Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 812783, 10 pages
doi:10.1155/2012/812783

Research Article

Approximation of Solutions of an Equilibrium Problem in a Banach Space

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Received 15 December 2011; Accepted 4 February 2012

Academic Editor: Rudong Chen

An equilibrium problem is investigated based on a hybrid projection iterative algorithm. Strong convergence theorems for solutions of the equilibrium problem are established in a strictly convex and uniformly smooth Banach space which also enjoys the Kadec-Klee property.

1. Introduction

Equilibrium problems which were introduced by Fan [1] and Blum and Oettli [2] have had a great impact and influence on the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. It has been shown [3–8] that equilibrium, problems include variational inequalities, fixed point, the Nash equilibrium, and game theory as special cases. A number of iterative algorithms have recently been studying for fixed point and equilibrium problems, see [9–26] and the references therein. However, there were few results established in the framework of the Banach spaces. In this paper, we suggest and analyze a projection iterative algorithm for finding solutions of equilibrium in a Banach space.

2. Preliminaries

In what follows, we always assume that E is a Banach space with the dual space E^*. Let C be a nonempty, closed, and convex subset of E. We use the symbol J to stand for the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad \forall x \in E, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of elements between E and E^*.

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in U_E$ with $x \neq y$. It is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta. \tag{2.2}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex; for details see [27] and the references therein.

Recall that a Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfies that $x_n \rightharpoonup x \in C$, where $\rightharpoonup$ denotes the weak convergence, and $\|x_n\| \to \|x\|$, where $\to$ denotes the strong convergence, and then $x_n \to x$. It is known that if E is uniformly convex, then E enjoys the Kadec-Klee property; for details see [26] and the references therein.

E is said to be smooth provided $\lim_{t \to 0}(\|x + ty\| - \|x\|)/t$ exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$.

It is well known that if E^* is strictly convex, then J is single valued; if E^* is reflexive, and smooth, then J is single valued and demicontinuous; for more details see [27, 28] and the references therein.

It is also well known that if D is a nonempty, closed, and convex subset of a Hilbert space H, and $P_D : H \to D$ is the metric projection from H onto D, then P_D is nonexpansive. This fact actually characterizes the Hilbert spaces, and consequently, it is not available in more general Banach spaces. In this connection, Alber [29] introduced a generalized projection operator Π_D in the Banach spaces which is an analogue of the metric projection in the Hilbert spaces.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.3}$$

Notice that, in a Hilbert space H, (2.3) is reduced to $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that is assigned to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the following minimization problem:

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.4}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and the strict monotonicity of the mapping J; see, for example, [27, 28]. In the Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E, \tag{2.5}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad \forall x, y, z \in E. \tag{2.6}$$

Let $T : C \to C$ be a mapping. Recall that a point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. T is said to be relatively nonexpansive if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ \forall p \in F(T). \tag{2.7}$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [27, 29, 30].

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. In this paper, we consider the following equilibrium problem. Find $p \in C$ such that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{2.8}$$

We use $\mathrm{EP}(f)$ to denote the solution set of the equilibrium problem (2.3). That is,

$$\mathrm{EP}(f) = \{p \in C : f(p, y) \geq 0,\ \forall y \in C\}. \tag{2.9}$$

Given a mapping $Q : C \to E^*$, let

$$f(x, y) = \langle Qx, y - x\rangle, \quad \forall x, y \in C. \tag{2.10}$$

Then $p \in \mathrm{EP}(f)$ if and only if p is a solution of the following variational inequality. Find p such that

$$\langle Qp, y - p\rangle \geq 0, \quad \forall y \in C. \tag{2.11}$$

To study the equilibrium problem (2.8), we may assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$, for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;

(A3)

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y), \quad \forall x, y, z \in C; \tag{2.12}$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and weakly lower semicontinuous.

In this paper, we study the problem of approximating solutions of equilibrium problem (2.8) based on a hybrid projection iterative algorithm in a strictly convex and uniformly smooth Banach space which also enjoys the Kadec-Klee property. To prove our main results, we need the following lemmas.

Lemma 2.1. *Let E be a strictly convex and uniformly smooth Banach space and C a nonempty, closed, and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in E$. Then*

(a) (see [2]). *There exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.13}$$

(b) (see [31]). *Define a mapping $T_r^f : E \to C$ by*

$$T_r^f x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle, \ \forall y \in C \right\}. \tag{2.14}$$

Then the following conclusions hold:

(1) T_r^f *is single valued;*

(2) T_r^f *is a firmly nonexpansive-type mapping; that is, for all $x, y \in E$,*

$$\left\langle T_r^f x - T_r^f y, JT_r^f x - JT_r^f y \right\rangle \leq \left\langle T_r^f x - T_r^f y, Jx - Jy \right\rangle; \tag{2.15}$$

(3) $F(T_r^f) = \mathrm{EP}(f)$;

(4) $\mathrm{EP}(f)$ *is closed and convex;*

(5) T_r^f *is relatively nonexpansive.*

Lemma 2.2 (see [29]). *Let E be a reflexive, strictly convex, and smooth Banach space and C a nonempty, closed, and convex subset of E. Let $x \in E$, and $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C. \tag{2.16}$$

Lemma 2.3 (see [29]). *Let E be a reflexive, strictly convex, and smooth Banach space and C a nonempty, closed, and convex subset of E, and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.17}$$

Lemma 2.4 (see [27]). *Let E be a reflexive, strictly convex, and smooth Banach space. Then one has the following*

$$\phi(x, y) = 0 \Longleftrightarrow x = y, \quad \forall x, \ y \in E. \tag{2.18}$$

3. Main Results

Theorem 3.1. *Let E be a strictly convex and uniformly smooth Banach space which also enjoys the Kadec-Klee property and C a nonempty, closed, and convex subset of E. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A4) such that $\mathrm{EP}(f)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n \in C, \text{such that } f(y_n, u) + \frac{1}{r_n}\langle u - y_n, Jy_n - Jx_n\rangle \geq 0, \quad \forall u \in C, \\ C_{n+1} = \{u \in C_n : 2\langle x_n - u, Jx_n - Jy_n\rangle \geq \phi(x_n, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{gathered} \tag{3.1}$$

where $\{r_n\}$ is a real number sequence in $[r,\infty)$, where r is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = \Pi_{\mathrm{EP}(f)}x_0$.

Proof. In view of Lemma 2.1, we see that $\mathrm{EP}(f)$ is closed and convex. Next, we show that C_n is closed and convex. It is not hard to see that C_n is closed. Therefore, we only show that C_n is convex. It is obvious that $C_1 = C$ is convex. Suppose that C_h is convex for some $h \in \mathbb{N}$. Next, we show that C_{h+1} is also convex for the same h. Let $a, b \in C_{h+1}$ and $c = ta + (1-t)b$, where $t \in (0,1)$. It follows that

$$\phi(x_h, y_h) \leq 2\langle x_h - a, Jx_h - Jy_h\rangle, \qquad \phi(x_h, y_h) \leq 2\langle x_h - b, Jx_h - Jy_h\rangle, \tag{3.2}$$

where $a, b \in C_h$. From the above two inequalities, we can get that

$$\phi(x_h, y_h) \leq 2\langle x_h - c, Jx_h - Jy_h\rangle, \tag{3.3}$$

where $c \in C_h$. It follows that C_{h+1} is closed and convex. This completes the proof that C_n is closed, and convex.

Next, we show that $\mathrm{EP}(f) \subset C_n$. It is obvious that $\mathrm{EP}(f) \subset C = C_1$. Suppose that $\mathrm{EP}(f) \subset C_h$ for some $h \in \mathbb{N}$. For any $z \in \mathrm{EP}(f) \subset C_h$, we see from Lemma 2.1 that

$$\phi(z, y_h) \leq \phi(z, x_h). \tag{3.4}$$

On the other hand, we obtain from (2.6) that

$$\phi(z, y_h) = \phi(z, x_h) + \phi(x_h, y_h) + 2\langle z - x_h, Jx_h - Jy_h\rangle. \tag{3.5}$$

Combining (3.4) with (3.5), we arrive at

$$2\langle x_h - z, Jx_h - Jy_h\rangle \geq \phi(x_h, y_h) \tag{3.6}$$

which implies that $z \in C_{h+1}$. This shows that $\mathrm{EP}(f) \subset C_{h+1}$. This completes the proof that $\mathrm{EP}(f) \subset C_n$.

Next, we show that $\{x_n\}$ is a convergent sequence and strongly converges to $\overline{x}$, where $\overline{x} \in \mathrm{EP}(f)$. Since $x_n = \Pi_{C_n} x_0$, we see from Lemma 2.2 that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n. \tag{3.7}$$

It follows from $\mathrm{EP}(f) \subset C_n$ that

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in \mathrm{EP}(f). \tag{3.8}$$

By virtue of Lemma 2.3, we obtain that

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0)\\ &\leq \phi(\Pi_{\mathrm{EP}(f)} x_0, x_0) - \phi(\Pi_{\mathrm{EP}(f)} x_0, x_n)\\ &\leq \phi(\Pi_{\mathrm{EP}(f)} x_0, x_0).\end{aligned} \tag{3.9}$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows from (2.5) that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \overline{x}$. Since C_n is closed and convex, we see that $\overline{x} \in C_n$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}\phi(\overline{x}, x_0) &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jx_0\rangle + \|x_0\|^2\\ &\leq \liminf_{n\to\infty}\left(\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\right)\\ &= \liminf_{n\to\infty}\phi(x_n, x_0)\\ &\leq \limsup_{n\to\infty}\phi(x_n, x_0)\\ &\leq \phi(\overline{x}, x_0),\end{aligned} \tag{3.10}$$

which implies that $\phi(x_n, x_0) \to \phi(\overline{x}, x_0)$ as $n \to \infty$. Hence, $\|x_n\| \to \|\overline{x}\|$ as $n \to \infty$. In view of the Kadec-Klee property of E, we see that $x_n \to \overline{x}$ as $n \to \infty$. Notice that $x_{n+1} = \Pi_{\mathrm{EP}(f)} x_0 \in C_{n+1} \subset C_n$. It follows that

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0)\\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.11}$$

Since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we arrive at $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$. This shows that $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from the boundedness that $\lim_{n\to\infty}\phi(x, x_0)$ exists. It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.12}$$

By virtue of $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we find that

$$\phi(x_n, y_n) \leq 2\langle x_n - x_{n+1}, Jx_n - Jy_n \rangle. \tag{3.13}$$

It follows that

$$\lim_{n\to\infty} \phi(x_n, y_n) = 0. \tag{3.14}$$

In view of (2.5), we see that

$$\lim_{n\to\infty} (\|x_n\| - \|y_n\|) = 0. \tag{3.15}$$

Since $x_n \to \overline{x}$, we find that

$$\lim_{n\to\infty} \|y_n\| = \|\overline{x}\|. \tag{3.16}$$

It follows that

$$\lim_{n\to\infty} \|Jy_n\| = \|J\overline{x}\|. \tag{3.17}$$

This implies that $\{Jy_n\}$ is bounded. Note that both E and E^* are reflexive. We may assume that $Jy_n \rightharpoonup y^* \in E^*$. In view of the reflexivity of E, we see that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned}\phi(x_n, y_n) &= \|x_n\|^2 - 2\langle x_n, Jy_n \rangle + \|y_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, Jy_n \rangle + \|Jy_n\|^2.\end{aligned} \tag{3.18}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned}0 &\geq \|\overline{x}\|^2 - 2\langle \overline{x}, y^* \rangle + \|y^*\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jy \rangle + \|Jy\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jy \rangle + \|y\|^2 \\ &= \phi(\overline{x}, y).\end{aligned} \tag{3.19}$$

That is, $\overline{x} = y$, which in turn implies that $y^* = J\overline{x}$. It follows that $Jy_n \rightharpoonup J\overline{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain from (3.17) that $\lim_{n\to\infty} Jy_n = J\overline{x}$. Since $J^{-1} : E^* \to E$ is demicontinuous, we find that $y_n \rightharpoonup \overline{x}$. This implies from (3.16) and the Kadec-Klee property of E that $\lim_{n\to\infty} y_n = \overline{x}$. This in turn implies that $\lim_{n\to\infty}\|y_n - x_n\| = 0$. Since J is uniformly norm-to-norm continuous on any bounded sets, we find that

$$\lim_{n\to\infty}\|Jy_n - Jx_n\| = 0. \tag{3.20}$$

Next, we show that $\overline{x} \in EF(f)$. In view of Lemma 2.1, we find from $y_n = T^f_{r_n}x_n$ that

$$f(y_n, u) + \frac{1}{r_n}\langle u - y_n, Jy_n - Jx_n\rangle \geq 0, \quad \forall u \in C. \tag{3.21}$$

It follows from condition (A2) and (3.20) that

$$\frac{1}{r_n}\|u - y_n\|\|Jy_n - Jx_n\| \geq f(u, y_n), \quad \forall u \in C. \tag{3.22}$$

In view of condition (A4), we obtain from (3.17) that

$$f(u, \overline{x}) \leq 0, \quad \forall u \in C. \tag{3.23}$$

For $0 < t < 1$ and $u \in C$, define $u_t = tu + (1-t)\overline{x}$. It follows that $u_t \in C$, which yields that $f(u_t, \overline{x}) \leq 0$. It follows from conditions (A1) and (A4) that

$$0 = f(u_t, u_t) \leq tf(u_t, u) + (1-t)f(u_t, \overline{x}) \leq tf(u_t, u). \tag{3.24}$$

That is,

$$f(u_t, u) \geq 0. \tag{3.25}$$

Letting $t \downarrow 0$, we find from condition (A3) that $f(\overline{x}, u) \geq 0$, for all $u \in C$. This implies that $\overline{x} \in \mathrm{EP}(f)$. This shows that $\overline{x} \in \mathrm{EP}(f)$.

Finally, we prove that $\overline{x} = \Pi_{\mathrm{EP}(f)}x_0$. Letting $n \to \infty$ in (3.8), we see that

$$\langle \overline{x} - w, Jx_0 - J\overline{x}\rangle \geq 0, \quad \forall w \in \mathrm{EP}(f). \tag{3.26}$$

In view of Lemma 2.2, we can obtain that $\overline{x} = \Pi_{\mathrm{EP}(f)}x_0$. This completes the proof. □

In the framework of the Hilbert spaces, we have the following.

Corollary 3.2. *Let E be a Hilbert space and C a nonempty, closed, and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) such that $\mathrm{EP}(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$
\begin{gathered}
x_0 \in E \text{ chosen arbitrarily}, \\
C_1 = C, \\
x_1 = P_{C_1} x_0, \\
y_n \in C, \text{ such that } f(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C, \\
C_{n+1} = \left\{u \in C_n : 2\langle x_n - u, x_n - y_n\rangle \geq \|x_n - y_n\|^2\right\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered}
\tag{3.27}
$$

where $\{r_n\}$ is a real number sequence in $[r, \infty)$, where r is some positive real number. Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{\mathrm{EP}(f)} x_0$.

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