Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 813060, 7 pages
doi:10.1155/2012/813060

Research Article

On the Bankruptcy Situations and the Alexia Value

S. Z. Alparslan Gök[1, 2] and A. Sarıarslan[1]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Süleyman Demirel University, 32 260 Isparta, Turkey*
[2] *Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey*

Correspondence should be addressed to S. Z. Alparslan Gök, zeynepalparslan@yahoo.com

Received 27 March 2012; Revised 22 June 2012; Accepted 15 July 2012

Academic Editor: Maurizio Porfiri

The main result of this paper is to show that the three ancient bankruptcy situations from the 2000-year-old Babylonian Talmud can be solved by using the average lexicographic value (Alexia) from cooperative game theory.

1. Introduction

This paper is based on the results of Guiasu [1] who solves the three ancient problems, namely, the bankruptcy problem, the contested garment problem, and the rights arbitration problem, from the 2000-year-old Babylonian Talmud by using the Shapley value [2] from cooperative game theory. We also refer to such games as a TU (transferable utility) game. The main objective of Guiasu [1] is to show that the Shapley value can be used for justifying the ancient solutions given to all these three problems viewed as cooperative n-person games.

Recently, Tijs et al. [3] introduce the average lexicographic (Alexia) value, a value which averages the lexicographic maxima of the core [4], for games with a nonempty core and show that the Alexia value coincides with the Shapley value for the class of convex games. Further, Curiel et al. [5] show that bankruptcy games are convex.

We notice that two solution concepts are dominant in game theory. One is the Nash equilibrium set [6], and the other is the core. There are remarkable similarities if one looks at the role of the Nash equilibrium set in noncooperative game theory and the role of the core in cooperative game theory. Noncooperative games without Nash equilibria as well as cooperative games with an empty core are not very attractive for a game theorist and also not in practice.

In the sequel we show that the Alexia value can also be an alternative to the Shapley value for the solution of these problems, and note that in general the most popular one-point solution concepts such as the Shapley value for general TU-games do not provide a core element as a solution, but the Alexia value provides a core element as a solution for all games with a nonempty core.

Bankruptcy situations and the problems from the Talmud have been intensively studied in literature [7–10]. In this study we aim to give a uniform method by using the Alexia value for solving these problems which is done by Guiasu [1] by using the Shapley value.

We first consider the bankruptcy problem which is known as Talmudic problem of three wives. In the story a man married with three women and promised them in their marriage contract the sum of $d_1 = 100$, $d_2 = 200$, and $d_3 = 300$ units of money after his death. But, the estate E was less than 600 units. The division of the estate among the three wives is as follows: for $E = 100$, the wives get 33(1/3), and 33(1/3), 33(1/3), for $E = 200$ the wives get 50, 75, and 75, and for $E = 300$ the wives get 50, 100, and 150, respectively (cf. [11]).

The second story is called the contested garment problem. The two hold a garment and one of them wants all and the other half of it. Then the division of the garment is three quarters for the former one and one quarter for the latter one (cf. [7]).

Finally, we look at the right arbitration problem. The story is based on a father and his four sons. The father Jacob willed different units of money from his estate to his four sons. So after his death, the four sons produced different deeds, and all of them bear the same date. The son Reuben produced a deed duly witnessed that Jacob willed to him the entire estate on his death, the son Simeon produced a deed that his father willed to him half of the estate, the son Levi produced a deed giving him one third, and the son Judah produced a deed giving him one forth. Assuming that the estate is $E = 120$ the division between the sons is 7(1/2), 10(5/6), 20(5/6), and 80(5/6), respectively (cf. [7]).

Following the steps of Guiasu [1], we show that the division of the estate for these three ancient problems may be obtained by using the Alexia value if the bankruptcy problem is viewed as being a cumulative game and the rights arbitration problem is viewed as being a maximal game. In a cumulative TU game the members are not willing to reach a compromise and share their claims. So the values of the characteristic function for the coalitions are calculated additively. On the other hand in a maximal TU game the members of the coalition are willing to reach a compromise and share their claims. For this reason the values of the characteristic function for the proper coalitions are calculated by taking into account the maximum claim.

The paper is organized as follows. In Section 2, we give some preliminaries from cooperative game theory which are necessary in the following sections. The model and the general solution is given in Section 3. Section 4 gives solutions to the three Talmudic problems by using the Alexia value. Finally, we conclude in Section 5.

2. Preliminaries

A cooperative game in coalitional form is an ordered pair $\langle N, v\rangle$, where $N = \{1, 2, \ldots, n\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. We identify a game $\langle N, v\rangle$ with its characteristic function v and denote by G^N the set of cooperative games with player set N. In this paper we assume that $v \in G^N$ is monotonic, that is, $v(S) \geq 0$ and $v(S_1) \leq v(S_2)$ if $S_1 \leq S_2$.

A map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ is called a balanced map if $\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)e^S = e^N$. Here, e^S is the characteristic vector for coaliton S with

$$e_i^S := \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \in N \setminus S. \end{cases} \tag{2.1}$$

An n-person game $\langle N, v\rangle$ is called a balanced game if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have $\sum_{S\in 2^N\setminus\{\emptyset\}} \lambda(S)v(S) \le v(N)$.

The core of a game $\langle N, v\rangle$ is the set

$$C(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \ge v(S)\ \forall\ S \in 2^N \setminus \{\emptyset\} \right\}. \tag{2.2}$$

A game is called balanced if its core is nonempty [12, 13].

An order (permutation) σ is a bijective function $\sigma : N \to N = \{1, 2, \ldots, n\}$, where $\sigma(k)$ denotes the player at position $k \in \{1, 2, \ldots, n\}$ in the order (permutation) σ. The set of all orders (permutations) of N is denoted with $\Pi(N)$.

The set $P^\sigma(i) = \{r \in N \mid \sigma^{-1}(r) < \sigma^{-1}(i)\}$ consists of all predecessors of i with respect to the order (permutation) σ.

Let $v \in G^N$ and $\sigma \in \Pi(N)$. The marginal contribution vector with respect to σ and v is denoted by $m^\sigma(v) \in \mathbb{R}^n$. This vector has the value $m_i^\sigma(v) = v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))$ for each $i \in N$ in it's ith coordinate.

The Shapley value $\phi(v)$ of a game $v \in G^N$ is the average of the marginal vectors of the game, that is, $\phi(v) = (1/n!) \sum_{\sigma\in\Pi(N)} m^\sigma(v)$.

For a balanced game $\langle N, v\rangle$ and an order $\sigma \in \Pi(N)$, the lexinal $\lambda^\sigma(v) \in \mathbb{R}^N$ is defined as the lexicographic maximum on $C(v)$ with respect to σ, that is,

$$\lambda^\sigma_{\sigma(k)}(v) = \max\left\{ x_{\sigma(k)} \mid x \in C(v), x_{\sigma(1)} = \lambda^\sigma_{\sigma(l)}(v)\ \forall\ l \in \{1, 2, \ldots, k-1\} \right\}, \tag{2.3}$$

for all $k \in \{1, 2, \ldots, n\}$.

The lexinal is recursively defined such that every player gets the maximum he can obtain inside the core under the restriction that the players before him in the corresponding order obtain their restricted maxima.

For a balanced game $\langle N, v\rangle$, the Alexia value $\alpha(v)$ is defined as the average over the lexinals as follows:

$$\alpha(v) = \frac{1}{n!} \sum_{\sigma\in\Pi(N)} \lambda^\sigma(v). \tag{2.4}$$

A game $v \in G^N$ is called convex if and only if $v(S \cup T) + v(S \cap T) \ge v(S) + v(T)$ for each $S, T \in 2^N$.

The following example demonstrates the definitions above.

Example 2.1 (See [14]). Let $v \in G^N$ be a two-person-balanced game. Then, $v(1,2) \ge v(1)+v(2)$ and $C(v) = \operatorname{conv}\{v(N) - v(2), v(2)\}$.

3. The Model and the Solution

A bankruptcy situation with set of claimants N is a pair (E, d), where $E \geq 0$ is the estate to be divided, and $d \in \mathbb{R}_+^N$ is the vector of claims such that $\sum_{i \in N} d_i \geq E$. We assume without loss of generality that $d_1 < d_2 < \cdots < d_n$.

If the values $(v(\{1\}), \ldots, v(\{n\}))$ of v are given, then for each subset of players $\{i_1, i_2, \ldots, i_k\}$ the characteristic function v of the corresponding cooperative cumulative game is defined by

$$v(\{i_1, i_2, \ldots, i_k\}) = \min\{v(\{i_1\}) + v(\{i_2\}) + \cdots + v(\{i_k\}), E\}, \tag{3.1}$$

for all $k = 2, 3, \ldots, n$.

If the values $(v(\{1\}), \ldots, v(\{n\}))$ of v are given, then for each subset of players $\{i_1, i_2, \ldots, i_k\}$ the characteristic function v of the corresponding cooperative maximal game is defined by

$$v(\{i_1, i_2, \ldots, i_k\}) = \min\{\max\{v(\{i_1\}), v(\{i_2\}), \ldots, v(\{i_k\})\}, E\}, \tag{3.2}$$

for each $k = 2, 3, \ldots, n$.

To each bankruptcy situation (E, d) one can associate a bankruptcy game v defined by $v(\{i\}) = \min\{d_i, E\}$ for each $i = 1, 2, \ldots, n$ which can be considered as an arbitrary cumulative or maximal TU game.

Then the Alexia value for each player is $\alpha_i(v) = E/n$ for each $i = 1, 2, \ldots, n$. Accordingly for a cumulative game

$$\begin{aligned} v(\{i\}) &= \min\{d_i, E\} = E \quad \text{for each } i = 1, 2, \ldots, n, \\ v(\{i_1, i_2, \ldots, i_k\}) &= \min\{v(\{i_1\}) + v(\{i_2\}) + \cdots + v(\{i_k\}), E\} = E, \end{aligned} \tag{3.3}$$

for each $k = 2, 3, \ldots, n$.

On the other hand for a maximal game

$$\begin{aligned} v(\{i\}) &= \min\{d_i, E\} = E \quad \text{for each } i = 1, 2, \ldots, n, \\ v(\{i_1, i_2, \ldots, i_k\}) &= \min\{\max\{v(\{i_1\}), v(\{i_2\}), \ldots, v(\{i_k\})\}, E\} = E, \end{aligned} \tag{3.4}$$

for each $k = 2, 3, \ldots, n$.

Then the Alexia value is also $\alpha_i(v) = E/n$ for each $i = 1, 2, \ldots, n$.

Remark 3.1. As it is stated more detailed in Guiasu [1] the division of the estate in the bankruptcy problem can be interpreted as the judge looking at the game as a cumulative game with the wives that does not want to compromise and share parts of their claims. The division of the estate in the rights arbitration problem can be interpreted as the judge looking at the game as a maximal game supposing that the four brothers want to reach a compromise and share parts of their claims when they form a coalition. Conversely in the contested garment problem since there are no proper coalitions in the total set $N = \{1, 2\}$, the two claims d_1, d_2, and the estate E completely determine the game.

Remark 3.2. Guiasu [1] show that the three bankruptcy situations from the Talmud can be modeled by using the cumulative or maximal TU games, and their solution can be found by using the Shapley value. Further, Tijs et al. [3] show that if a game is convex, then $\phi(v) = \alpha(v)$, and Curiel et al. [5] states that bankruptcy games are convex. In view of these results we conclude that the solution of the three TU games that we have constructed can also be calculated by using the Alexia value.

Before closing this section, we note that we also have to consider the generalized situation $d_1 < d_2 < \cdots < d_{m-1} < E \leq d_m < \cdots < d_n$. Then the game is constructed as follows:

$$\begin{gathered} v(\emptyset) = 0, \qquad v(\{i\}) = d_i, \quad \text{for each } i = 1, 2, \ldots, m-1, \\ v(\{i\}) = d_{m-1} + \frac{(E - d_{m-1})}{(n - m + 1)}, \quad \text{for each } i = m, \ldots, n. \end{gathered} \tag{3.5}$$

As an interpretation the claim of the first $m-1$ players whose claims are less than the estate E can claim d_i, but the other $n-m+1$ players whose claims are more than the estate E cannot claim more than d_{m-1} plus the excess of the remaining estate $E-d_{m-1}$ equally divided between them.

4. Solution of the Bankruptcy Situations from the Talmud

We first consider the solution of the Talmudic problem with three wives. In this situation the estate will be divided between the three wives so $N = \{1, 2, 3\}$. We assume that the game to be constructed is cumulative. We give solutions according to the following cases.

(i) For $n = 3$, $m = 1$ we have the situation $E \leq d_1 < d_2 < d_3$. The game is constructed as follows: $v(\emptyset) = 0$, $v(S) = d_1$ otherwise, and the solution is $\alpha(v) = (1/3)(E, E, E)$. So for $E = 100$, $d_1 = 100$, $d_2 = 200$, and $d_3 = 300$, the solution is $\alpha(v) = (33(1/3), 33(1/3), 33(1/3))$.

(ii) For $n = 3$, $m = 2$ we have the situation $d_1 < E \leq d_2 < d_3$. Then the game is constructed as follows:

$$v(\emptyset) = 0,\, v(\{1\}) = d_1,\, v(\{2\}) = d_1 + \frac{(E - d_1)}{2},\, v(S) = E \quad \text{otherwise}, \tag{4.1}$$

and $\alpha(v) = (1/12)(2(E + d_1), 5E - d_1, 5E - d_1)$. So for $E = 200$, $d_1 = 100$, $d_2 = 200$, and $d_3 = 300$, the solution is $\alpha(v) = (50, 75, 75)$.

(iii) For $n = 3$, $m = 3$ we have the situation $d_1 < d_2 < E \leq d_3$. Then the game is constructed as follows: $v(\emptyset) = 0$, $v(\{1\}) = d_1$, $v(\{2\}) = d_2$,

$$v(\{1, 2\}) = \min\{d_1 + d_2, E\} = \begin{cases} d_1 + d_2, & d_1 + d_2 < E \\ E, & d_1 + d_2 \geq E, \end{cases} \tag{4.2}$$

$v(S) = E$ otherwise.

For $d_1+d_2 < E$ the solution is $\alpha(v) = (d_1/2,\ d_2/2,\ E-((d_1+d_2)/2))$, and for $d_1+d_2 \geq E$, $\alpha(v) = (1/6)(E+2d_1-d_2, E-2d_2-d_1, 4E-d_1-d_2)$. So for $E = 300$, $d_1 = 100$, $d_2 = 200$, and $d_3 = 300$, the solution is $\alpha(v) = (50, 100, 150)$.

Next we consider the solution of the congested garment problem. In this situation $N = \{1,2\}$ and $d_1 < E = d_2$, that is, $n = m = 2$. We have a two-person game, so the game is same if we take either maximal or cumulative. Then the game is constructed as follows: $v(\emptyset) = 0$, $v(\{1\}) = d_1$, $v(\{2\}) = E$, and $v(N) = E$. The Alexia value $\alpha(v)$ is $(d_1/2,\ E-(d_1/2))$. So for $E = 1$, $d_1 = 1/2$, $d_2 = 1$, the solution is $\alpha(v) = (1/4, 3/4)$.

Finally, we look at the solution of the rights arbitration problem. Here the estate will be divided between the four sons, so $N = \{1,2,3,4\}$. The claims of the sons can be written as $d_1 < d_2 < d_3 < E = d_4$, that is, $n = m = 4$. We assume that the game is maximal. Then the game is constructed as follows: $v(\emptyset) = 0$, $v(\{1\}) = d_1$, $v(\{2\}) = d_2$, $v(\{3\}) = d_3$, $v(\{4\}) = d_4$, $v(\{1,2\}) = d_2$, $v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = d_3$, $v(S) = E$, otherwise.

The solution is $\alpha(v) = (1/12)(3d_1, 4d_2-d_1, 6d_3-d_1-2d_2, 12d_4-d_1-2d_2-6d_3)$. So for $E = 120$, $d_1 = 30$, $d_2 = 40$, $d_3 = 60$, $d_4 = 120$, $\alpha(v) = (7(1/2), 10(5/6), 20(5/6), 80(5/6))$.

In the sequel we show that the three ancient problems from Babylonian Talmud can be solved by using the Alexia value from cooperative game theory. The calculations for the Talmudic problems above are done by using the Alexia value, and it is seen that they are equal to the Shapley value. We also notice that each situation is modeled by using bankruptcy games. We know that each bankruptcy game is convex (see Curiel et al. [5]), and for each convex game the Shapley value and the Alexia value are equal (see Tijs et al. [3]). Finally, we notice that the iterative solutions of the three ancient problems by using the Alexia value are straightforward by following the procedure given by Guiasu [1].

5. Concluding Remarks

In this paper we give an alternative solution, the Alexia value from cooperative game theory, to solve the three ancient bankruptcy situations from the 2000-year-old Babylonian Talmud. Our intuition and results are based on Guiasu [1] who used the Shapley value for the solution of these problems. Since each situation is modeled by using bankruptcy games, we used the result of Curiel et al. [5] that each bankruptcy game is convex and the result of Tijs et al. [3] that for each convex game the Shapley value and the Alexia value are equal.

Notice that the Alexia value is an interesting alternative to the Shapley value because the Alexia value provides a core element as a solution for all games with a nonempty core. Moreover it can be seen as a run-to-the-core rule for games with a nonempty core. To be more precise every lexinal player is running to the core according to a certain order where every player takes the maximum he can obtain within the subset of the core that remains after the players before him have made their respective choices. Further, the Alexia value combines two often applied arguments with respect to choosing an allocation: using orderings of the players, and at the same time respecting the fairness criterion of the core. Hence, the Alexia value combines the attractive properties of the Shapley value.

Acknowledgment

The authors gratefully acknowledge two anonymous referees.

References

[1] S. Guiasu, "Three ancient problems solved by using the game theory logic based on the Shapley value," *Synthese*, vol. 181, supplement 1, pp. 65–79, 2011.

[2] L. S. Shapley, "A value for n-person games," *Annals of Mathematics Studies*, vol. 28, pp. 307–317, 1953.

[3] S. Tijs, P. Borm, E. Lohmann, and M. Quant, "An average lexicographic value for cooperative games," *European Journal of Operational Research*, vol. 213, no. 1, pp. 210–220, 2011.

[4] D. B. Gillies, "Solutions to general non-zero-sum games," in *Contributions to the Theory of Games*, A. W. Tucker and R. D. Luce, Eds., vol. 4 of *Annals of Mathematics Studies, no. 40*, pp. 47–85, Princeton University Press, Princeton, NJ, USA, 1959.

[5] I. J. Curiel, M. Maschler, and S. H. Tijs, "Bankruptcy games," *Zeitschrift für Operations Research*, vol. 31, no. 5, pp. A143–A159, 1987.

[6] J. F. Nash Jr., "The bargaining problem," *Econometrica*, vol. 18, pp. 155–162, 1950.

[7] B. O'Neill, "A problem of rights arbitration from the Talmud," *Mathematical Social Sciences*, vol. 2, no. 4, pp. 345–371, 1982.

[8] R. J. Aumann and M. Maschler, "Game theoretic analysis of a bankruptcy problem from the Talmud," *Journal of Economic Theory*, vol. 36, no. 2, pp. 195–213, 1985.

[9] H. P. Young, "On dividing an amount according to individual claims or liabilities," *Mathematics of Operations Research*, vol. 12, no. 3, pp. 398–414, 1987.

[10] W. Thomson, "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey," *Mathematical Social Sciences*, vol. 45, no. 3, pp. 249–297, 2003.

[11] E. Y. Gura, "Insights into the game theory: an alternative mathematical experience," in *Quaderni Di Ricerca Didattica*, pp. 172–183, G.R.I.M. (Department of Mathematics, University of Palermo), Palermo, Italy, 2009.

[12] O. N. Bondareva, "Some applications of the methods of linear programming to the theory of cooperative games," *Problemly Kibernetiki*, vol. 10, pp. 119–139, 1963 (Russian).

[13] L. S. Shapley, "On balanced sets and cores," *Naval Research Logistics Quarterly*, vol. 14, pp. 453–460, 1967.

[14] S. Tijs, *The First Steps with Alexia, the Average Lexicographic Value*, vol. 123, Tilburg University, Center for Economic Research, CentER DP, Tilburg, The Netherlands, 2005.