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Research Article

Expected Residual Minimization Method for a Class of Stochastic Quasivariational Inequality Problems

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We consider the expected residual minimization method for a class of stochastic quasivariational inequality problems (SQVIP). The regularized gap function for quasivariational inequality problem (QVIP) is in general not differentiable. We first show that the regularized gap function is differentiable and convex for a class of QVIPs under some suitable conditions. Then, we reformulate SQVIP as a deterministic minimization problem that minimizes the expected residual of the regularized gap function and solve it by sample average approximation (SAA) method. Finally, we investigate the limiting behavior of the optimal solutions and stationary points.

1. Introduction

The quasivariational inequality problem is a very important and powerful tool for the study of generalized equilibrium problems. It has been used to study and formulate generalized Nash equilibrium problem in which a strategy set of each player depends on the other players' strategies (see, for more details, [1–3]).

QVIP is to find a vector $x^* \in S(x^*)$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in S(x^*), \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping, the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$, and $S : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued mapping of which $S(x)$ is a closed convex set in $\mathbb{R}^n$ for each x.

In particular, if S is a closed convex set and $S(x) \equiv S$ for each x, then QVIP (1.1) becomes the classical variational inequality problem (VIP): find a vector $x^* \in S$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in S. \tag{1.2}$$

In most important practical applications, the function F always involves some random factors or uncertainties. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Taking the randomness into account, we get stochastic quasivariational inequality problem (SQVIP): find an $x^* \in S(x^*)$ such that

$$P\{\omega \in \Omega : \langle F(x^*, \omega), x - x^* \rangle \geq 0, \ \forall x \in S(x^*)\} = 1, \tag{1.3}$$

or equivalently,

$$\langle F(x^*, \omega), x - x^* \rangle \geq 0, \quad \forall x \in S(x^*), \ \omega \in \Omega \text{ a.s.}, \tag{1.4}$$

where $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a mapping and a.s. is abbreviation for "almost surely" under the given probability measure P.

Due to the introduction of randomness, SQVIP (1.4) becomes more practical and also evokes more and more attentions in the recent literature [4–16]. However, to our best knowledge, most publications in the existing literature discuss the stochastic complementarity problems and the stochastic variational inequality problems, which are two special cases of (1.4). It is well known that quasivariational inequalities are more complicated than variational inequalities and complementarity problems and that they have widely applications. Therefore, it is meaningful and interesting to study the general problem (1.4).

Because of the existence of a random element ω, we cannot generally find a vector $x^* \in S(x^*)$ such that (1.4) holds almost surely. That is, (1.4) is not well defined if we think of solving (1.4) before knowing the realization ω. Therefore, in order to get a reasonable resolution, an appropriate deterministic reformulation for SQVIP becomes an important issue in the study of the considered problem.

Recently, one of the mainstreaming research methods on the stochastic variational inequality problem is expected residual minimization method (see [4, 5, 7, 11–13, 16] and the references therein). Chen and Fukushima [5] formulated the stochastic linear complementarity problem (SLCP) as a minimization problem which minimizes the expectation of gap function (also called residual function) for SLCP. They regarded the optimal solution of this minimization problem as a solution to SLCP. This method is the so-called expected residual minimization method (ERM). Following the ideas of Chen and Fukushima [5], Zhang and Chen [16] considered the stochastic nonlinear complementary problems. Luo and Lin [12, 13] generalized the expected residual minimization method to solve stochastic variational inequality problem.

In this paper, we focus on ERM method for SQVIP. We first show that the regularized gap function for QVIP is differentiable and convex under some suitable conditions. Then, we formulate SQVIP (1.4) as an optimization problem and solve this problem by SAA method.

The rest of this paper is organized as follows. In Section 2, some preliminaries and the reformulation for SQVIP are given. In Section 3, we give some suitable conditions under which the regularized gap function for QVIP is differentiable and convex. In Section 4, we show that the objective function of the reformulation problem is convex and differentiable under some suitable conditions. Finally, the convergence results of optimal solutions and stationary points are given in Section 5.

2. Preliminaries

Throughout this paper, we use the following notations. $\|\cdot\|$ denotes the Euclidean norm of a vector. For an $n \times n$ symmetric positive-definite matrix G, $\|\cdot\|_G$ denotes the G-norm defined by $\|x\|_G = \sqrt{\langle x, Gx\rangle}$ for $x \in \mathbb{R}^n$ and $\mathrm{Proj}_{S,G}(x)$ denotes the projection of the point x onto the closed convex set S with respect to the norm $\|\cdot\|_G$. For a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, $\nabla_x F(x)$ denotes the usual gradient of $F(x)$ in x. It is easy to verify that

$$\sqrt{\lambda_{\min}}\|x\| \leq \|x\|_G \leq \sqrt{\lambda_{\max}}\|x\|, \tag{2.1}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of G, respectively.

The regularized gap function for the QVIP (1.1) is given as follows:

$$f_\alpha(x) := \max_{y \in S(x)} \left\{ -\langle F(x), y - x\rangle - \frac{\alpha}{2}\|y - x\|_G^2 \right\}, \tag{2.2}$$

where α is a positive parameter. Let $X \subseteq \mathbb{R}^n$ be defined by $X = \{x \in \mathbb{R}^n : x \in S(x)\}$. This is called a feasible set of QVIP (1.1). For the relationship between the regularized gap function (2.2) and QVIP (1.1), the following result has been shown in [17, 18].

Lemma 2.1. *Let $f_\alpha(x)$ be defined by* (2.2). *Then $f_\alpha(x) \geq 0$ for all $x \in X$. Furthermore, $f_\alpha(x^*) = 0$ and $x^* \in X$ if and only if x^* is a solution to QVIP* (1.1). *Hence, problem* (1.1) *is equivalent to finding a global optimal solution to the problem:*

$$\min_{x \in X} f_\alpha(x). \tag{2.3}$$

Though the regularized gap function $f_\alpha(x)$ is directional differentiable under some suitable conditions (see, [17, 18]), it is in general nondifferentiable.

The regularized gap function (or residual function) for SQVIP (1.4) is as follows:

$$f_\alpha(x, \omega) := \max_{y \in S(x)} \left\{ -\langle F(x, \omega), y - x\rangle - \frac{\alpha}{2}\|y - x\|_G^2 \right\}, \tag{2.4}$$

and the deterministic reformulation for SQVIP is

$$\min_{x \in X} \Theta(x) := \mathbb{E} f_\alpha(x, \omega), \tag{2.5}$$

where $\mathbb{E}$ denotes the expectation operator.

Note that the objective function $\Theta(x)$ contains mathematical expectation. Throughout this paper, we assume that $\mathbb{E} f_\alpha(x, \omega)$ cannot be calculated in a closed form so that we will have to approximate it through discretization. One of the most well-known discretization approaches is sample average approximation method. In general, for an integrable function $\phi : \Omega \to \mathbb{R}$, we approximate the expected value $\mathbb{E}[\phi(\omega)]$ with sample average $(1/N_k)\sum_{\omega_i\in\Omega_k}\phi(\omega_i)$, where $\omega_1,\ldots,\omega_{N_k}$ are independently and identically distributed random samples of ω and $\Omega_k := \{\omega_1,\ldots,\omega_{N_k}\}$. By the strong law of large numbers, we get the following lemma.

Lemma 2.2. *If $\phi(\omega)$ is integrable, then*

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\phi(\omega_i)=\mathbb{E}[\phi(\omega)] \tag{2.6}$$

holds with probability one.

Let

$$\Theta_k(x) := \frac{1}{N_k}\sum_{\omega_i\in\Omega_k} f_\alpha(x,\omega_i). \tag{2.7}$$

Applying the above techniques, we can get the following approximation of (2.5):

$$\min_{x\in X}\Theta_k(x). \tag{2.8}$$

3. Convexity and Differentiability of $f_\alpha(x)$

In the remainder of this paper, we restrict ourself to a special case, where $S(x) = S + m(x)$. Here, S is a closed convex set in $\mathbb{R}^n$ and $m(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping. In this case, we can show that $f_\alpha(x)$ is continuously differentiable whenever so are the functions $F(x)$ and $m(x)$. In order to get this result, we need the following lemma (see [19, Chapter 4, Theorem 1.7]).

Lemma 3.1. *Let $S \in \mathbb{R}^n$ be a nonempty closed set and $U \in \mathbb{R}^m$ be an open set. Assume that $f : \mathbb{R}^n \times U \to \mathbb{R}$ be continuous and the gradient $\nabla_u f(\cdot,\cdot)$ is also continuous. If the problem $\min_{x\in S} f(x,u)$ is uniquely attained at $x(u)$ for any fixed $u \in U$, then the function $\phi(u) := \min_{x\in S} f(x,u)$ is continuously differentiable and $\nabla_u\phi(u)$ is given by $\nabla_u\phi(u) = \nabla_u f(x(u),u)$.*

For any $y \in S(x) = S + m(x)$, we can find a vector $z \in S$ such that $y = z + m(x)$. Thus, we can rewrite (2.2) as follows:

$$\begin{aligned} f_\alpha(x) &= \max_{z\in S}\left\{-\langle F(x), z+m(x)-x\rangle - \frac{\alpha}{2}\|z-(x-m(x))\|_G^2\right\} \\ &= -\min_{z\in S}\left\{\langle F(x), z-(x-m(x))\rangle + \frac{\alpha}{2}\|z-(x-m(x))\|_G^2\right\}. \end{aligned} \tag{3.1}$$

The minimization problem in (3.1) is essentially equivalent to the following problem:

$$\min_{z\in S}\left\|z-\left[x-m(x)-\alpha^{-1}G^{-1}F(x)\right]\right\|_G^2. \tag{3.2}$$

It is easy to know that problem (3.2) has a unique optimal solution $\mathrm{Proj}_{S,G}(x - m(x) - \alpha^{-1}G^{-1}F(x))$. Thus, $\mathrm{Proj}_{S,G}(x - m(x) - \alpha^{-1}G^{-1}F(x))$ is also a unique solution of problem (3.1). The following result is a natural extension of [20, Theorem 3.2].

Theorem 3.2. *If S is a closed convex set in $\mathbb{R}^n$ and $m(x)$ and $F(x)$ are continuously differentiable, then the regularized gap function $f_\alpha(x)$ given by (2.2) is also continuously differentiable and its gradient is given by*

$$\nabla f_\alpha(x) = [I - \nabla m(x)]F(x) - [\nabla F(x) - \alpha(I - \nabla m(x))G][z_\alpha(x) - (x - m(x))], \tag{3.3}$$

where $z_\alpha(x) = Proj_{S,G}(x - m(x) - \alpha^{-1}G^{-1}F(x))$ and I denotes the $n \times n$ identity matrix.

Proof. Let us define the function $h : \mathbb{R}^n \times S \to \mathbb{R}$ by

$$h(x, z) = \langle F(x), z - (x - m(x)) \rangle + \frac{\alpha}{2}\|z - (x - m(x))\|_G^2. \tag{3.4}$$

It is obviously that if $F(x)$ and $m(x)$ are continuous, then $h(x, z)$ is continuous in (x, z). If $F(x)$ and $m(x)$ are continuously differentiable, then

$$\nabla_x h(x, z) = -[I - \nabla m(x)]F(x) + [\nabla F(x) - \alpha(I - \nabla m(x))G][z - (x - m(x))] \tag{3.5}$$

is continuous in (x, z). By (3.1), we have

$$f_\alpha(x) = -\min_{z \in S} h(x, z). \tag{3.6}$$

Since the minimum on the right-hand side of (3.6) is uniquely attained at $z = z_\alpha(x)$, it follows from Lemma 3.1 that $f_\alpha(x)$ is differentiable and its gradient is given by

$$\begin{aligned} \nabla f_\alpha(x) &= -\nabla_x h(x, z_\alpha(x)) \\ &= [I - \nabla m(x)]F(x) - [\nabla F(x) - \alpha(I - \nabla m(x))G][z_\alpha(x) - (x - m(x))]. \end{aligned} \tag{3.7}$$

This completes the proof. □

Remark 3.3. When $m(x) \equiv 0$, we have $S(x) \equiv S$ and so QVIP (1.1) reduces to VIP (1.2). In this case

$$\nabla f_\alpha(x) = F(x) - [\nabla F(x) - \alpha G][z_\alpha(x) - x], \tag{3.8}$$

where

$$z_\alpha(x) = \mathrm{Proj}_{S,G}\left(x - \alpha^{-1}G^{-1}F(x)\right). \tag{3.9}$$

Moreover, when $\alpha = 1$, we have

$$\begin{aligned} \nabla f_\alpha(x) &= F(x) - [\nabla F(x) - G][z_\alpha(x) - x], \\ z_\alpha(x) &= \mathrm{Proj}_{S,G}\left(x - G^{-1}F(x)\right), \end{aligned} \tag{3.10}$$

which is the same as [20, Theorem 3.2].

Now we investigate the conditions under which $f_\alpha(x)$ is convex.

Theorem 3.4. *Suppose that $F(x) = Mx + q$ and $m(x) = Nx$, where M and N are $n \times n$ matrices and $q \in \mathbb{R}^n$ is a vector. Denote $\beta_{\min}$ and $\mu_{\max}$ by the smallest and largest eigenvalues of $M^T(I - N) + (I - N)^T M$ and $(N - I)^T G(N - I)$, respectively. We have the following statements.*

(i) *If $\mu_{\max} > 0$, $\beta_{\min} \geq 0$ and $\alpha \leq (\beta_{\min}/\mu_{\max})$, then the function $f_\alpha(x)$ is convex. Moreover, if there exists a constant $\beta > 0$ such that $\alpha \leq (\beta_{\min}/\mu_{\max}(1 + \beta))$, then $f_\alpha(x)$ is strongly convex with modulus $\alpha\beta\mu_{\max}$.*

(ii) *If $\mu_{\max} = 0$ and $\beta_{\min} \geq 0$, then the function $f_\alpha(x)$ is convex. Moreover, if $\beta_{\min} > 0$, then $f_\alpha(x)$ is strongly convex with modulus $\beta_{\min}$.*

Proof. Substituting $F(x) = Mx + q$ and $m(x) = Nx$ into (3.1), we have

$$f_\alpha(x) = \max_{z \in S}\left\{-\langle Mx + q, z + (N - I)x\rangle - \frac{\alpha}{2}\|z - (I - N)x\|_G^2\right\}. \tag{3.11}$$

Define

$$H(x, z) = -\langle Mx + q, z + (N - I)x\rangle - \frac{\alpha}{2}\|z - (I - N)x\|_G^2. \tag{3.12}$$

Noting that

$$\nabla_x^2 H(x, z) = M^T(I - N) + (I - N)^T M - \alpha(N - I)^T G(N - I), \tag{3.13}$$

we have, for any $y \in \mathbb{R}^n$,

$$\begin{aligned} y^T \nabla_x^2 H(x, z) y &= y^T\left[M^T(I - N) + (I - N)^T M\right]y - \alpha y^T(N - I)^T G(N - I)y \\ &\geq (\beta_{\min} - \alpha\mu_{\max})\|y\|^2. \end{aligned} \tag{3.14}$$

If $\mu_{\max} > 0$, $\beta_{\min} \geq 0$ and $\alpha \leq (\beta_{\min}/\mu_{\max})$, we have

$$y^T \nabla_x^2 H(x,z) y \geq (\beta_{\min} - \alpha\mu_{\max}) \|y\|^2 \geq 0. \tag{3.15}$$

This implies that the Hessen matrix $\nabla_x^2 H(x,z)$ is positive semidefinite and hence $H(x,z)$ is convex in x for any $z \in S$. In consequence, by (3.11), the regularized gap function $f_\alpha(x)$ is convex. Moreover, if $\alpha \leq (\beta_{\min}/\mu_{\max}(1+\beta))$, then

$$y^T \nabla_x^2 H(x,z) y \geq (\beta_{\min} - \alpha\mu_{\max}) \|y\|^2 \geq \alpha\beta\mu_{\max} \|y\|^2, \tag{3.16}$$

which means that $H(x,z)$ is strongly convex with modulus $\alpha\beta\mu_{\max}$ in x for any $z \in S$. From (3.11), we know that the regularized gap function $f_\alpha(x)$ is strongly convex.

If $\mu_{\max} = 0$ and $\beta_{\min} \geq 0$, we have

$$y^T \nabla_x^2 H(x,z) y \geq \beta_{\min} \|y\|^2 \geq 0. \tag{3.17}$$

Thus, the regularized gap function $f_\alpha(x)$ is convex. Moreover, if $\beta_{\min} > 0$, then the regularized gap function $f_\alpha(x)$ is strongly convex with modulus $\beta_{\min}$. This completes the proof. □

Remark 3.5. When $N = 0$, QVIP (1.1) reduces to VIP (1.2). Denote $\overline{\beta}_{\min}$ and $\overline{\mu}_{\max}$ by the smallest and largest eigenvalues of $M^T + M$ and G, respectively. In this case, the function

$$\overline{f}_\alpha(x) = \max_{z \in S} \left\{ -\langle F(x), z - x \rangle - \frac{\alpha}{2} \|z - x\|_G^2 \right\} \tag{3.18}$$

is convex when $\overline{\mu}_{\max} > 0$, $\overline{\beta}_{\min} \geq 0$ and $\alpha \leq (\overline{\beta}_{\min}/\overline{\mu}_{\max})$.

Remark 3.6. When $N = 0$ and $G = I$, we have that $\overline{\mu}_{\max} = 1$. In this case, the function

$$\widehat{f}_\alpha(x) = \max_{z \in S} \left\{ -\langle F(x), z - x \rangle - \frac{\alpha}{2} \|z - x\|^2 \right\} \tag{3.19}$$

is convex when $\overline{\beta}_{\min} \geq 0$ and $\alpha \leq \overline{\beta}_{\min}$. This is consistent with [4, Theorem 2.1].

4. Properties of Function Θ

In this section, we consider the properties of the objective function $\Theta(x)$ of problem (2.5). In what follows we show that $\Theta(x)$ is differentiable under some suitable conditions.

Theorem 4.1. *Suppose that $F(x,\omega) := M(\omega)x + Q(\omega)$, where $M : \Omega \to \mathbb{R}^{n\times n}$ and $Q : \Omega \to \mathbb{R}^n$ with*

$$\mathbb{E}\left[\|M(\omega)\|^2 + \|Q(\omega)\|^2 \right] < +\infty. \tag{4.1}$$

Let $S(x) = S + Nx$. Then the function $\Theta(x)$ is differentiable and

$$\nabla_x \Theta(x) = \mathbb{E}\nabla_x f_\alpha(x,\omega). \tag{4.2}$$

Proof. Since $S(x) = S + Nx$, it is easy to know that

$$f_\alpha(x,\omega) = -\langle F(x,\omega), y_\alpha(x,\omega) - (x - Nx)\rangle - \frac{\alpha}{2}\left\|y_\alpha(x,\omega) - (x - Nx)\right\|_G^2, \tag{4.3}$$

where

$$y_\alpha(x,\omega) = \mathrm{Proj}_{S,G}\left(x - Nx - \alpha^{-1}G^{-1}F(x,\omega)\right). \tag{4.4}$$

It follows from Lemma 2.1 that $f_\alpha(x,\omega) \geq 0$ and so

$$\begin{aligned}\frac{\alpha}{2}\left\|y_\alpha(x,\omega) - x + Nx\right\|_G^2 &\leq -\langle F(x,\omega), y_\alpha(x,\omega) - x + Nx\rangle \\ &\leq \|F(x,\omega)\|\left\|y_\alpha(x,\omega) - x + Nx\right\| \\ &\leq \frac{1}{\sqrt{\lambda_{\min}}}\|F(x,\omega)\|\left\|y_\alpha(x,\omega) - x + Nx\right\|_G.\end{aligned} \tag{4.5}$$

Thus,

$$\begin{gathered}\left\|y_\alpha(x,\omega) - x + Nx\right\|_G \leq \frac{2}{\alpha\sqrt{\lambda_{\min}}}\|F(x,\omega)\|, \\ \left\|y_\alpha(x,\omega) - x + Nx\right\| \leq \frac{1}{\sqrt{\lambda_{\min}}}\left\|y_\alpha(x,\omega) - x + Nx\right\|_G \leq \frac{2}{\alpha\lambda_{\min}}\|F(x,\omega)\|.\end{gathered} \tag{4.6}$$

In a similar way to Theorem 3.2, we can show that $f_\alpha(x,\omega)$ is differentiable with respect to x and

$$\nabla_x f_\alpha(x,\omega) = (I - N)F(x,\omega) - [M(\omega) - \alpha(I - N)G]\left[y_\alpha(x,\omega) - (I - N)x\right]. \tag{4.7}$$

It follows that

$$\begin{aligned}\left\|\nabla_x f_\alpha(x,\omega)\right\| &\le \|I-N\|\|F(x,\omega)\| - \|M(\omega)-\alpha(I-N)G\|\left\|y_\alpha(x,\omega)-(I-N)x\right\| \\ &\le \left\{\|I-N\| + \frac{2}{\alpha\lambda_{\min}}\|M(\omega)-\alpha(I-N)G\|\right\}\|F(x,\omega)\| \\ &\le \left(1+\frac{2\|G\|}{\lambda_{\min}}\right)\|I-N\|\|F(x,\omega)\| + \frac{2}{\alpha\lambda_{\min}}\|M(\omega)\|\|F(x,\omega)\| \\ &\le \left(1+\frac{2\|G\|}{\lambda_{\min}}\right)\|I-N\|(1+\|x\|)(\|M(\omega)\|+\|Q(\omega)\|) \\ &\quad + \frac{2}{\alpha\lambda_{\min}}(1+\|x\|)(\|M(\omega)\|+\|Q(\omega)\|)^2 \\ &\le \left(1+\frac{2\|G\|}{\lambda_{\min}}\right)\|I-N\|(1+\|x\|)(\|M(\omega)\|+\|Q(\omega)\|) \\ &\quad + \frac{4}{\alpha\lambda_{\min}}(1+\|x\|)\left(\|M(\omega)\|^2+\|Q(\omega)\|^2\right).\end{aligned} \tag{4.8}$$

By [21, Theorem 16.8], the function Θ is differentiable and $\nabla_x\Theta(x) = \mathbb{E}\nabla_x f_\alpha(x,\omega)$. This completes the proof. □

The following theorem gives some conditions under which $\Theta(x)$ is convex.

Theorem 4.2. *Suppose that the assumption of Theorem 4.1 holds. Let*

$$\beta_0 := \inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{\min}\left(M(\omega)^T(I-N) + (I-N)^T M(\omega)\right), \tag{4.9}$$

where Ω_0 is a null subset of Ω and $\lambda_{\min}(G)$ denotes the smallest eigenvalue of G. We have the following statements.

(i) *If $\mu_{\max} > 0$, $\beta_0 > 0$ and $\alpha \le (\beta_0/\mu_{\max})$, then the function $\Theta(x)$ is convex. Moreover, if $\alpha \le (\beta_0/\mu_{\max}(1+\beta))$ with $\beta > 0$, then $\Theta(x)$ is strongly convex with modulus $\alpha\beta\mu_{\max}$.*

(ii) *If $\mu_{\max} = 0$ and $\beta_0 \ge 0$, then the function $\Theta(x)$ is convex. Moreover, if $\beta_0 > 0$, then $\Theta\alpha(x)$ is strongly convex with modulus β_0.*

Proof. Define

$$H(x,z,\omega) = -\langle M(\omega)x + Q(\omega), z + (N-I)x\rangle - \frac{\alpha}{2}\|z-(I-N)x\|_G^2. \tag{4.10}$$

Noting that

$$\nabla_x^2 H(x,z,\omega) = M(\omega)^T(I-N) + (I-N)^T M(\omega) - \alpha(N-I)^T G(N-I), \tag{4.11}$$

we have, for any $y \in \mathbb{R}^n$,

$$\begin{aligned} y^T \nabla_x^2 H(x,z,\omega) y &= y^T \left[M(\omega)^T (I-N) + (I-N)^T M(\omega) \right] y - \alpha y^T (N-I)^T G(N-I) y \\ &\geq (\beta_0 - \alpha \mu_{\max}) \ \|y\|^2, \end{aligned} \tag{4.12}$$

where the inequality holds almost surely.

If $\mu_{\max} > 0$, $\beta_0 > 0$ and $\alpha \leq (\beta_0 / \mu_{\max})$, then

$$y^T \nabla_x^2 H(x,z,\omega) y \geq 0. \tag{4.13}$$

This implies that the Hessen matrix $\nabla_x^2 H(x,z,\omega)$ is positive semidefinite and hence $H(x,z,\omega)$ is convex in x for any $z \in S$. Since

$$f_\alpha(x,\omega) = \max_{y \in S(x)} \left\{ -\langle F(x,\omega), y - x \rangle - \frac{\alpha}{2} \|y - x\|_G^2 \right\} = \max_{z \in S} H(x,z,\omega), \tag{4.14}$$

the regularized gap function $f_\alpha(x,\omega)$ is convex and so is $\Theta(x)$. Moreover, if $\alpha \leq (\beta_0 / \mu_{\max}(1 + \beta))$, then

$$y^T \nabla_x^2 H(x,z,\omega) y \geq \alpha \beta \mu_{\max} \|y\|^2, \tag{4.15}$$

which means that $H(x,z,\omega)$ is strongly convex in x for any $z \in S$. From the definitions of $H(x,z,\omega)$ and $f_\alpha(x,\omega)$, we know that $f_\alpha(x,\omega)$ is strongly convex with modulus $\alpha\beta\mu_{\max}$ and so is $\Theta(x)$.

If $\mu_{\max} = 0$ and $\beta_0 \geq 0$, then

$$y^T \nabla_x^2 H(x,z,\omega) y \geq \beta_0 \|y\|^2 \geq 0, \tag{4.16}$$

which implies that the regularized gap function $f_\alpha(x,\omega)$ is convex and so is $\Theta(x)$. Moreover, if $\beta_0 > 0$, then $\Theta(x)$ is strongly convex with modulus β_0. This completes the proof. □

It is easy to verify that $X = \{x \in \mathbb{R}^n : x \in S(x)\}$ is a convex subset when $S(x) = S + Nx$. Thus, Theorem 4.2 indicates that problem (2.5) is a convex program. From the proof details of Theorem 4.2, we can also get that problem (2.8) is a convex program. Hence we can obtain a global optimal solution using existing solution methods.

5. Convergence of Solutions and Stationary Points

In this section, we will investigate the limiting behavior of the optimal solutions and stationary points of (2.8).

Note that if the conditions of Theorem 4.1 are satisfied, then the set X is closed, and

$$\mathbb{E}\|M(\omega)\| < \infty, \qquad \mathbb{E}[\|M(\omega)\| + \|Q(\omega)\| + c]^2 < \infty, \tag{5.1}$$

where c is a constant.

Theorem 5.1. *Suppose that the conditions of Theorem 4.1 are satisfied. Let x^k be an optimal solution of problem* (2.8) *for each k. If x^* is an accumulation point of $\{x^k\}$, then it is an optimal solution of problem* (2.5)*.*

Proof. Without loss of generality, we assume that x^k itself converges to x^* as k tends to infinity. It is obvious that $x^* \in X$.

We first show that

$$\lim_{k\to\infty}\left|\Theta_k\left(x^k\right) - \Theta_k(x^*)\right| = 0. \tag{5.2}$$

It follows from mean-value theorem that

$$\begin{aligned}
\left|\Theta_k\left(x^k\right) - \Theta_k(x^*)\right| &= \left|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left[f_\alpha\left(x^k,\omega_i\right) - f_\alpha(x^*,\omega_i)\right]\right| \\
&\le \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left|f_\alpha\left(x^k,\omega_i\right) - f_\alpha(x^*,\omega_i)\right| \\
&\le \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left\|\nabla_x f_\alpha\left(y_i^k,\omega_i\right)\right\|\left\|x^k - x^*\right\|,
\end{aligned} \tag{5.3}$$

where $y_i^k = \gamma_i^k x^k + (1-\gamma_i^k)x^*$ and $\gamma_i^k \in [0,1]$. From the proof details of Theorem 4.1, we have

$$\begin{aligned}
\left\|\nabla_x f_\alpha\left(y_i^k,\omega_i\right)\right\| \le{}& \left(1 + \frac{2\|G\|}{\lambda_{\min}}\right)\left(1 + \left\|y_i^k\right\|\right)(\|M(\omega_i)\| + \|Q(\omega_i)\|)\|I - N\| \\
&+ \frac{2}{\alpha\lambda_{\min}}\left(1 + \left\|y_i^k\right\|\right)(\|M(\omega_i)\| + \|Q(\omega_i)\|)^2.
\end{aligned} \tag{5.4}$$

Since $\lim_{k\to+\infty} x^k = x^*$, there exists a constant C such that $\|x^k\| \le C$ for each k. By the definition of y_i^k, we know that $\|y_i^k\| \le C$. Hence,

$$\begin{aligned}
\left\|\nabla_x f_\alpha\left(y_i^k,\omega_i\right)\right\| &\le \left(1 + \frac{2\|G\|}{\lambda_{\min}}\right)(1+C)(\|M(\omega_i)\| + \|Q(\omega_i)\|)\|I - N\| \\
&\quad + \frac{2}{\alpha\lambda_{\min}}(1+C)(\|M(\omega_i)\| + \|Q(\omega_i)\|)^2 \\
&\le C'(\|M(\omega_i)\| + \|Q(\omega_i)\| + 1)^2,
\end{aligned} \tag{5.5}$$

where

$$C' = \max\left\{\left(1 + \frac{2\|G\|}{\lambda_{\min}}\right)(1+C)\|I - N\|,\ \frac{2}{\alpha\lambda_{\min}}(1+C)\right\}. \tag{5.6}$$

It follows that

$$\left|\Theta_k\left(x^k\right) - \Theta_k(x^*)\right| \le C' \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} (\|M(\omega_i)\| + \|Q(\omega_i)\| + 1)^2 \left\|x^k - x^*\right\| \longrightarrow 0, \tag{5.7}$$

which means that (5.2) holds.

Now, we show that x^* is an optimal solution of problem (2.5). It follows from (5.2) and

$$\left|\Theta_k\left(x^k\right) - \Theta(x^*)\right| \le \left|\Theta_k\left(x^k\right) - \Theta_k(x^*)\right| + |\Theta_k(x^*) - \Theta(x^*)|, \tag{5.8}$$

that $\lim_{k\to+\infty}\Theta_k(x^k) = \Theta(x^*)$. Since x^k is an optimal solution of problem (2.8) for each k, we have that, for any $x \in X$,

$$\Theta_k\left(x^k\right) \le \Theta_k(x). \tag{5.9}$$

Letting $k \to \infty$ above, we get from (5.2) and Lemma 2.2 that

$$\Theta(x^*) \le \Theta(x), \tag{5.10}$$

which means x^* is an optimal solution of problem (2.5). This completes the proof. □

In general, it is difficult to obtain a global optimal solution of problem (2.8), whereas computation of stationary points is relatively easy. Therefore, it is important to study the limiting behavior of stationary points of problem (2.8).

Definition 5.2. x^k is said to be stationary to problem (2.8) if

$$\left\langle \nabla_x \Theta_k\left(x^k\right), y - x^k \right\rangle \ge 0, \quad \forall y \in X, \tag{5.11}$$

and x^* is said to be stationary to problem (2.5) if

$$\langle \nabla_x \Theta(x^*), y - x^* \rangle \ge 0, \quad \forall y \in X. \tag{5.12}$$

Theorem 5.3. *Let x^k be stationary to problem* (2.8) *for each k. If the conditions of Theorem 4.1 are satisfied, then any accumulation point x^* of $\{x^k\}$ is a stationary point of problem* (2.5).

Proof. Without loss of generality, we assume that $\{x^k\}$ itself converges to x^*.

At first, we show that

$$\lim_{k\to\infty} \left\|\nabla_x \Theta_k\left(x^k\right) - \nabla_x \Theta_k(x^*)\right\| = 0. \tag{5.13}$$

It follows from (2.1) and the nonexpansivity of the projection operator that

$$
\begin{aligned}
&\left\| y_\alpha\left(x^k,\omega\right) - y_\alpha(x^*,\omega)\right\| \\
&\quad\le \frac{1}{\sqrt{\lambda_{\min}}}\left\| y_\alpha(x^k,\omega) - y_\alpha(x^*,\omega)\right\|_G \\
&\quad= \frac{1}{\sqrt{\lambda_{\min}}}\Big\| \mathrm{Proj}_{S,G}\left(x^k - Nx^k - \alpha^{-1}G^{-1}F\left(x^k,\omega\right)\right) \\
&\qquad\qquad - \mathrm{Proj}_{S,G}\left(x^* - Nx^* - \alpha^{-1}G^{-1}F(x^*,\omega)\right)\Big\|_G \\
&\quad\le \frac{1}{\sqrt{\lambda_{\min}}}\left\| x^k - Nx^k - \alpha^{-1}G^{-1}F\left(x^k,\omega\right) - \left[x^* - Nx^* - \alpha^{-1}G^{-1}F(x^*,\omega)\right]\right\|_G \\
&\quad\le \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left\| x^k - Nx^k - \alpha^{-1}G^{-1}F\left(x^k,\omega\right) - \left[x^* - Nx^* - \alpha^{-1}G^{-1}F(x^*,\omega)\right]\right\| \\
&\quad\le \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left(\|I-N\| + \alpha^{-1}\left\|G^{-1}\right\|\|M(\omega)\|\right)\left\|x^k - x^*\right\|.
\end{aligned}
\tag{5.14}
$$

Thus,

$$
\begin{aligned}
&\left\|\nabla_x\Theta_k\left(x^k\right) - \nabla_x\Theta_k(x^*)\right\| \\
&\quad\le \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left\|\nabla_x f_\alpha\left(x^k,\omega_i\right) - \nabla_x f_\alpha(x^*,\omega_i)\right\| \\
&\quad= \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\Big\|(I-N)\left[M(\omega_i)x^k + Q(\omega_i)\right] - \left[M(\omega_i) - \alpha(I-N)G\right] \\
&\qquad\qquad \times\left[y_\alpha\left(x^k,\omega_i\right) - (I-N)x^k\right] \\
&\qquad\qquad - \{(I-N)[M(\omega_i)x^* + Q(\omega_i)] - [M(\omega_i) - \alpha(I-N)G] \\
&\qquad\qquad\quad \times[y_\alpha(x^*,\omega_i) - (I-N)x^*]\}\Big\| \\
&\quad\le 2\left\|x^k - x^*\right\|\|I-N\|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)\| + \alpha\|I-N\|^2\|G\|\left\|x^k - x^*\right\| \\
&\qquad + \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i) - \alpha(I-N)G\|\left\|y_\alpha\left(x^k,\omega_i\right) - y_\alpha(x^*,\omega_i)\right\| \\
&\quad\le 2\left\|x^k - x^*\right\|\|I-N\|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)\| + \alpha\|I-N\|^2\|G\|\left\|x^k - x^*\right\| \\
&\qquad + \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i) - \alpha(I-N)G\|\left[\|I-N\| + \alpha^{-1}\left\|G^{-1}\right\|\|M(\omega_i)\|\right]\left\|x^k - x^*\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \left[2+\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left(\|G\|\left\|G^{-1}\right\|+1\right)\right]\|I-N\|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)\|\left\|x^k-x^*\right\| \\
&\quad+\left(1+\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\right)\alpha\|I-N\|^2\|G\|\left\|x^k-x^*\right\| \\
&\quad+\alpha^{-1}\left\|G^{-1}\right\|\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\|M(\omega_i)\|^2\left\|x^k-x^*\right\| \\
&\longrightarrow 0,
\end{aligned}
\tag{5.15}
$$

which means that (5.13) is true.

Next, we show that

$$
\lim_{k\to\infty}\nabla_x\Theta_k\left(x^k\right)=\nabla_x\Theta(x^*). \tag{5.16}
$$

It follows from Lemma 2.2 and Theorem 4.1 that

$$
\lim_{k\to\infty}\nabla_x\Theta_k(x^*)=\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\nabla_x f_\alpha(x^*,\omega_i)=\mathbb{E}\nabla_x f_\alpha(x^*,\omega)=\nabla_x\Theta(x^*). \tag{5.17}
$$

By (5.13), we have

$$
\begin{aligned}
\left\|\nabla_x\Theta_k\left(x^k\right)-\nabla_x\Theta(x^*)\right\| &\le \left\|\nabla_x\Theta_k\left(x^k\right)-\nabla_x\Theta_k(x^*)\right\|+\|\nabla_x\Theta_k(x^*)-\nabla_x\Theta(x^*)\| \\
&\longrightarrow 0,
\end{aligned}
\tag{5.18}
$$

which implies that (5.16) is true.

Now we show that x^* is a stationary point of problem (2.5). Since x^k is stationary to problem (2.8), that is, for any $y\in X$,

$$
\left\langle\nabla_x\Theta_k\left(x^k\right),y-x^k\right\rangle\ge 0. \tag{5.19}
$$

Letting $k\to\infty$ above, we get from (5.16) that

$$
\langle\nabla_x\Theta(x^*),y-x^*\rangle\ge 0. \tag{5.20}
$$

Thus, x^* is a stationary point of problem (2.5). This completes the proof. □

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