1,$$
$$y(0) = 0, \qquad y(1) = \frac{3}{2}y\left(\frac{1}{3}\right). \tag{5.9}$$

(i) If

$$f(t,y,z) = \begin{cases} \dfrac{\sin t}{3} + \dfrac{10y^2+8}{400} + \left(\dfrac{z}{300}\right)^2, & y \le 4; \\ \dfrac{\sin t}{3} + \dfrac{21}{50} + \left(\dfrac{z}{300}\right)^2, & y \ge 4; \end{cases} \tag{5.10}$$

then the boundary value problem (5.9) has at least one nonnegative solution.

(ii) Further, if

$$f(t,y,z) = \begin{cases} \dfrac{\sin t}{3} + \dfrac{10y^2+8}{400} + \left(\dfrac{z}{4000}\right)^2, & y \le 4 \\ \dfrac{\sin t}{3} + \dfrac{9y^2+24}{400} + \left(\dfrac{z}{4000}\right)^2, & 4 \le y \le 191 \\ \dfrac{\sin t}{3} + \dfrac{328353}{400} + \left(\dfrac{z}{4000}\right)^2, & y \ge 191, \end{cases} \tag{5.11}$$

then the boundary value problem (5.9) has at least three nonnegative solutions.

Proof. Here, $\alpha_1 = 3/2$ and $\eta = 1/3$. After simple calculation, we get $L = 5.464$, $N = 3.732$, $C = 1.5396$, and $\lambda = 0.1786$.

(i) At least one nonnegative solution: we choose $d = 3$ and $L_1 = 10$. Here, $\min\{d/L, L_1/N\} = 0.549$;

$$f(t,y,z) < 0.549, \quad \text{for } 0 \le t \le 1,\ 0 \le y \le 3,\ -10 \le z \le 10. \tag{5.12}$$

Thus, condition (H_1) is satisfied. Now from Theorem 4.2 the problem has at least one nonnegative solution y_1 such that $\sup_{t\in[0,1]}|y_1(t)| < 3$ with $\sup_{t\in(0,1]}|p(t)y_1'(t)| < 10$.

(ii) At least three nonnegative solutions: we choose constants $d = 4$, $a = 34$, $c = 4800$, $L_1 = 10$, and $L_2 = 3500$. Here, $\min\{d/L, L_1/N\} = 0.732$, $\min\{c/L, L_2/N\} = 878.45$, $a/\lambda = 190.34$, and $a/C = 22.084$;

$$f(t,y,z) < 0.732, \quad \text{for } 0 \le t \le 1,\ 0 \le y \le 4,\ -10 \le z \le 10;$$
$$f(t,y,z) > 22.084 \quad \text{for } \frac{1}{3} \le t \le 1,\ 34 \le y \le 190.34,\ -3500 \le z \le 3500; \tag{5.13}$$
$$f(t,y,z) < 878.45 \quad \text{for } 0 \le t \le 1,\ 0 \le y \le 4800,\ -3500 \le z \le 3500.$$

Thus, conditions (H_1), (H_2), and (H_3) are satisfied. Now from Theorem 4.3 the problem has at least three nonnegative solutions y_1, y_2, and y_3 such that

$$\begin{gathered}\sup_{t\in[0,1]}\left|y_1(t)\right| < 4, \qquad \sup_{t\in(0,1]}\left|p(t)y_1(t)\right| < 10;\\ 34 < \min_{t\in[\eta,1]}\left|y_2(t)\right| \le \sup_{t\in[0,1]}\left|y_2(t)\right| < 4800, \qquad \sup_{t\in(0,1]}\left|p(t)y_2'(t)\right| \le 3500,\\ \sup_{t\in[0,1]}\left|y_3(t)\right| \le 190.34, \qquad \sup_{t\in(0,1]}\left|p(t)y_3'(t)\right| \le 3500.\end{gathered} \tag{5.14}$$

□

Remark 5.3. For $\alpha_1 = 1$, the problem (1.1)-(1.2) can be regarded as two-point boundary value problem with boundary conditions as

$$y(0) = 0, \qquad y'(1) = 0, \tag{5.15}$$

in the limiting case $\eta \to 1^-$. Thus the results established in this work are also valid for the two-point boundary value problem [21, 22].

Remark 5.4. Theorem 4.3 and Corollary 4.4 extend Theorem 3.1 and Corollary 3.1 of [19] to doubly singular three-point boundary value problem.

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