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Research Article

Existence of Multiple Solutions for a Singular Elliptic Problem with Critical Sobolev Exponent

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We consider the existence of multiple solutions of the singular elliptic problem $-\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)+|u|^{p-2}u/|x|^{(a+1)p}=f|u|^{r-2}u+h|u|^{s-2}u+|x|^{-bp^*}|u|^{p^*-2}u$, $u(x)\to 0$ as $|x|\to +\infty$, where $x\in\mathbb{R}^N$, $1<p<N$, $a<(N-p)/p$, $a\le b\le a+1$, $r, s>1$, $p^*=Np/(N-pd)$, $d=a+1-b$. By the variational method and the theory of genus, we prove that the above-mentioned problem has infinitely many solutions when some conditions are satisfied.

1. Introduction and Main Results

In this paper, we consider the existence of multiple solutions for the singular elliptic problem

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)+\frac{|u|^{p-2}u}{|x|^{(a+1)p}}=f|u|^{r-2}u+h|u|^{s-2}u+|x|^{-bp^*}|u|^{p^*-2}u,\quad x\in\mathbb{R}^N,$$
$$u(x)\longrightarrow 0\quad \text{as } |x|\longrightarrow +\infty, \tag{1.1}$$

where $1<p<N$, $a<(N-p)/p$, $a\le b\le a+1$, $r>1$, $p^*=Np/(N-pd)$, $d=a+1-b$. $f(x)$ and $h(x)$ are nonnegative functions in $\mathbb{R}^N$.

In recent years, the existence of multiple solutions on elliptic equations has been considered by many authors. In [1], Assunção et al. considered the following quasilinear degenerate elliptic equation:

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)+\lambda|x|^{-(a+1)p}|u|^{p-2}u=|x|^{-bq}|u|^{q-2}u+f, \tag{1.2}$$

where $x \in \mathbb{R}^N$, $1 < p < N$, $q = Np/[N-p(a+1-b)]$. When $\lambda = 0$, $f = \varepsilon g$, where $0 < \varepsilon \le \varepsilon_0$ and $0 \le g \in (L_b^q(\mathbb{R}^N))^*$; the authors proved that problem (1.2) has at least two positive solutions. Rodrigues in [2] studied the following critical problem on bounded domain $\Omega \in \mathbb{R}^N$:

$$\begin{gathered} -\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bp^*}|u|^{p^*-2}u + |x|^{-\beta} f|u|^{r-2}u, \quad x \in \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

By the variational method on Nehari manifolds [3, 4], the author proved the existence of at least two positive solutions and the nonexistence of solutions when some certain conditions are satisfied. When $p = 2$ and $a = -1$, Miotto and Miyagaki in [5] considered the semilinear Dirichlet problem in infinite strip domains

$$\begin{gathered} -\Delta u + u = \lambda f(x)|u|^{q-1} + h(x)|u|^{p-1}, \quad x \in \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.4}$$

The authors also proved that problem (1.4) has at least two positive solutions by the methods of Nehari manifold. For other references, we refer to [6–11] and the reference therein. In fact, motivated by [1, 2, 5], we consider the problem (1.1). Since our problem is singular and is studied in the whole space $\mathbb{R}^N$, the loss of compactness of the Sobolev embedding renders a variational technique that is more delicate. By the variational method and the theory of genus, we prove that problem (1.1) has infinitely many solutions when some suitable conditions are satisfied.

In order to state our result, we introduce some weighted Sobolev spaces. For $r, s \ge 1$ and $g = g(x) > 0$ in R^N, we define the spaces $L^r(\mathbb{R}^N, g)$ and $L^s(\mathbb{R}^N, g)$ as being the set of Lebesgue measurable functions $u : \mathbb{R}^N \to \mathbb{R}^1$, which satisfy

$$\begin{gathered} \|u\|_{r,g} = \|u\|_{L^r(\mathbb{R}^N,g)} = \left(\int_{\mathbb{R}^N} g(x)|u|^r dx\right)^{1/r} < \infty, \\ \|u\|_{s,g} = \|u\|_{L^s(\mathbb{R}^N,g)} = \left(\int_{\mathbb{R}^N} g(x)|u|^s dx\right)^{1/s} < \infty. \end{gathered} \tag{1.5}$$

Particularly, when $g(x) \equiv 1$, we have

$$\|u\|_r = \|u\|_{L^r(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^r dx\right)^{1/r} < \infty. \tag{1.6}$$

We denote the completion of $C_0^\infty(\mathbb{R}^N)$ by $X = W_a^{1,p}(\mathbb{R}^N)$ with the norm of

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |x|^{-ap}|u|^{\mathrm{p}} dx\right)^{1/p}, \tag{1.7}$$

where $1 < p < N$ and $a < (N-p)/p$. It is easy to find that X is a reflexive and separable Banach space with the norm $\|u\|_X$.

The following Hardy-Sobolev inequality is due to Caffarelli et al. [12], which is called Caffarelli-Kohn-Nirenberg inequality. There exist constants S_1, $S_2 > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}dx\right)^{p/p^*} \leq S_1 \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx, \quad \forall u \in C_0^\infty\left(\mathbb{R}^N\right), \tag{1.8}$$

$$\int_{\mathbb{R}^N} |x|^{-(a+1)p}|u|^p dx \leq S_2 \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx, \quad \forall u \in C_0^\infty\left(\mathbb{R}^N\right), \tag{1.9}$$

where $p^* = Np/(N - pd)$ is called the Sobolev critical exponent.

In the present paper, we make the following assumptions:

(A_1) $f(x) \in L^{\sigma_1}(\mathbb{R}^N, g_1) \bigcap L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ for $1 < r < p$, where $g_1 = |x|^{(a+1)r\sigma_1}$, $\sigma_1 = p/(p - r)$;

(A_2) $f(x) \in L^{\sigma_2}(\mathbb{R}^N, g_2) \bigcap L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ for $p < r < p^*$, where $g_2 = |x|^{br\sigma_2}$, $\sigma_2 = p^*/(p^* - r)$.

(A_3) $h(x) \in L^{\mu}(\mathbb{R}^N, g_3) \bigcap L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ for $p < s < p^*$, where $g_3 = |x|^{\mu bp^*}$, $\mu = p^*/(p^* - s)$.

Then, we give some basic definitions.

Definition 1.1. $u \in X$ is said to be a weak solution of (1.1) if for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ there holds

$$\begin{aligned}\int_{\mathbb{R}^N}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \frac{|u|^{p-2}u\varphi}{|x|^{(a+1)p}}\right)dx &= \int_{\mathbb{R}^N} f|u|^{r-2}u\varphi\, dx + \int_{\mathbb{R}^N} h|u|^{s-2}u\varphi\, dx \\ &\quad + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*-2}u\varphi\, dx.\end{aligned} \tag{1.10}$$

Let $I(u) : X \to \mathbb{R}^1$ be the energy functional corresponding to problem (1.1), which is defined as

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N}\left(|x|^{-ap}|\nabla u|^p + \frac{|u|^p}{|x|^{(a+1)p}}\right)dx - \frac{1}{r}\int_{\mathbb{R}^N} f|u|^r dx - \frac{1}{s}\int_{\mathbb{R}^N} h|u|^s dx - \frac{1}{p^*}\int_{\mathbb{R}^N} \frac{|u|^{p^*}}{|x|^{bp^*}}dx, \tag{1.11}$$

for all $u \in X$. Then the functional $I \in C^1(X, R^1)$ and for all $\varphi \in X$, there holds

$$\begin{aligned}\langle I'(u), \varphi\rangle &= \int_{\mathbb{R}^N}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\varphi + \frac{|u|^{p-2}u\varphi}{|x|^{(a+1)p}}\right)dx - \int_{\mathbb{R}^N} f(x)|u|^{r-2}u\varphi dx \\ &\quad - \int_{\mathbb{R}^N} h(x)|u|^{s-2}u\varphi\, dx - \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*-2}u\varphi\, dx.\end{aligned} \tag{1.12}$$

It is well known that the weak solutions of problem (1.1) are the critical points of the functional $I(u)$, see [13]. Thus, to prove the existence of weak solutions of (1.1), it is sufficient to show that $I(u)$ admits a sequence of critical points in X.

Our main result in this paper is the following.

Theorem 1.2. *Let* $1 < p < N$, $a < (N-p)/p$, $a \leq b \leq a+1$, $r > 1$, $p^* = Np/(N-pd)$, $d = a+1-b$, $\max\{r,p\} < s < p^*$. *Assume* (A_1)–(A_3) *are fulfilled. Then problem* (1.1) *has infinitely many solutions in* X.

2. Preliminary Results

Our proof is based on variational method. One important aspect of applying this method is to show that the functional $I(u)$ satisfies $(PS)_c$ condition which is introduced in the following definition.

Definition 2.1. Let $c \in R^1$ and X be a Banach space. The functional $I(u) \in C^1(X,R)$ satisfies the $(PS)_c$ condition if for any $\{u_n\} \subset X$ such that

$$I(u_n) \longrightarrow c, \quad I'(u_n) \longrightarrow 0 \quad \text{in } X^* \text{ as } n \longrightarrow \infty \tag{2.1}$$

contains a convergent subsequence in X.

The following embedding theorem is an extension of the classical Rellich-Kondrachov compactness theorem, see [14].

Lemma 2.2. *Suppose* $\Omega \subset \mathbb{R}^N$ *is an open bounded domain with* C^1 *boundary and* $0 \in \Omega$. $N \geq 3$, $a < (N-p)/p$. *Then the embedding* $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ *is continuous if* $1 \leq r \leq Np/(N-p)$ *and* $0 \leq \alpha \leq (1+a)r + N(1-r/p)$, *and is compact if* $1 \leq r < Np/(N-p)$ *and* $0 \leq \alpha < (1+a)r + N(1-r/p)$.

Now we prove an embedding theorem, which is important in our paper.

Lemma 2.3. *Assume* (A_1)-(A_2) *and* $1 < r < p^*$. *Then the embedding* $X \hookrightarrow L^r(\mathbb{R}^N, f)$ *is compact.*

Proof. We split our proof into two cases.

(i) Consider $1 < r < p$.

By the Hölder inequality and (1.9) we have that

$$\begin{aligned} \|u\|_{L^r(\mathbb{R}^N,f)}^r &= \int_{\mathbb{R}^N} f(x)|u|^r dx \leq \left(\int_{\mathbb{R}^N} |u|^p |x|^{-(a+1)p} dx\right)^{r/p} \left(\int_{\mathbb{R}^N} f^{\sigma_1} |x|^{(a+1)r\sigma_1} dx\right)^{1/\sigma_1} \\ &= \left(\int_{\mathbb{R}^N} |u|^p |x|^{-(a+1)p} dx\right)^{r/p} \|f\|_{L^{\sigma_1}(\mathbb{R}^N,g_1)} \\ &\leq S_2^{r/p} \|u\|_X^r \|f\|_{L^{\sigma_1}(\mathbb{R}^N,g_1)}, \end{aligned} \tag{2.2}$$

where $g_1 = |x|^{(a+1)r\sigma_1}$, $\sigma_1 = p/(p-r)$. Then the embedding is continuous. Next, we will prove that the embedding is compact.

Let B_R be a ball center at origin with the radius $R > 0$. For the convenience, we denote $L^r(\mathbb{R}^N, f)$ by Z, that is, $Z = L^r(\mathbb{R}^N, f)$. Assume $\{u_n\}$ is a bounded sequence in X. Then $\{u_n\}$ is bounded in $X(B_R)$. We choose $\alpha = 0$ in Lemma 2.2, then there exist $u \in Z(B_R)$ and a subsequence, still denoted by $\{u_n\}$, such that $\|u_n - u\|_{L^r(B_R)} \to 0$ as $n \to \infty$. We want to prove that

$$\lim_{R\to\infty} \sup_{u\in X\setminus\{0\}} \frac{\|u\|_{Z(B_R^c)}}{\|u\|_X} = 0, \tag{2.3}$$

where $B_R^c = \mathbb{R}^N \setminus B_R$. In fact, we obtain from (2.2) that

$$\|u\|_{Z(B_R^c)}^r \le S_2^{r/p} \|u\|_X^r \|f\|_{L^{\sigma_1}(B_R^c, g_1)}. \tag{2.4}$$

The fact $f \in L^{\sigma_1}(\mathbb{R}^N, g_1)$ shows that

$$\lim_{R\to\infty} \int_{B_R^c} f^{\sigma_1} g_1 dx = 0. \tag{2.5}$$

Then (2.4) and (2.5) imply that

$$\frac{\|u\|_{Z(B_R^c)}}{\|u\|_X} \le S_2^{1/p} \|f\|_{L^{\sigma_1}(B_R^c, g_1)}^{1/r}, \tag{2.6}$$

which gives (2.3).

In the following, we will prove that $u_n \to u$ strongly in $Z(R^N)$.

Since X is a reflexive Banach space and $\{u_n\}$ is bounded in X. Then we may assume, up to a subsequence, that

$$u_n \rightharpoonup u \quad \text{in X}. \tag{2.7}$$

In view of (2.3), we get that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ large enough such that

$$\|u_n\|_{Z(B_{R_\varepsilon}^c)} \le \varepsilon \|u_n\|_X \quad (n = 1, 2, \ldots). \tag{2.8}$$

On the other hand, due to the compact embedding $X(B_{R_\varepsilon}) \hookrightarrow Z(R_\varepsilon)$ in Lemma 2.2, we have that

$$\lim_{n\to\infty} \|u_n - u\|_{Z(B_{R_\varepsilon})} = 0. \tag{2.9}$$

Therefore, there is $N_0 > 0$ such that

$$\|u_n - u\|_{Z(B_{R_\varepsilon})} < \varepsilon, \tag{2.10}$$

for $n > N_0$. Thus, the inequalities (2.8) and (2.10) show that

$$\begin{aligned}\|u_n - u\|_Z &\le \|u_n - u\|_{Z(B_{R_\varepsilon})} + \|u_n - u\|_{Z(B^c_{R_\varepsilon})} \\ &\le \|u_n - u\|_{Z(B_{R_\varepsilon})} + \|u_n\|_{Z(B^c_{R_\varepsilon})} + \|u\|_{Z(B^c_{R_\varepsilon})} \\ &\le (1 + \|u_n\|_X + \|u\|_X)\varepsilon.\end{aligned} \tag{2.11}$$

This shows that $\{u_n\}$ is convergent in $Z = L^r(\mathbb{R}^N, f)$.

(ii) Consider $p \le r < p^*$.

It follows from (1.8) and the Hölder inequality that

$$\begin{aligned}\|u\|^r_{L^r(\mathbb{R}^N,f)} &= \int_{\mathbb{R}^N} f(x)|u|^r dx \le \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}dx\right)^{r/p^*}\left(\int_{\mathbb{R}^N} f^{\sigma_2}|x|^{br\sigma_2}dx\right)^{1/\sigma_2} \\ &\le S_1^{r/p}\left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right)^{r/p}\left(\int_{\mathbb{R}^N} f^{\sigma_2}|x|^{br\sigma_2}dx\right)^{1/\sigma_2} \\ &\le S_1^{r/p}\|u\|^r_X \|f\|_{L^{\sigma_2}(\mathbb{R}^N,g_2)},\end{aligned} \tag{2.12}$$

where $g_2 = |x|^{br\sigma_2}$, $\sigma_2 = p^*/(p^* - r)$. Thus, the fact of $f \in L^{\sigma_2}(\mathbb{R}^N, g_2)$ and (2.12) imply that the embedding is continuous. Similar to the proof of (i) we can also prove that the embedding $X \hookrightarrow L^r(\mathbb{R}^N, f)$ is compact for $p \le r < p^*$. □

Similarly, we have the following result of compact embedding.

Lemma 2.4. *Assume $1 < p < s < p^*$ and (A_3), then the embedding $X \hookrightarrow L^s(\mathbb{R}^N, h)$ is compact.*

The following concentration compactness principle is a weighted version of the Concentration Compactness Principle II due to Lions [15–18], see also [19, 20].

Lemma 2.5. *Let $1 < p < N$, $-\infty < a < (N-p)/p$, $a \le b \le a+1$, $p^* = Np/(N-pd)$, $d = a+1-b$. Suppose that $\{u_n\} \subset W^{1,p}_a(\mathbb{R}^N)$ is a sequence such that*

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W^{1,p}_a\left(\mathbb{R}^N\right), \\ |x|^{-ap}|\nabla u_n|^p &\rightharpoonup \mu \quad \text{in } \mathcal{M}\left(\mathbb{R}^N\right), \\ |x|^{-bp^*}|u_n|^{p^*} &\rightharpoonup \eta \quad \text{in } \mathcal{M}\left(\mathbb{R}^N\right), \\ u_n &\longrightarrow u \quad \text{a.e. on } \mathbb{R}^N,\end{aligned} \tag{2.13}$$

where μ, η are measures supported on Ω and $\mathcal{M}(\mathbb{R}^N)$ is the space of bounded measures in $\mathbb{R}^N$. Then there are the following results.

(1) *There exists some at most countable set* J*, a family* $\{x_j \in \Omega \mid j \in J\}$ *of distinct points in* $\mathbb{R}^N$*, and a family* $\{\eta_j \mid j \in J\}$ *of positive numbers such that*

$$\eta = |x|^{-bp^*}|u|^{p^*} + \sum_{j\in J}\eta_j\delta_{x_j}, \tag{2.14}$$

where δ_{x_j} *is the Dirac measure at* x_j*.*

(2) *The following equality holds*

$$\mu \geq |x|^{-ap}|\nabla u|^p + \sum_{j\in J}\mu_j\delta_{x_j}, \tag{2.15}$$

for some family $\{\mu_j > 0 \mid j \in J\}$ *satisfying*

$$S_1(\eta_j)^{p/p^*} \leq \mu_j \quad \forall j \in J, \qquad \sum_{j\in J}(\eta_j)^{p/p^*} \leq \infty. \tag{2.16}$$

(3) *There hold*

$$\begin{gathered}\lim_{n\to+\infty}\sup\int_\Omega |x|^{-ap}|\nabla u_n|^p dx = \int_\Omega d\mu + \mu_\infty,\\ \lim_{n\to+\infty}\sup\int_\Omega |x|^{-bp^*}|\nabla u_n|^{p^*} dx = \int_\Omega d\eta + \eta_\infty,\end{gathered} \tag{2.17}$$

where

$$\begin{gathered}\mu_\infty := \lim_{R\to\infty}\lim_{n\to+\infty}\sup\int_{\Omega\bigcap B_R^c} |x|^{-ap}|\nabla u_n|^p dx,\\ \eta_\infty := \lim_{R\to\infty}\lim_{n\to+\infty}\sup\int_{\Omega\bigcap B_R^c} |x|^{-bp^*}|\nabla u_n|^{p^*} dx.\end{gathered} \tag{2.18}$$

Lemma 2.6. *Let* $1 < p < r < s < p^*$*. Then* $I(u)$ *satisfies the* $(PS)_c$ *condition with* $c \leq (1/r - 1/p^*)S_1^{p^*/(p^*-p)}$*, where* S_1 *is as in* (1.8).

Proof. We will split the proof into three steps.
Step 1. $\{u_n\}$ is bounded in X.

Let $\{u_n\}$ be a $(PS)_c$ sequence of $I(u)$ in X, that is,

$$I(u_n) \longrightarrow c, \quad I'(u_n) \longrightarrow 0 \quad \text{in } X^* \text{ as } n \longrightarrow \infty. \tag{2.19}$$

Then, we have

$$\begin{aligned} 1 + c + \|u_n\|_X &\geq I(u_n) - \frac{1}{r}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|_X^p + \left(\frac{1}{r} - \frac{1}{s}\right)\|u_n\|_{L^s(\mathbb{R}^N,h)}^s + \left(\frac{1}{r} - \frac{1}{p^*}\right)\|u_n\|_{L^{p^*}(\mathbb{R}^N)}^{p^*} \\ &\geq \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|_X^p. \end{aligned} \tag{2.20}$$

Since $p > 1$, (2.20) shows that $\{u_n\}$ is bounded in X.
Step 2. There exists $\{u_n\}$ in X such that $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$.

The inequality (1.8) shows that $\{u_n\}$ is bounded in $L^{p^*}(\mathbb{R}^N, |x|^{-bp^*})$. Then the above argument and the compactness embedding in Lemma 2.2 mean that the following convergence hold:

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } W_0^{1,p}\left(\mathbb{R}^N\right), \\ u_n \rightharpoonup u \quad \text{in } L^{p^*}\left(\mathbb{R}^N, |x|^{-bp^*}\right), \\ u_n \longrightarrow u \quad \text{a.e. in } \mathbb{R}^N. \end{gathered} \tag{2.21}$$

It follows from Lemma 2.5 that there exist nonnegative measures μ and η such that

$$|x|^{-bp^*}|u_n|^{p^*} \rightharpoonup \eta = |x|^{-bp^*}|u|^{p^*} + \sum_{j\in J}\eta_j\delta_{x_j}, \tag{2.22}$$

$$|x|^{-ap}|\nabla u_n|^p \geq |x|^{-ap}|\nabla u|^p + \sum_{j\in J}\mu_j\delta_{x_j}. \tag{2.23}$$

Thus, in order to prove $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$ it is sufficient to prove that $\eta_j = \eta_\infty = 0$.

For the proof of $\eta_j = 0$, we define the functional $\varphi \in C_0^\infty(R^{\mathbb{N}})$ such that

$$\varphi \equiv 1, \quad \text{in } B(x_j, \varepsilon), \quad \varphi \equiv 0, \quad \text{in } B(x_j, 2\varepsilon)^c, \quad |\nabla\varphi| \leq \frac{2}{\varepsilon}, \tag{2.24}$$

where x_j belongs to the support of $d\eta$. It follows from (2.1) that

$$\lim_{n\to\infty}\langle I'(u_n), u_n\varphi\rangle = 0. \tag{2.25}$$

Since $\|u_n\|_X$ is bounded, we can get from (1.8)-(1.9), Lemmas 2.3 and 2.5 that

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\psi u_n dx &= \lim_{n\to\infty}\Bigg(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}\psi\,dx-\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u_n|^p\psi\,dx\\ &\quad-\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p\psi\,dx+\int_{\mathbb{R}^N}f(x)|u_n|^r\psi\,dx\\ &\quad+\int_{\mathbb{R}^N}h(x)|u_n|^s\psi\,dx\Bigg)\\ &\longrightarrow\int_{\mathbb{R}^N}\psi d\eta-\int_{\mathbb{R}^N}\psi d\mu=\eta_j-\mu_j\quad(\text{as }\varepsilon\longrightarrow 0).\end{aligned}\tag{2.26}$$

On the other hand,

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\psi u_n dx\\ &\quad\le\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p dx\right)^{(p-1)/p}\left(\int_{\mathbb{R}^N}|x|^{-ap}|u_n|^p\left|\nabla\psi\right|^p dx\right)^{1/p}\\ &\quad\le c\left(\int_{B_{2\varepsilon}}\left|\nabla\psi\right|^N dx\right)^{1/N}\left(\int_{B_{2\varepsilon}}|x|^{(-aNp)/(N-p)}|u|^{Np/(N-p)}dx\right)^{(N-p)/Np}\\ &\quad\le c\left(\int_{B_{2\varepsilon}}|x|^{(-aNp)/(N-p)}|u|^{Np/(N-p)}dx\right)^{(N-p)/Np}\longrightarrow 0\quad(\varepsilon\longrightarrow 0),\end{aligned}\tag{2.27}$$

where $B_{2\varepsilon}\triangleq B(x_j,2\varepsilon)$. Then $\mu_j=\eta_j$; furthermore, (2.16) implies that $\mu_j=\eta_j=0$ or $\eta_j>S_1^{p^*/(p^*-p)}$. We will prove that the later does not hold. Suppose otherwise, there exists some $j_0\in J$ such that $\eta_{j_0}>S_1^{p^*/(p^*-p)}$. Then (2.19) and Lemma 2.4 show that

$$\begin{aligned}c+o(1)&=I(u_n)-\frac{1}{r}\left\langle I'(u_n),u_n\right\rangle\\ &=\left(\frac{1}{p}-\frac{1}{r}\right)\|u_n\|_X^p+\left(\frac{1}{r}-\frac{1}{p^*}\right)\int_{\Omega}|x|^{-bp^*}|u_n|^{p^*}dx\\ &\ge\left(\frac{1}{r}-\frac{1}{p^*}\right)\eta_{j_0}>\left(\frac{1}{r}-\frac{1}{p^*}\right)S_1^{p^*/(p^*-p)},\end{aligned}\tag{2.28}$$

which contradicts the hypothesis of c. Then $\mu_j=\eta_j=0$.

Similarly, we define the functional $\psi_1\in C_0^\infty(R^{\mathbb{N}})$ as

$$\psi_1\equiv 0,\quad |x|<R,\quad \psi_1\equiv 1,\quad |x|>2R,\quad \left|\nabla\psi_1\right|\le\frac{2}{R}.\tag{2.29}$$

Then, the similar proof as above shows that $\eta_\infty = \mu_\infty = 0$. Thus, we can deduce from (2.22) that

$$\int_{R^N} |x|^{-bp^*}|u_n|^{p^*}dx \longrightarrow \int_{R^N} |x|^{-bp^*}|u|^{p^*}dx \quad \text{as } n \longrightarrow \infty, \tag{2.30}$$

which implies that $u_n \to u$ in $L^{p^*}(R^N, |x|^{-bp^*})$.

Step 3. $\{u_n\}$ converges strongly in X.

The following inequalities [21] play an important role in our proof:

$$|\xi - \zeta|^p \le \begin{cases} c\left\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta \right\rangle & \text{for } p \ge 2, \\ c\left\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta \right\rangle^{p/2} \left(|\xi|^p + |\zeta|^p\right)^{(2-p)/2} & \text{for } 1 < p < 2. \end{cases} \tag{2.31}$$

Our aim is to prove that $\{u_n\}$ is a Cauchy sequence of X. In fact, let $\varphi = u_n - u_m$ in (1.12), it follows from (2.19) that

$$\begin{aligned} A_{mn} &+ \int_{R^N} |x|^{-(a+1)p}\left(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right)(u_n - u_m)dx \\ &= \langle I'(u_n) - I'(u_m), u_n - u_m\rangle \\ &\quad + \int_{R^N} f(x)\left(|u_n|^{r-2}u_n - |u_m|^{r-2}u_m\right)(u_n - u_m)dx \\ &\quad + \int_{R^N} h(x)\left(|u_n|^{s-2}u_n - |u_m|^{s-2}u_m\right)(u_n - u_m)dx \\ &\quad + \int_{R^N} |x|^{-bp^*}\left(|u_n|^{p^*-2}u_n - |u_m|^{p^*-2}u_m\right)(u_n - u_m)dx, \end{aligned} \tag{2.32}$$

where

$$A_{mn} = \int_{R^N} |x|^{-ap}\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right) \cdot \nabla(u_n - u_m)dx. \tag{2.33}$$

Using the inequalities (2.31), we can get by direct computation that

$$A_{mn} \ge \begin{cases} c\displaystyle\int_{R^N} |x|^{-ap}|\nabla(u_n - u_m)|^p dx, & p \ge 2 \\ c\left(\displaystyle\int_{R^N} |x|^{-ap}|\nabla(u_n - u_m)|^p dx\right)^{2/p}, & 1 < p < 2, \end{cases} \tag{2.34}$$

with some constant $c > 0$, independent of n and m.

Then the Hölder inequality together with (1.8) and (2.30) yield that

$$\int_{R^N} |x|^{-bp^*}\left(|u_n|^{p^*-2}u_n - |u_m|^{p^*-2}u_m\right)(u_n - u_m)dx \longrightarrow 0 \quad (\text{as } n, m \longrightarrow \infty). \tag{2.35}$$

Similarly, we have from the Hölder inequality, Lemmas 2.3 and 2.4 that

$$\int_{R^N} f(x)\left(|u_n|^{r-2}u_n - |u_m|^{r-2}u_m\right)(u_n-u_m)dx \longrightarrow 0 \quad (\text{as } n,m \longrightarrow \infty),$$
$$\int_{R^N} h(x)\left(|u_n|^{s-2}u_n - |u_m|^{s-2}u_m\right)(u_n-u_m)dx \longrightarrow 0 \quad (\text{as } n,m \longrightarrow \infty). \tag{2.36}$$

Therefore, the above estimates imply that $\|u_n-u_m\|_X \to 0$ $(n,m \to \infty)$, that is, $\{u_n\}$ is a Cauchy sequence of X. Then $\{u_n\}$ converges strongly in X and we complete the proof. □

Similarly, we have the following lemma.

Lemma 2.7. *Let $1<r<p<s<p^*$. Then $I(u)$ satisfies the $(PS)_c$ condition with $c \le (1/s - 1/p^*)S_1^{p^*/(p^*-p)} + (((s-r)/(s-p))S_2)^{r/(p-r)}((r-p)(s-r)/prs)\|f\|^{p/(p-r)}_{L^{\sigma_1}(\mathbb{R}^N,g_1)}$, where S_1, S_2 are as in* (1.8)*, and* (1.9) *respectively.*

Proof. Step 1. $\{u_n\}$ is bounded in X.

Let $\{u_n\}$ be a $(PS)_c$ sequence of $I(u)$ in X. Then we have from Lemma 2.3 that

$$\begin{aligned} c+1+\|u_n\|_X &\ge I(u_n) - \frac{1}{s}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{p}-\frac{1}{s}\right)\|u_n\|_X^p - \left(\frac{1}{r}-\frac{1}{s}\right)\|u_n\|^r_{L^r(\mathbb{R}^N,f)} + \left(\frac{1}{s}-\frac{1}{p^*}\right)\|u_n\|^{p^*}_{L^{p^*}(\mathbb{R}^N)} \\ &\ge \left(\frac{1}{p}-\frac{1}{s}\right)\|u_n\|_X^p - \left(\frac{1}{r}-\frac{1}{s}\right)S_2^{r/p}\|u_n\|_X^r\|f\|_{L^{\sigma_1}(\mathbb{R}^N,g_1)}. \end{aligned} \tag{2.37}$$

Since $1<r<p<s$, (2.37) shows that $\|u_n\|$ is bounded in X.

Step 2. There exists $\{u_n\}$ in X such that $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$.

Similar to the proof of Lemma 2.5, we can get that $\mu_j = \eta_j = 0$ or $\eta_j > S_1^{p^*/(p^*-p)}$ by applying the functional φ. Now we prove that there is no $j_0 \in J$ such that $\eta_{j_0} > S_1^{p^*/(p^*-p)}$. Suppose otherwise, then

$$\begin{aligned} c+o(1) &= I(u_n) - \frac{1}{s}\langle I'(u), u_n\rangle \\ &= \left(\frac{1}{p}-\frac{1}{s}\right)\|u_n\|_X^p - \left(\frac{1}{r}-\frac{1}{s}\right)\|u_n\|^r_{L^r(\mathbb{R}^N,f)} + \left(\frac{1}{s}-\frac{1}{p^*}\right)\|u_n\|^{p^*}_{L^{p^*}(\mathbb{R}^N)} \\ &\ge \left(\frac{1}{p}-\frac{1}{s}\right)\|u_n\|_X^p - \left(\frac{1}{r}-\frac{1}{s}\right)S_2^{r/p}\|u_n\|_X^r\|f\|_{L^{\sigma_1}(\mathbb{R}^N,g_1)} + \left(\frac{1}{s}-\frac{1}{p^*}\right)S_1^{p^*/(p^*-p)}. \end{aligned} \tag{2.38}$$

Let

$$q(t) = \left(\frac{1}{p}-\frac{1}{s}\right)t^p - \left(\frac{1}{r}-\frac{1}{s}\right)S_2^{r/p}\|f\|_{L^{\sigma_1}(\mathbb{R}^N,g_1)}t^r, \quad t\ge 0. \tag{2.39}$$

Then $q(t)$ has the unique minimum point at

$$t_0 = \left[\frac{s-r}{s-p} S_2^{r/p} \|f\|_{L^{\sigma_1}(\mathbb{R}^N, g_1)}\right]^{1/(p-r)},$$
$$q(t_0) = \left(\frac{s-r}{s-p} S_2\right)^{r/(p-r)} \frac{(r-p)(s-r)}{prs} \|f\|^{p/(p-r)}_{L^{\sigma_1}(\mathbb{R}^N, g_1)}. \tag{2.40}$$

Then it follows from (2.38) that

$$c + o(1) \geq \left(\frac{1}{s} - \frac{1}{p^*}\right) S_1^{p^*/(p^*-p)} + \left(\frac{s-r}{s-p} S_2\right)^{(r/p-r)} \frac{(r-p)(s-r)}{prs} \|f\|^{p/(p-r)}_{L^{\sigma_1}(\mathbb{R}^N, g_1)}, \tag{2.41}$$

which contradicts the hypothesis of c.

Step 3. $\{u_n\}$ converges strongly in X.

By Lemma 2.4, this result can be similarly obtained by the method in Lemma 2.6, so we omit the proof. □

3. Existence of Infinitely Solutions

In this section, we will use the minimax procedure to prove the existence of infinity many solutions of problem (1.1). Let $\mathcal{A}$ denotes the class of $A \subset X \setminus \{0\}$ such that A is closed in X and symmetric with respect to the origin. For $A \in \mathcal{A}$, we recall the genus $\gamma(A)$ which is defined by

$$\gamma(A) := \min\{m \in N : \exists \phi \in C(A, \mathbb{R}^m \setminus \{0\}), \phi(x) = -\phi(-x)\}. \tag{3.1}$$

If there is no mapping ϕ as above for any $m \in N$, then $\gamma(A) = +\infty$, and $\gamma(\emptyset) = 0$. The following proposition gives some main properties of the genus, see [13, 22].

Proposition 3.1. *Let $A, B \in \mathcal{A}$. Then*

(1) *if there exists an odd map $g \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$,*

(2) *if $A \subset B$, then $\gamma(A) \leq \gamma(B)$,*

(3) *$\gamma(A \bigcup B) \leq \gamma(A) + \gamma(B)$.*

(4) *if S is a sphere centered at the origin in $\mathbb{R}^N$, then $\gamma(S) = N$,*

(5) *if A is compact, then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that $N_\delta(A) \in \mathcal{A}$ and $\gamma(N_\delta(A)) = \gamma(A)$, where $N_\delta(A) = \{x \in X : \|x - A\| \leq \delta\}$.*

Lemma 3.2. *Assume (A_1)–(A_3). Then for any $m \in N$, there exists $\varepsilon = \varepsilon(m) > 0$ such that*

$$\gamma(\{u \in X : I(u) \leq -\varepsilon\}) \geq m. \tag{3.2}$$

Proof. For given $m \in \mathbb{N}^+$, let X_m be a m-dimensional subspace of X. If $p < r < s < p^*$, then for $u \in X_m$ we have

$$I(u) = \frac{1}{p}\|u\|_X^p - \frac{1}{r}\|u\|_{L^r(\mathbb{R}^N,f)}^r - \frac{1}{s}\|u\|_{L^s(\mathbb{R}^N,h)}^s - \frac{1}{p^*}\|u\|_{L^{p^*}}^{p^*} \le \frac{1}{p}\|u\|_X^p - \frac{1}{r}\|u\|_{L^r(\mathbb{R}^N,f)}^r. \tag{3.3}$$

The fact that all the norms on finite dimensional space are equivalent implies that for all $u \in X_m$

$$I(u) \le \frac{1}{p}\|u\|_X^p - c\|u\|_X^r, \tag{3.4}$$

for some constant $c > 0$. Then there exist large $\rho > 0$ and small $\varepsilon > 0$ such that

$$I(u) \le -\varepsilon, \qquad \|u\|_{X_m} = \rho. \tag{3.5}$$

Denote

$$S_\rho = \left\{u \in X_m : \|u\|_{X_m} = \rho\right\}. \tag{3.6}$$

Then S_ρ is a sphere centered at the origin with radius of ρ and

$$S_\rho \subset \{u \in X : I(u) \le -\varepsilon\} \triangleq I^{-\varepsilon}. \tag{3.7}$$

Therefore, Proposition 3.1 shows that $\gamma(I^{-\varepsilon}) \ge \gamma(S_\rho) = m$.

If $r < p < s < p^*$, we have

$$I(u) = \frac{1}{p}\|u\|_X^p - \frac{1}{r}\|u\|_{L^r(\mathbb{R}^N,f)}^r - \frac{1}{s}\|u\|_{L^s(\mathbb{R}^N,h)}^s - \frac{1}{p^*}\|u\|_{L^{p^*}}^{p^*} \le \frac{1}{p}\|u\|_X^p - \frac{1}{s}\|u\|_{L^s(\mathbb{R}^N,h)}^s. \tag{3.8}$$

Since $\|u\|_{L^s(\mathbb{R}^N,h)}^s$ is also a norm and all norms on the finite dimensional space X_m are equivalent, we have

$$I(u) \le \frac{1}{p}\|u\|_X^p - c\|u\|_X^s. \tag{3.9}$$

Then there exist large $\sigma > 0$ and small $\varepsilon > 0$ such that

$$I(u) \le -\varepsilon, \qquad \|u\|_{X_m} = \sigma. \tag{3.10}$$

Denote

$$S_\sigma = \left\{u \in X_m : \|u\|_{X_m} = \sigma\right\}. \tag{3.11}$$

Then S_σ is a sphere centered at the origin with radius of σ and

$$S_\sigma \subset \{u \in X : I(u) \le -\varepsilon\} \triangleq I^{-\varepsilon}. \tag{3.12}$$

Therefore, Proposition 3.1 shows that $\gamma(I^{-\varepsilon}) \ge \gamma(S_\sigma) = m$. □

Let $\mathcal{A}_m = \{A \in \mathcal{A} : \gamma(A) \ge m\}$. It is easy to check that $\mathcal{A}_{m+1} \subset \mathcal{A}_m (m = 1, 2, \ldots)$. We define

$$c_m = \inf_{A \in \mathcal{A}_m} \sup_{u \in A} I(u). \tag{3.13}$$

It is not difficult to find that

$$c_1 \le c_2 \le \cdots \le c_m \le \cdots . \tag{3.14}$$

and $c_m > -\infty$ for any $m \in \mathbb{N}$ since $I(u)$ is coercive and bounded below. Furthermore, we define the set

$$K_c = \left\{u \in X : I(u) = c, I'(u) = 0\right\}. \tag{3.15}$$

Then, K_c is compact and we have the following important lemma, see [22].

Lemma 3.3. *All the c_m are critical values of $I(u)$. Moreover, if $c = c_m = c_{m+1} = \cdots = c_{m+\tau}$, then $\gamma(K_c) \ge 1 + \tau$.*

Proof of Theorem 1.2. In view of Lemmas 2.6 and 2.7, $I(u)$ satisfies the $(PS)_c$ condition in X. Furthermore, as the standard argument of [13, 22, 23], Lemma 3.3 gives that $I(u)$ has infinity many critical points with negative values. Thus, problem (1.1) has infinitely many solutions in X, and we complete the proof. □

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