Research Article

Dynamical Models for Infectious Diseases with Varying Population Size and Vaccinations

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We formulate and discuss models for the spread of infectious diseases with variable population sizes and vaccinations on the susceptible individuals. First, we assume that the susceptible individuals are vaccinated continuously. We establish the threshold-like results for the existence and global stability of the disease-free and the endemic equilibriums for these systems. Especially, we prove the global stability of the endemic equilibriums by converting the systems into integrodifferential equations. Second, we suppose that vaccinations occur once per time period. We obtain the existence and global stability of the disease-free periodic solutions for such systems with impulsive effects. By a useful bifurcation theorem, we acquire the existence of the endemic periodic solutions when the disease-related deaths do not occur. At last, we compare the results with vaccinations and without vaccinations and illustrate our results by numerical simulations.

1. Introduction

Confidence that the infectious diseases would soon be eliminated was created by the improved sanitation, effective antibiotics and vaccination programs in the 1960s, but it collapsed now. Human and animal invasions of new ecosystems, global warming, environmental degradation, and increased international travels provide many opportunities for the spread and the eruption of infectious diseases. It is clear that new infectious diseases are emerging, and some eliminated diseases are reemerging since the infectious agents' evolvement and adaptation to the environment. Moreover, these infectious diseases lead to terrible suffering and mortality. Consequently, infectious diseases are receiving more and more attention in developing countries, even in the developed countries.

The emerging and reemerging of infectious diseases have been studied by many scientists in different fields. Mathematical models are important tools to analyze and control the spread of infectious diseases. Hethcote [1] gives a review on the mathematics of infectious diseases. Most models for the transmission of infectious diseases descend from the

pioneering work of Kermark-Mckerdrick on SIR (susceptible-infectious-removal), in which vital dynamics (birth and death) is negligible for the short incubation of infectious diseases. The possible and realistic situations would be to discuss epidemic models with varying population size, which may refer to Mena-Lorca and Hethcote [2], Anderson and May [3], Gao and Hethcote [4], Li and Graef [5], and Brauer and Driessche [6]. Thresholds are obtained which determine whether the diseases die out or break out. The existence and stability of equilibrium points are investigated for each model, but there is still some work to do. Such as in [4], the global stability of the endemic equilibrium points was not obtained. We complete this in the present paper.

Vaccination programs have been applied to prevent and control the yield and spread of infectious diseases, which achieved a lot. Models with vaccination are constructed and analyzed by Shulgin et al. [7], Stone et al. [8], Li and Ma [9], Greenhalgh and Das [10] and Greenhalgh [11]. Some useful results are obtained, and results with vaccination and without vaccination are compared. But they assumed that the susceptible is vaccinated continuously. In fact, it would be more realistic and reasonable that the susceptible is assumed to be vaccinated in a single pulse or at fixed moments. In this paper, we consider not only a constant flow of new members into the susceptible but also vaccinating continuously and impulsively on the susceptible. We will investigate the dynamical behaviors of these epidemic models, which are described by continuous or impulsive differential equations. The models of infectious diseases with impulsive effects had been discussed in [12], where the birth, rather than the vaccination, is assumed to be impulsive.

We denote by $S(t)$ the number of members of a population who are susceptible to an infection at time t, $I(t)$ the number of members who are infective at time t, and $R(t)$ the number of members at time t who have been removed as the result of recovery from the infection with temporary immunity against reinfection. The total population size at time t is represented by $N(t)$ with $N(t) = S(t) + I(t) + R(t)$. In addition, basic hypotheses are needed to formulate our models:

(1) there is a constant flow of A new members into the susceptible in unit time;

(2) a fraction $p \geq 0$ of the susceptible is vaccinated in unit time or in a single pulse once per time period, which will enter directly into the removal owing to obtaining the immunity;

(3) there is a constant per capita natural death rate $d > 0$ in each group;

(4) a fraction $\alpha \geq 0$ of the infective dies from the infection, and a fraction $\gamma \geq 0$ of the infective recovers in unit time;

(5) a fraction $\delta \geq 0$ of the removal loses their immunity and becomes the susceptible in unit time;

(6) the force of the infection is βI, where β is the effective per capita contact rate of the infective individuals and the incidence rate is $\beta S I$.

In the next section, an SIR model with variable population size and continuous vaccination is analyzed. The existence and stability of the equilibrium points for this model are discussed. The global stability of the disease-free equilibrium is proved by differential comparison theorem, and the global stability of the endemic equilibrium is obtained by converting the system into an integrodifferential equation. In Section 3, we consider an impulsive differential epidemic model, of which the stability and the existence of disease-free periodic solution are discussed. Further, the existence of endemic periodic solution is also studied for such a system

with $\alpha = 0$, which implies that the disease-related deaths are not considered. At last, we give some examples to illustrate our results by numerical simulation.

2. An SIR Model with Continuous Vaccination

In this section, we discuss the disease transmission model, which is described by

$$
\begin{aligned}
\dot{S}(t) &= A - dS - \beta SI + \delta R - pS, \\
\dot{I}(t) &= \beta SI - (\gamma + \alpha + d)I, \\
\dot{R}(t) &= \gamma I - (\delta + d)R + pS.
\end{aligned}
\tag{2.1}
$$

Denoting $N = S + I + R$, and adding these three equations, we have

$$
\dot{N} = A - dN - \alpha I.
\tag{2.2}
$$

Therefore, we may obtain system (2.3), which is equivalent to system (2.1), and consider the following system:

$$
\begin{aligned}
\dot{N}(t) &= A - dN - \alpha I, \\
\dot{I}(t) &= \beta(N - I - R)I - (\gamma + \alpha + d)I, \\
\dot{R}(t) &= pN + (\gamma - p)I - (d + \delta + p)R.
\end{aligned}
\tag{2.3}
$$

For nonnegative initial conditions $(N(0), I(0), R(0))$, it is easily known that $N(t), I(t)$, and $R(t)$ remain nonnegative, and the total population size $N(t)$ is ultimately upper bounded by A/d. Moreover, we have

$$
N(t) = N(0)e^{-dt} + \frac{A}{d}\left(1 - e^{-dt}\right) - \alpha \int_0^t e^{-d(t-s)}I(s)ds,
$$

$$
\begin{aligned}
R(t) &= R(0)e^{-(d+\delta+p)t} + (\gamma - p)\int_0^t e^{-(d+\delta+p)(t-s)}I(s)ds + p\int_0^t e^{-(d+\delta+p)(t-s)}N(s)ds \\
&= \frac{Ap}{d(d+\delta+p)} + \left[\frac{pN(0)}{p+\delta} - \frac{Ap}{d(p+\delta)}\right]e^{-dt} + \left[R(0) - \frac{pN(0)}{p+\delta} - \frac{Ap}{d(d+\delta+p)} + \frac{Ap}{d(p+\delta)}\right] \\
&\quad \times e^{-(d+\delta+p)t} - \frac{\alpha p}{p+\delta}\int_0^t e^{-d(t-s)}I(s)ds + \left(\gamma - p + \frac{\alpha p}{p+\delta}\right)\int_0^t e^{-(d+\delta+p)(t-s)}I(s)ds.
\end{aligned}
\tag{2.4}
$$

Substituting them into the second equation of (2.3), then

$$
\begin{aligned}
\dot{I}(t) = \beta I \Bigg\{ & \frac{A(d+\delta)}{d(d+\delta+p)} + \left[\frac{\delta N(0)}{p+\delta} - \frac{A\delta}{d(p+\delta)} \right] e^{-dt} - \left(\gamma - p + \frac{\alpha p}{p+\delta} \right) \int_0^t e^{-(d+\delta+p)(t-s)} I(s)\,ds \\
& -I(t) - \frac{\alpha\delta}{p+\delta} \int_0^t e^{-d(t-s)} I(s)\,ds - \left[R(0) - \frac{pN(0)}{p+\delta} - \frac{Ap}{d(d+\delta+p)} + \frac{Ap}{d(p+\delta)} \right] e^{-(d+\delta+p)t} \Bigg\} \\
& - (\gamma + \alpha + d) I.
\end{aligned}
\tag{2.5}
$$

With respect to equilibriums of (2.3), we have the following.

Theorem 2.1. *There only exists the disease-free equilibrium $E^0(N^0, 0, R^0)$, which is globally asymptotically stable if $A\beta(\delta+d) < d(\gamma+\alpha+d)(d+\delta+p)$. Here, $N^0 = A/d$ and $R^0 = Ap/(d(d+\delta+p))$.*

Proof. The existence and uniqueness of the disease-free equilibrium E^0 for system (2.3) are easily obtained if inequality $A\beta(\delta + d) < d(\gamma + \alpha + d)(d + \delta + p)$ holds. Next, we first discuss the local stability of E^0. The Jacobian matrix of system (2.3) at a point $E(N, I, R)$ is

$$
J(E) = \begin{bmatrix} -d & -\alpha & 0 \\ \beta I & \beta(N - R - 2I) - (\gamma + \alpha + d) & -\beta I \\ p & \gamma - p & -(d + \delta + p) \end{bmatrix}.
\tag{2.6}
$$

Thus,

$$
J(E^0) = \begin{bmatrix} -d & -\alpha & 0 \\ 0 & \dfrac{A\beta(d+\delta)}{d(d+\delta+p)} - (\gamma + \alpha + d) & 0 \\ p & \gamma - p & -(d+\delta+p) \end{bmatrix},
\tag{2.7}
$$

and $J(E^0)$ has three eigenvalues with negative real part if inequality $A\beta(\delta + d) < d(\gamma + \alpha + d)(d + \delta + p)$ is true, which shows that the disease-free equilibrium E^0 is locally stable.

Further, we assume $\gamma \le p$. It is easily known that for any $\varepsilon_1 > 0$, that there exists $T_1 > 0$ such that

$$
N(t) \le \frac{A}{d} + \varepsilon_1, \quad t > T_1.
\tag{2.8}
$$

Hence, we obtain from the third equation of (2.3) that

$$
\dot{R}(t) < p \left[\frac{A}{d} + \varepsilon_1 \right] - (d + \delta + p) R, \quad t > T_1.
\tag{2.9}
$$

Then, for any $\varepsilon_2 > 0$, there exists $T_2 > T_1$ such that

$$R(t) \leq \frac{Ap}{d(\delta + d + p)} + \varepsilon_1 + \varepsilon_2, \quad t > T_2. \tag{2.10}$$

Therefore, we have

$$\dot{S}(t) < \frac{A(d + \delta)(d + p)}{d(d + \delta + p)} + \delta(\varepsilon_1 + \varepsilon_2) - (d + p)S, \quad t > T_2. \tag{2.11}$$

This shows that for any $\varepsilon_3 > 0$, there exists $T_3 > T_2$ such that

$$S(t) < \frac{A(d + \delta)}{d(d + \delta + p)} + \frac{\delta(\varepsilon_1 + \varepsilon_2)}{d + p} + \varepsilon_3, \quad t > T_3. \tag{2.12}$$

Moreover, we have

$$\dot{I}(t) \leq I\left[\frac{A\beta(d + \delta)}{d(d + \delta + p)} + \frac{\delta(\varepsilon_1 + \varepsilon_2) + (d + p)\varepsilon_3}{d + p} - (\gamma + \alpha + d)\right], \quad t > T_3. \tag{2.13}$$

By the assumption of $A\beta(\delta + d) < d(\gamma + \alpha + d)(d + \delta + p)$, we may choose $\varepsilon_1, \varepsilon_2$, and ε_3, are small enough such that

$$\frac{\beta A(d + \delta)}{d(d + \delta + p)} + \frac{\delta(\varepsilon_1 + \varepsilon_2) + (d + p)\varepsilon_3}{d + p} - (\gamma + \alpha + d) < 0, \quad t > T_3. \tag{2.14}$$

Therefore, we obtain

$$\lim_{t \to +\infty} I(t) = 0. \tag{2.15}$$

This leads to

$$\lim_{t \to +\infty} N(t) = \frac{A}{d} = N^0, \qquad \lim_{t \to +\infty} R(t) = \frac{Ap}{d(d + \delta + p)} = R^0. \tag{2.16}$$

If $\gamma \geq p$, we obtain from (2.5)

$$\dot{I}(t) \leq \beta I\left\{\frac{A(d + \delta)}{d(d + \delta + p)} + \left[\frac{\delta N(0)}{p + \delta} - \frac{A\delta}{d(p + \delta)}\right]e^{-dt}\right.$$
$$\left. - \left[R(0) - \frac{pN(0)}{p + \delta} - \frac{Ap}{d(d + \delta + p)} + \frac{Ap}{d(p + \delta)}\right]e^{-(d+\delta+p)t} - \frac{(\gamma + \alpha + d)}{\beta}\right\}. \tag{2.17}$$

Since $\lim_{t \to +\infty} e^{-dt} = 0$, $\lim_{t \to +\infty} e^{-(d+\delta+p)t} = 0$, and $A\beta(\delta + d) < d(\gamma + \alpha + d)(d + \delta + p)$, then there exists $\varrho > 0$ and $T > 0$ such that

$$\dot{I}(t) \leq -\varrho\beta I, \quad t > T. \tag{2.18}$$

Thus, we prove that (2.15) is held. Further, (2.16) is obtained.   □

Theorem 2.2. *The unstable disease-free equilibrium E^0 and the local stable endemic equilibrium $E^*(N^*, I^*, R^*)$ of system (2.3) coexist if $A\beta(\delta + d) > d(\gamma + \alpha + d)(d + \delta + p)$ holds. Here,*

$$N^* = \frac{A - \alpha I^*}{d}, \quad I^* = \frac{A(\delta + d) - d(d + \delta + p)S^*}{(\gamma + \alpha + d)(d + \delta) - \gamma\delta}, \quad R^* = \frac{pS^* + \gamma I^*}{d + \delta}, \quad S^* = \frac{\gamma + \alpha + d}{\beta}. \tag{2.19}$$

Proof. It is easily known that system (2.3) has unique positive equilibrium $E^*(N^*, I^*, R^*)$ except the disease-free equilibrium $E^0(N^0, 0, R^0)$ if inequality $A\beta(\delta+d) > d(\gamma+\alpha+d)(d+\delta+p)$ holds.

Moreover, we may obtain the Jacobian matrix of (2.3) at equilibrium E^* as

$$J(E^*) = \begin{bmatrix} -d & -\alpha & 0 \\ \beta I^* & -\beta I^* & -\beta I^* \\ p & \gamma - p & -(d + \delta + p) \end{bmatrix}. \tag{2.20}$$

The characteristic equation of $J(E^*)$ is given by

$$\lambda^3 + Q_1\lambda^2 + Q_2\lambda + Q_3 = 0, \tag{2.21}$$

and the coefficients Q_i $(i = 1, 2, 3)$ are

$$\begin{aligned} Q_1 &= \beta I^* + 2d + \delta + p, \\ Q_2 &= \beta I^*(2d + \gamma + \alpha + \delta) + d(d + \delta + p), \\ Q_3 &= \beta I^*[d(\gamma + d + \delta) + \alpha(d + \delta)]. \end{aligned} \tag{2.22}$$

Clearly, $Q_i > 0$ $(i = 1, 2, 3)$ and $Q_1 Q_2 > Q_3$. By Routh-Hurwitz criterion, there exist three eigenvalues with negative real part for Jacobian matrix $J(E^*)$, which shows that equilibrium E^* is locally stable.   □

Further, we study the global stability of E^*.

Theorem 2.3. *If $(\gamma - p)(p + \delta) + \alpha p > 0$ is held besides $A\beta(\delta + d) > d(\gamma + \alpha + d)(d + \delta + p)$, then the positive equilibrium E^* of (2.3) is globally asymptotically stable.*

Proof. Since $I^* > 0$, we may make the change of variable $I(t) = I^* e^{y(t)}$, thus

$$\dot{y}(t) = \frac{\dot{I}(t)}{I(t)}, \qquad y(t) = \ln\frac{I(t)}{I^*}, \tag{2.23}$$

and (2.5) is equivalent to

$$
\begin{aligned}
\dot{y}(t) = \beta \Bigg\{ & \frac{A(d+\delta)}{d(d+\delta+p)} + \left[\frac{\delta N(0)}{p+\delta} - \frac{A\delta}{d(p+\delta)} \right] e^{-dt} - \left(\gamma - p + \frac{\alpha p}{p+\delta} \right) I^* \int_0^t e^{-(d+\delta+p)(t-s)} e^{y(s)} ds \\
& - I^* e^{y(t)} - \frac{\alpha\delta}{p+\delta} I^* \int_0^t e^{-d(t-s)} e^{y(s)} ds - \left[R(0) - \frac{pN(0)}{p+\delta} - \frac{Ap}{d(d+\delta+p)} + \frac{Ap}{d(p+\delta)} \right] \\
& \times e^{-(d+\delta+p)t} \Bigg\} - (\gamma + \alpha + d).
\end{aligned}
$$

(2.24)

Further, we define $g(y) = e^y - 1$ and

$$
a(s) = \begin{cases} 0, & s \le 0, \\ \beta I^* \left[1 + \dfrac{\alpha\delta}{p+\delta} \displaystyle\int_0^s e^{-du} du + \left(\gamma - p + \dfrac{\alpha p}{p+\delta} \right) \displaystyle\int_0^s e^{-(d+\delta+p)u} du \right], & s > 0, \end{cases}
$$

(2.25)

so that $a(s)$ has a jump βI^* at $s = 0$ and $\dot{a}(s) = \beta I^*[\alpha\delta/(p+\delta)e^{-ds} + (\gamma - p + \alpha p/(p+\delta))$ $e^{-(d+\delta+p)s}] > 0$ for $s > 0$ if $(\gamma - p)(p+\delta) + \alpha p > 0$, then

$$
\begin{aligned}
\int_0^t g(y(t-s)) da(s) = \beta I^* & \left[\frac{\alpha\delta}{p+\delta} \int_0^t e^{-d(t-u)} e^{y(u)} du + \frac{(\gamma-p)(p+\delta)+\alpha p}{p+\delta} \int_0^t e^{-(d+\delta+p)(t-u)} e^{y(u)} du \right] \\
& - \beta I^* \left[\frac{\alpha\delta}{p+\delta} \frac{1-e^{-dt}}{d} + \frac{(\gamma-p)(p+\delta)+\alpha p}{p+\delta} \frac{1-e^{-(d+\delta+p)t}}{d+\delta+p} \right].
\end{aligned}
$$

(2.26)

Hence, system (2.3) is reduced to a single integrodifferential equation

$$
\dot{y}(t) = - \int_0^t g(y(t-s)) da(s) - h(y(t)) + f(t).
$$

(2.27)

Here, $h(y) = \beta I^*(e^y - 1)$ and

$$
\begin{aligned}
f(t) = \beta \Bigg\{ & \left[\frac{\delta N(0)}{p+\delta} - \frac{A\delta}{d(p+\delta)} + \frac{\alpha\delta I^*}{d(p+\delta)} \right] e^{-dt} \\
& - \left[R(0) - \frac{pN(0)}{p+\delta} - \frac{Ap}{d(d+\delta+p)} + \frac{Ap}{d(p+\delta)} + \frac{((\gamma-p)(p+\delta)+\alpha p)I^*}{(d+\delta+p)(p+\delta)} \right] e^{-(d+\delta+p)t} \Bigg\} \\
& + \left\{ \frac{A\beta(d+\delta)}{d(d+\delta+p)} - \frac{\alpha\delta\beta I^*}{d(p+\delta)} - \frac{((\gamma-p)(p+\delta)+\alpha p)\beta I^*}{(d+\delta+p)(p+\delta)} - \beta I^* - (\gamma+\alpha+d) \right\}.
\end{aligned}
$$

(2.28)

With the equilibrium condition of (2.3), $f(t)$ can be simplified as

$$
\begin{aligned}
f(t) = \beta \Bigg\{ &\left[\frac{\delta N(0)}{p+\delta} - \frac{A\delta}{d(p+\delta)} + \frac{\alpha \delta I^*}{d(p+\delta)} \right] e^{-dt} \\
&- \left[R(0) - \frac{pN(0)}{p+\delta} - \frac{Ap}{d(d+\delta+p)} + \frac{Ap}{d(p+\delta)} + \frac{((\gamma-p)(p+\delta)+\alpha p)I^*}{(d+\delta+p)(p+\delta)} \right] e^{-(d+\delta+p)t} \Bigg\},
\end{aligned}
$$

$$(2.29)$$

which is negative exponential.

Obviously, the equilibrium I^* of (2.5) corresponds to the equilibrium $y(t) \equiv 0$ of (2.27). According to Theorem 18.2.3 of Gripenberg et al. [13], since $a(s)$ is of strong positive type ([14, 15]), $g(y)$ is continuous and $\int_0^y g(y)dy \to \infty$ as $|y| \to \infty$, $h(y)$ is also continuous and $g(y)h(y) \geq 0$ for $-\infty < y < +\infty$, $f(t)$, and $\dot{f}(t)$ are in $L^2(0,\infty)$, it follows that every bounded solution of (2.27) satisfies $\lim_{t \to \infty} g(y(t)) = 0$. Owing to $g(y) = 0$ only for $y = 0$, this implies that every solution of (2.27) tends to zero as $t \to \infty$ and therefore the equilibrium I^* of (2.5) is globally asymptotically stable. As (2.5) is equivalent to system (2.3), the unique positive equilibrium (N^*, I^*, R^*) of (2.3) is globally asymptotically stable. □

It is similar to the classical SIR models that there exists the threshold quantity and it is given by $R_1 = A/d \cdot \beta/(\gamma+\alpha+d) \cdot (\delta+d)/(\delta+d+p)$ for the model (2.1). If $R_1 < 1$, then system (2.1) has only the globally asymptotically stable disease-free equilibrium. This shows that the epidemic disease will die out. Otherwise, if $R_1 > 1$, system (2.1) has a locally stable positive equilibrium except disease-free equilibrium. Moreover, the global asymptotical stability of positive equilibrium is obtained under the assumption of $(\gamma-p)(p+\delta)+\alpha p > 0$. This reveals a fact that the disease may be "invaded" or always exists in the population forever. According to the expression of R_1, it is the convenient and important policy to control the occurrence of the disease that the flow of the members decreases, the vaccinated members increase, and the infectious period shortens.

3. An SIR Model with Impulsive Vaccinations

In this section, the assumption that the susceptible is vaccinated continuously is replaced by the assumption that the susceptible is vaccinated impulsively, that is, the susceptible is vaccinated at the fixed moments. At the moment $n\tau$, the vaccinated susceptible will enter directly to the removal owing to acquiring the temporary immunity, which leads to the following system:

$$
\dot{S}(t) = A - dS - \beta SI + \delta R, \quad \dot{I}(t) = \beta SI - (\gamma+\alpha+d)I, \quad \dot{R}(t) = \gamma I - (\delta+d)R, \quad t \neq n\tau,
$$

$$
S(n\tau^+) = (1-p)S(n\tau), \qquad I(n\tau^+) = I(n\tau), \qquad R(n\tau^+) = R(n\tau) + pS(n\tau),
$$

$$(3.1)$$

where $\tau > 0$ is the vaccinated period. Clearly, system (3.1) is equivalent to the system

$$\dot{S}(t) = A - dS - \beta SI + \delta(N - S - I), \quad \dot{I}(t) = \beta SI - (\gamma + \alpha + d)I, \quad \dot{N}(t) = A - dN - \alpha I,$$
$$t \neq n\tau,$$

$$S(n\tau^+) = (1 - p)S(n\tau), \qquad I(n\tau^+) = I(n\tau), \qquad N(n\tau^+) = N(n\tau).$$
$$(3.2)$$

Let $I = 0$, then system (3.2) is simplified as

$$\dot{S}(t) = A - (d + \delta)S + \delta N, \quad \dot{N}(t) = A - dN, \quad t \neq n\tau,$$
$$S(n\tau^+) = (1 - p)S(n\tau), \qquad N(n\tau^+) = N(n\tau).$$
$$(3.3)$$

Lemma 3.1. *Consider the following system:*

$$\dot{x}(t) = a - bx(t), \quad t \neq n\tau, \qquad x(n\tau^+) = (1 - p)x(n\tau). \tag{3.4}$$

Then there exists a positive periodic solution $x^*(t)$, which is globally attractive. Here $x^*(t) = a/b(1 - (p(e^{-b(t-n\tau)})/(1 - (1 - p)e^{-b\tau})))$ for $n\tau < t \leq (n + 1)\tau$.

The proof is so simple that we omit it.

By Lemma 3.1, we can easily know that there exists a positive periodic solution $(\widetilde{S}(t), \widetilde{N}(t))$ for system (3.3), and their expression is

$$\widetilde{S}(t) = \frac{A}{d}\left(1 - \frac{pe^{-(d+\delta)(t-n\tau)}}{1 - (1 - p)e^{-(d+\delta)\tau}}\right), \quad \widetilde{N}(t) = \frac{A}{d}, \quad t \neq n\tau,$$

$$\widetilde{S}(n\tau^+) = \frac{A}{d}\left[\frac{(1 - p)(1 - e^{-(d+\delta)\tau})}{1 - (1 - p)e^{-(d+\delta)\tau}}\right].$$
$$(3.5)$$

This indicates that system (3.2) has a disease-free periodic solution $(\widetilde{S}(t), 0, \widetilde{N}(t))$ with period τ, and its local stability may be determined by considering the behavior of small-amplitude perturbation of the solution. Define $(S(t), I(t), N(t)) = (\widetilde{S}(t) + x(t), y(t), \widetilde{N}(t) + z(t))$, than these may be written

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \Phi(t)\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}, \tag{3.6}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{bmatrix} -(d + \delta) & -\left(\delta + \beta\widetilde{S}(t)\right) & \delta \\ 0 & \beta\widetilde{S}(t) - (\gamma + \alpha + d) & 0 \\ 0 & -\alpha & -d \end{bmatrix}\Phi(t) \tag{3.7}$$

with $\Phi(0) = I$, the identity matrix. The resetting impulsive conditions of (3.2) become

$$
\begin{bmatrix} x(n\tau^+) \\ y(n\tau^+) \\ z(n\tau^+) \end{bmatrix} = \begin{bmatrix} 1-p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(n\tau) \\ y(n\tau) \\ z(n\tau) \end{bmatrix}.
\tag{3.8}
$$

Hence, if all eigenvalues $\mu_j (j = 1, 2, 3)$ of the monodromy matrix M,

$$
M = \begin{bmatrix} 1-p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Phi(\tau) = \begin{bmatrix} (1-p)e^{-(d+\delta)\tau} & (1-p)a_{12}(\tau) & (1-p)a_{13}(\tau) \\ 0 & e^{-(\gamma+\alpha+d)\tau+\beta\int_0^\tau \widetilde{S}(t)dt} & 0 \\ 0 & a_{32}(\tau) & e^{-d\tau} \end{bmatrix},
\tag{3.9}
$$

have absolute value less than one, then the τ-periodic solution $(\widetilde{S}(t), 0, \widetilde{N}(t))$ is locally stable. Actually, $a_{12}(\tau)$, $a_{13}(\tau)$, and $a_{32}(\tau)$ need not to be solved out, and we have $\mu_1 = (1-p)e^{-(d+\delta)\tau}$, $\mu_2 = e^{-(\gamma+\alpha+d)\tau+\beta\int_0^\tau \widetilde{S}(t)dt}$, and $\mu_3 = e^{-d\tau}$. Therefore, if $-(\gamma + \alpha + d)\tau + \beta\int_0^\tau \widetilde{S}(t)dt < 0$, which is equivalent to $A/d \cdot \beta/(\gamma + \alpha + d) \cdot (1 - p(e^{(d+\delta)\tau} - 1)/((d + \delta)(e^{(d+\delta)\tau} - 1 + p)\tau)) < 1$, then $|\mu_j| < 1$ $(j = 1, 2, 3)$ holds. And we have the following.

Theorem 3.2. *If inequality $A/d \cdot \beta/(\gamma + \alpha + d) \cdot (1 - p(e^{(d+\delta)\tau} - 1)/((d+\delta)(e^{(d+\delta)\tau} - 1 + p)\tau)) < 1$ holds, then there exists a disease-free periodic solution $(\widetilde{S}(t), 0, \widetilde{N}(t))$ for (3.2), which is locally stable. Moreover, it is globally asymptotically stable.*

Proof. Denote that $R_2 = A/d \cdot \beta/(\gamma + \alpha + d) \cdot (1 - p(e^{(d+\delta)\tau} - 1)/((d+\delta)(e^{(d+\delta)\tau} - 1 + p)\tau))$. In fact, we only prove that $\lim_{t\to\infty} S(t) = \widetilde{S}(t), \lim_{t\to\infty} I(t) = 0$, and $\lim_{t\to\infty} \widetilde{N}(t) = A/d$ for $R_2 < 1$.

For every solution $(S(t), I(t), N(t))$ of (3.2) with positive initial value $(S(0^+), I(0^+), N(0^+))$, it is clear that $\dot{N}(t) \leq A - dN$, which leads to

$$
N(t) \leq \frac{A}{d} + \varepsilon_1, \quad t > m\tau.
\tag{3.10}
$$

Here, $\varepsilon_1 > 0$ may be arbitrary small, and a positive integer m may be large enough. Hence, we have

$$
\dot{S}(t) \leq \frac{A}{d}(d + \delta) + \delta\varepsilon_1 - (d + \delta)S, \quad m\tau \leq n\tau < t \leq (n + 1)\tau,
$$
$$
S(n\tau^+) = (1 - p)S(n\tau).
\tag{3.11}
$$

Furthermore, we consider the system

$$
\dot{S}_1(t) = \frac{A}{d}(d + \delta) + \delta\varepsilon_1 - (d + \delta)S_1, \quad m\tau \leq n\tau < t \leq (n + 1)\tau,
$$
$$
S_1(n\tau^+) = (1 - p)S_1(n\tau).
\tag{3.12}
$$

Therefore, we have $S(t) \leq S_1(t)$, and $S_1(t)$ is the solution of (3.12) with the initial value $S_1(m\tau^+) = S(m\tau^+)$. In addition, for system (3.12), it is easily known that there exists a positive

periodic solution $\widetilde{S_1}(t)$, and every solution with positive initial value $S_1(0^+) > 0$ is globally attractive by Lemma 3.1. Here,

$$\widetilde{S_1}(t) = \left(\frac{A}{d} + \frac{\delta\varepsilon_1}{d+\delta}\right)\left(1 - \frac{pe^{-(d+\delta)(t-n\tau)}}{1-(1-p)e^{-(d+\delta)\tau}}\right), \quad n\tau < t \le (n+1)\tau,$$

$$\widetilde{S_1}(n\tau^+) = \left(\frac{A}{d} + \frac{\delta\varepsilon_1}{d+\delta}\right)\frac{(1-p)e^{-(d+\delta)\tau}}{1-(1-p)e^{-(d+\delta)\tau}}. \tag{3.13}$$

Thus, for an arbitrary small $\varepsilon_2 > 0$, there exists a positive integer $m_1 > m$ such that

$$S_1(t) < \widetilde{S_1}(t) + \varepsilon_2, \quad t \ge m_1\tau. \tag{3.14}$$

As a result, we have

$$\dot{I}(t) < \left[\beta\left(\widetilde{S_1}(t) + \varepsilon_2\right) - (\gamma + \alpha + d)\right]I, \quad t \ge m_1\tau. \tag{3.15}$$

So,

$$0 \le I(t) \le I(m_1\tau^+)e^{\int_{m_1\tau}^{t}[\beta(\widetilde{S_1}(t)+\varepsilon_2)-(\gamma+\alpha+d)]dt}, \quad t \ge m_1\tau. \tag{3.16}$$

Since $R_2 < 1$ holds, we may choose ε_1 and ε_2 small enough such that $\int_0^{\tau}[\beta(\widetilde{S_1}(t) + \varepsilon_2) - (\gamma + \alpha + d)]dt < 0$, which leads to

$$\lim_{t\to\infty} I(t) = 0. \tag{3.17}$$

And since

$$N(t) = N(0)e^{-dt} + \frac{A}{d}\left(1 - e^{-dt}\right) - \alpha\int_0^{t} e^{-d(t-s)}I(s)ds, \tag{3.18}$$

by (3.17), we have

$$\lim_{t\to\infty} N(t) = \frac{A}{d}. \tag{3.19}$$

Therefore, for an arbitrary small $\varepsilon_3 > 0$, there exists a positive integer $m_2(>m_1)$ such that

$$0 < I(t) < \varepsilon_3, \quad 0 < \frac{A}{d} - \varepsilon_3 < N(t) < \frac{A}{d} + \varepsilon_3, \quad t \ge m_2\tau. \tag{3.20}$$

Then, we have

$$\dot{S}(t) > \left[\frac{A(d+\delta)}{d} - 2\delta\varepsilon_3\right] - (d + \delta + \beta\varepsilon_3)S, \quad m_2\tau < n\tau < t \le (n+1)\tau,$$

$$S(n\tau^+) = (1-p)S(n\tau), \quad n\tau > m_2\tau.$$

$$(3.21)$$

Considering the system

$$\dot{S}_2(t) = \left[\frac{A(d+\delta)}{d} - 2\delta\varepsilon_3\right] - (d + \delta + \beta\varepsilon_3)S_2, \quad m_2\tau < n\tau < t \le (n+1)\tau,$$

$$S_2(n\tau^+) = (1-p)S_2(n\tau), \quad n\tau > m_2\tau,$$

$$(3.22)$$

we have $S(t) \ge S_2(t)$, where $S_2(t)$ is the solution of (3.22) with initial value $S_2(m_2\tau^+) = S(m_2\tau^+)$. Denote that $\widetilde{S_2}(t)$ is the globally asymptotically attractive and positive periodic solution; thus, for an arbitrary small number $\varepsilon_4 > 0$, there is a positive integer m_3 such that

$$\widetilde{S_2}(t) - \varepsilon_4 < S_2(t) < \widetilde{S_2}(t) + \varepsilon_4, \quad t \ge m_3\tau, \tag{3.23}$$

and since $S(t) > S_2(t)$, we may get

$$\widetilde{S_2}(t) - \varepsilon_4 < S(t), \quad t \ge m_3\tau. \tag{3.24}$$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and $M = \max\{m_2, m_3\}$, then for $t > M\tau$, we have

$$\widetilde{S_2}(t) - \varepsilon < S(t) < \widetilde{S_1}(t) + \varepsilon. \tag{3.25}$$

And since $\widetilde{S_1}(t)$ and $\widetilde{S_2}(t)$ will approach the positive periodic solution $\widetilde{S}(t)$ of system (3.3) for $\varepsilon \to 0$, therefore, we have

$$\lim_{t\to\infty} S(t) = \widetilde{S}(t). \tag{3.26}$$

Thus, the global stability of the boundary periodic solution $(\widetilde{S}(t), 0, A/d)$ has been proven. This indicates that the disease-free periodic solution $(\widetilde{S}(t), 0, \widetilde{R}(t))$ of (3.1) is also globally asymptotically stable. □

Clearly, if $A/d \cdot \beta/(\gamma + \alpha + d) < 1$ holds, then $R_2 < 1$ is true. So, we have the following.

Corollary 3.3. *If $A/d \cdot \beta/(\gamma + \alpha + d) < 1$ holds, then system (3.2) has a unique disease-free periodic solution $(\widetilde{S}(t), 0, \widetilde{N}(t))$ for arbitrary $p > 0$ and $\tau > 0$, which is globally asymptotically stable.*

In the following, one only considers the case: $A/d \cdot \beta/(\gamma + \alpha + d) > 1$.

Denote $f(\tau) \triangleq (A\beta/d - (\gamma+\alpha+d))\tau + A\beta/d \cdot p/(d+\delta) \cdot (1-\exp((d+\delta)\tau))/(\exp((d+\delta)\tau) - 1 + p)$. Notice that the function satisfies $f(\tau)$ is continuous on $[0, +\infty)$, $f(0) = 0$, $f(+\infty) = +\infty$, $f'(0) = -(\gamma+\alpha+d) < 0$, $f'(+\infty) = A\beta/d - (\gamma+\alpha+d) > 0$; moreover, $f''(\tau) = (d+\delta)p^2 e^{(d+\delta)\tau} A\beta/d \cdot (e^{(d+\delta)\tau} + 1 - p)/(e^{(d+\delta)\tau} - 1 + p)^3 > 0$. Then one has the following.
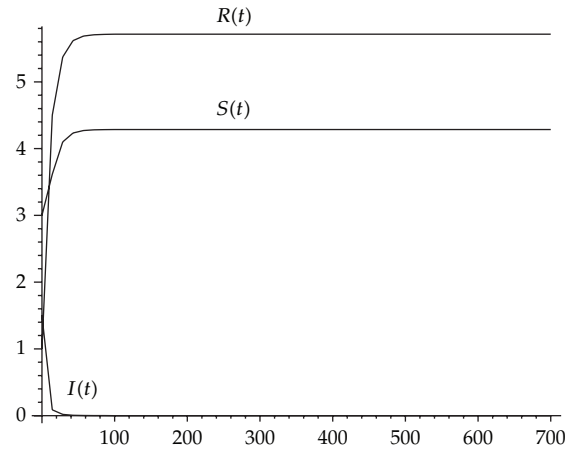
Figure 1

Lemma 3.4. *If $A/d \cdot \beta/(\gamma + \alpha + d) > 1$, then there exists and only exists a root τ_0 in $(0, +\infty)$ for function $y = f(\tau)$, and $f(\tau) < 0$ for $\tau < \tau_0$ and $f(\tau) > 0$ for $\tau > \tau_0$.*

Noting that $f(\tau) < 0$ is equivalent to $R_2 < 1$, one has the following.

Corollary 3.5. *For system (3.2), if $A/d \cdot \beta/(\gamma + \alpha + d) > 1$, then there exists $\tau_0 > 0$ such that system (3.2) has a unique disease-free periodic solution $(\widetilde{S}(t), 0, \widetilde{N}(t))$ for $\tau < \tau_0$, which is globally asymptotically stable.*

Now, we are in the position of studying the behaviors of (3.1) under the assumption of $R_2 > 1$. Here, we only discuss the existence of endemic periodic solution $(S^*(t), I^*(t), R^*(t))$ of system (3.1) with $\alpha = 0$. From the viewpoint of biology, we neglect the death-related disease. At this time, system (3.1) can be rewritten as follows:

$$\dot{S}(t) = A - dS - \beta SI + \delta(N - S - I), \quad \dot{I}(t) = \beta SI - (\gamma + d)I \quad \dot{N}(t) = A - dN, \quad t \neq n\tau,$$
$$S(n\tau^+) = (1 - p)S(n\tau), \quad I(n\tau^+) = I(n\tau), \quad N(n\tau^+) = N(n\tau).$$
$$(3.27)$$

If (3.27) has a periodic solution, it is certain that $N(t) \equiv A/d$. Then, we may change to consider the two-dimensional system (3.28), which is equivalent to (3.27).

$$\dot{S}(t) = \frac{A}{d}(d + \delta) - (d + \delta)S - \delta I - \beta SI, \quad \dot{I}(t) = \beta SI - (\gamma + d)I, \quad t \neq n\tau,$$
$$S(n\tau^+) = (1 - p)S(n\tau), \quad I(n\tau^+) = I(n\tau).$$
$$(3.28)$$

Note that there exists a positive periodic solution $S^\circ(t)$ of system

$$\dot{S} = (d + \delta)\left(\frac{A}{d} - S\right), \quad t \neq n\tau,$$
$$S(n\tau^+) = (1 - p)S(n\tau),$$
$$(3.29)$$

where

$$S^\circ(t) = \frac{A}{d}\left[1 - \frac{pe^{-(d+\delta)(t-n\tau)}}{1 - (1-p)e^{-(d+\delta)\tau}}\right], \quad n\tau < t \leq (n+1)\tau,$$

$$S^\circ(n\tau^+) = \frac{A}{d}\left[\frac{(1-p)(e^{(d+\delta)\tau} - 1)}{e^{(d+\delta)\tau} - 1 + p}\right],$$

(3.30)

then system (3.28) has a boundary periodic solution $\zeta(t) = (S^\circ(t), 0)$. With bifurcation technique and the important Theorem [16], we may obtain the positive periodic solution $(S^*(t), I^*(t))$ of (3.28). As

$$d_0' = 1 - \exp\int_0^\tau (\beta S^\circ(t) - (\gamma + d))dt,$$

$$a_0' = 1 - (1-p)\exp\int_0^\tau (-(d+\delta))dt > 0,$$

$$b_0' = -(1-p)\int_0^\tau \exp\left(\int_u^\tau -(d+\delta)dr\right)(-\beta S^\circ(u) - \delta)\exp\left(\int_0^u (\beta S^\circ(u) - (\gamma + d))dr\right)$$

$$\times du > 0,$$

$$\frac{\partial^2\Phi_2(\tau, X_0)}{\partial x_1\partial x_2} = \beta\tau\exp\int_0^\tau \frac{\partial F_2(\varsigma(r))}{\partial x_2}dr > 0;$$

$$\frac{\partial^2\Phi_2(\tau, X_0)}{\partial x_2^2} = \beta\int_0^\tau\left\{\int_0^u\left[\exp\left(\int_p^u (-(d+\delta))dr\right)(-\delta - \beta S^\circ(p))\exp\left(\int_0^p (\beta S^\circ(r) - (\gamma + d))dr\right)\right]dp\right\}$$

$$\times\left\{\exp\left(\int_u^\tau (\beta S^\circ(r) - (\gamma + d))dr\right)\right\}du < 0,$$

$$\frac{\partial^2\Phi_2(\tau, X_0)}{\partial\overline{\tau}\partial x_2} = (\beta S^\circ(\tau) - (\gamma + d))\exp\left(\int_0^\tau (\beta S^\circ(r) - (\gamma + d))dr\right);$$

$$\frac{\partial\Phi_1(\tau, X_0)}{\partial\overline{\tau}} = \dot{S}^\circ(\tau) > 0,$$

(3.31)

and if we choose τ_0, the unique root of $d_0' = f(\tau) = 0$, to be the bifurcated parameter, then we can easily see that $\partial^2\Phi_2(\tau_0, X_0)/\partial\overline{\tau}\partial x_2 > 0$. Further, we have $B < 0$ and $C > 0$. According to the theorem of Lakmeche and Arino [16], the supercritical bifurcation occurs for system (3.28).

Theorem 3.6. *If $\tau > \tau_0$ and is close to τ_0 enough, then there exists positive periodic solution $(S^*(t), I^*(t))$ for system (3.28). Here, τ_0 is the root of $d_0' = 0$.*

Further, one has the following.

Corollary 3.7. *For system (3.1), if $\tau > \tau_0$ and is close to τ_0 enough, then there exists positive periodic solution $(S^*(t), I^*(t), A/d - S^*(t) - I^*(t))$. Here, τ_0 is defined in Theorem 3.6.*
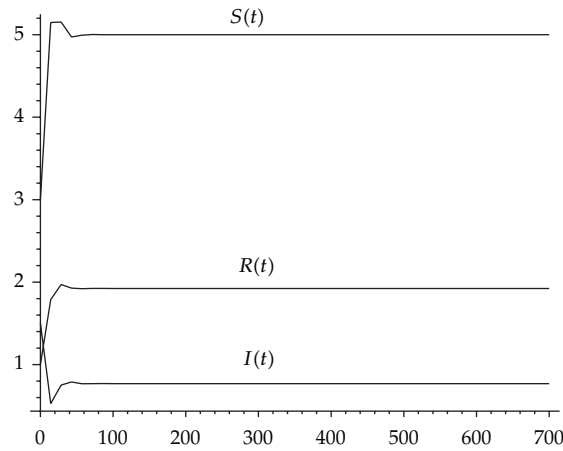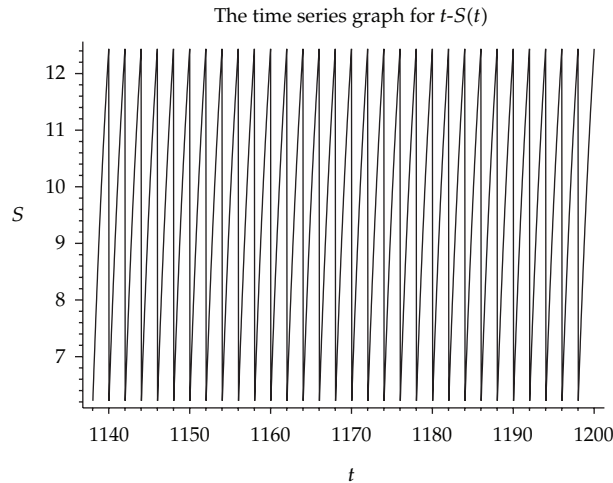
Figure 2

Figure 3

Thus, we complete the discussion on the epidemic model with impulsive vaccination. And we know that if the period of vaccination is smaller than τ_0, the disease will die out forever, but once it is larger than τ_0, the infectious disease is going to be the endemic disease.

4. Discussion and Numerical Simulation

In this paper, we first assume that the susceptible is vaccinated continuously. The model is formulated by a continuous differential system. Similar to most models for the spread of infectious diseases, there is a threshold parameter $R_1 = A/d \cdot \beta/(\gamma+\alpha+d) \cdot (\delta+d)/(\delta+d+p)$. If $R_1 < 1$, the disease-free equilibrium is approached by all solutions; if $R_1 > 1$, the disease-free equilibrium is unstable and the endemic equilibrium exists, which is locally stable. Especially, we prove that the endemic equilibrium is globally asymptotically stable if $(\gamma - p)(p + \delta) + \alpha p > 0$.
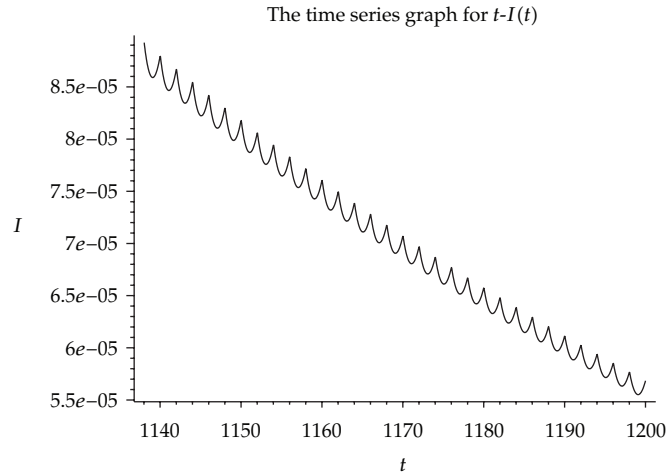
The time series graph for t-$I(t)$

Figure 4

The time series graph for t-$R(t)$
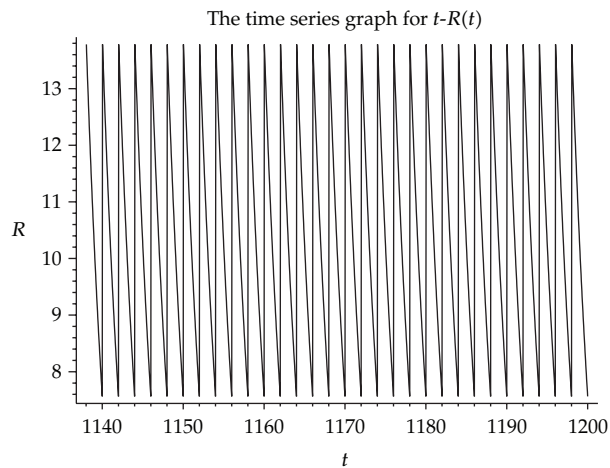

Figure 5

Obviously, there is a unique p_0 such that $R_1 = 1$. If p exceeds p_0, then $R_1 < 1$, and if p is less than p_0, then $R_1 > 1$. We show that increasing the vaccinated members is helpful to control the spread of the infectious diseases.

Let $A = 1$, $d = 0.1$, $\beta = 0.1$, $\delta = 0.2$, $p = 0.4$, $\gamma = 0.1$, and $\alpha = 0.3$, then we have $R_1 < 1$. By Theorem 2.1, we know that system (2.1) has a unique disease-free equilibrium $E^0(S^0, 0, R^0)$ with $S^0 = 4.285714286$, $R^0 = 5.714285714$, which is globally asymptotically stable. The time series of solutions with initial value $[S(0) = 3, I(0) = 1.5, R(0) = 1]$ are given in Figure 1.

If we choose $p := 0.1$, then $R_1 > 1$ and $(\gamma - p)(p + \delta) + \alpha p > 0$. By Theorem 2.2, there exists a globally asymptotically stable endemic equilibrium $E^*(S^*, I^*, R^*)$ with $S^* = 5.000000000$, $I^* = 0.7692307692$, and $R^* = 1.923076923$. Setting the initial values $S(0) = 3$, $I(0) = 1.5$, and $R(0) = 1$, we draw the time series graph of the solution of (2.1) in Figure 2.
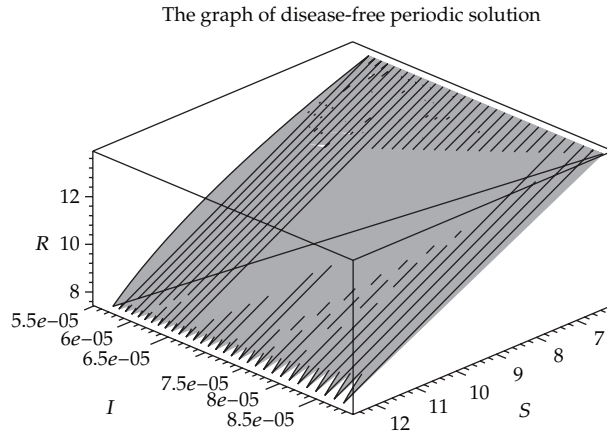
The graph of disease-free periodic solution

Figure 6

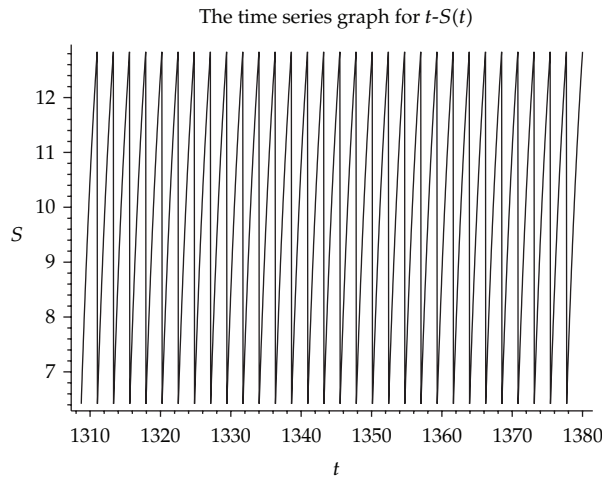The time series graph for t-$S(t)$

Figure 7

We continue to analyze our results. Let $p = 0$, that is, without vaccination on the susceptible. The model was discussed, and a threshold parameter $R = A/d \cdot \beta/(\gamma + \alpha + d)$ was obtained for such a model in [4]. It is clear that R is identical to R_1 with $p = 0$. In addition, in [4], the global stability of the endemic equilibrium was not proved. By our conclusion, we know that the endemic equilibrium is globally asymptotically stable if $R_1 > 1$, that is, we generalize the results in [4].

Second, the susceptible is supposed to be vaccinated at fixed moments. Such an epidemic model is described by an impulsive differential system, which also has a threshold parameter R_2. If $R_2 < 1$, the disease-free periodic solution is globally stable, while it is unstable for $R_2 > 1$. Clearly, if $A/d \cdot \beta/(\gamma + \alpha + d) < 1$, then $R_2 < 1$. Hence, there exists a unique globally asymptotically disease-free periodic solution. If $A/d \cdot \beta/(\gamma + \alpha + d) > 1$, then there exists a unique globally asymptotically disease-free periodic solution for $\tau < \tau_0$, and τ_0 is the root of $R_2 = 1$. With respect to the existence of positive periodic solution, we only consider
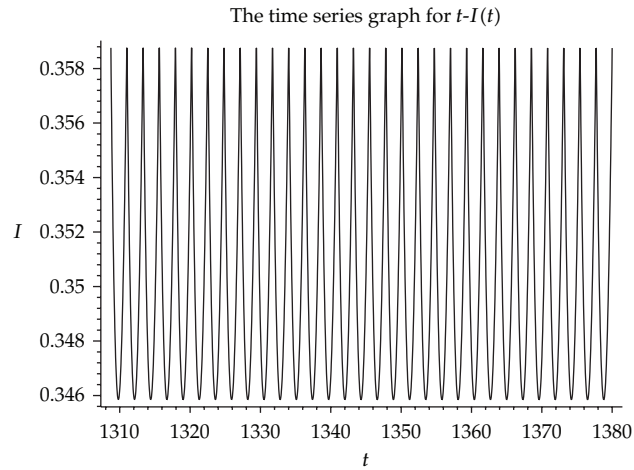
The time series graph for t-$I(t)$

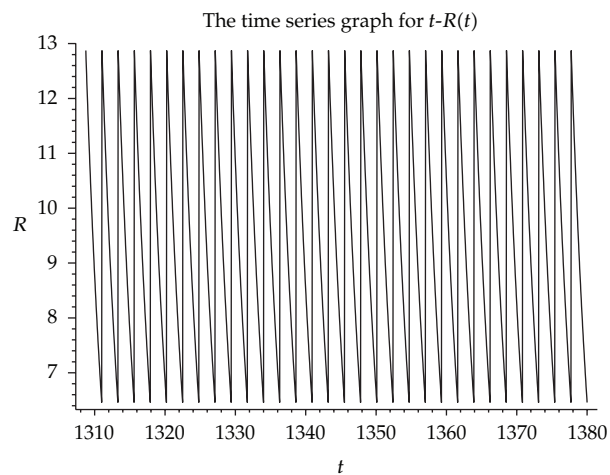**Figure** 8

The time series graph for t-$R(t)$

Figure 9

a simple model, in which $\alpha = 0$, that is, the disease-related death is neglected for the infective. The positive periodic solution exists if the vaccinated period τ satisfies $\tau > \tau_0$ and is close to the critical value τ_0 enough. It is implied that the epidemic disease may be controlled by shortening the vaccination period.

Let $A = 2$, $d = 0.1$, $\beta = 0.02$, $\delta = 0.2$, $p = 0.5$, $\gamma = 0.1$, $\alpha = 0$ and $\tau = 2$. Then $R_2 = 0.9636347352 < 1$ for impulsive system (3.1). Hence, there exists a disease-free periodic solution. Moreover, it is globally asymptotically stable. Figures 3, 4, and 5 are the time series of the solution with initial values $S(0) = 5, I(0) = 8$, and $R(0) = 5$, and Figure 6 is the trajectory phase for it, which implies that this solution tends to the disease-free periodic solution $(\widetilde{S}(t), 0, \widetilde{R}(t))$ with $\widetilde{S}(t) + \widetilde{R}(t) = A/d = 20$.

If we extend the period τ of the vaccination and choose $\tau = 2.3$, then $R_2 = 1.035848923 > 1$. By Corollary 3.7, we know that there exists a positive periodic solution
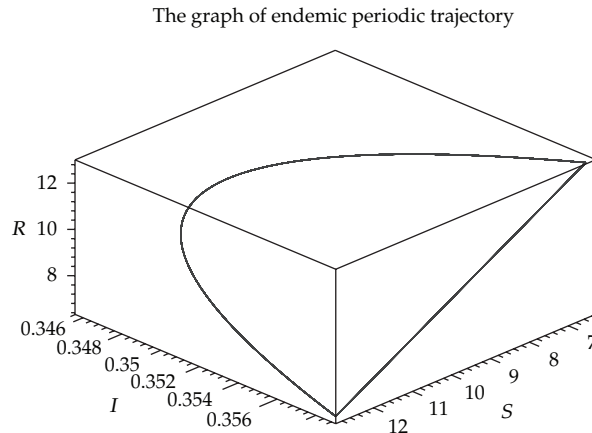
The graph of endemic periodic trajectory

Figure 10

$(S^*(t), I^*(t), R^*(t))$ with period $T = 2.3$. We draw its time-series in Figures 7, 8, and 9. And the graph of its trajectory is drawn in Figure 10.

Acknowledgments

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