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Research Article

A Remark on Myhill-Nerode Theorem for Fuzzy Languages

Shou-feng Wang

School of Mathematics, Yunnan Normal University, Yunnan, Kunming 650500, China

Correspondence should be addressed to Shou-feng Wang, wsf1004@163.com

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Regular fuzzy languages are characterized by some algebraic approaches. In particular, an extended version of Myhill-Nerode theorem for fuzzy languages is obtained.

1. Introduction

Fuzzy sets were introduced by Zadeh in [1] and since then have appeared in many fields of sciences. They have been studied within automata theory for the first time by Wee in [2]. More on recent development of algebraic theory of fuzzy automata and formal fuzzy languages can be found in the book Mordeson and Malik [3], the texts Malik et al. [4, 5], and Petković [6].

A fuzzy language is called *regular* if it can be recognized by a fuzzy automaton. In the texts Mordeson and Malik [3], Petković [6], Ignjatovic et al. [7], and Shen [8], regular fuzzy languages have been characterized by the *principal congruences* (*principal right congruences, principal left congruences*) determined by fuzzy languages, which are known as *Myhill-Nerode theorem* for fuzzy languages. Moreover, Petković [6] also considered the varieties of fuzzy languages and Ignjatovic and Ciric [9] considered regular operations of fuzzy languages.

Recently, Wang et al. [10] generalized the usual principal congruences (resp., principal right congruences, principal left congruences) to some kinds of *generalized principal congruences* (resp., *generalized principal right congruences, generalized principal left congruences*) determined by crisp languages by using *prefix-suffix-free subsets* (resp., *prefix-free languages, suffix-free languages*) and obtained some characterizations of regular crisp languages.

In this note, we will realize the idea of the text [10] for fuzzy languages. In other word, we characterize regular fuzzy languages by some kinds of generalized principal congruences (resp., generalized principal right congruences, generalized principal left congruences) determined by fuzzy languages. In particular, we obtain an extended version of Myhill-Nerode theorem for fuzzy languages.

2. Preliminaries

Throughout the paper, A is a finite set which is called an *alphabet* and A^* is the free monoid generated by A, that is, the set of all words with letters from A. The empty word is denoted by 1. The length of a word w in A^* is the number of letters appearing in w and is denoted by $|w|$. The *complement* of a subset L of A^* is the set $\overline{L} = \{w \in A^* \mid w \notin L\}$. A subset L of A^* is *cofinite* if $\overline{L}$ is finite. A nonempty subset S of A^* is called a *suffix-free language over* A if no element in S is a suffix of another element in S. *Prefix-free languages over* A can be defined dually. On the other hand, a nonempty subset L of A^* is called a *prefix-closed language over* A if any prefix of an element in L is also in L.

As an analogue of prefix-free languages and suffix-free languages over A, Wang et al. [10] introduced *prefix-suffix-free subsets* of $A^* \times A^*$. A subset Δ of the set $A^* \times A^*$ is called a *prefix-suffix-free subset* if for all words s, t, x, y in A^*, the following holds: if both (s, t) and (sx, yt) are in Δ, then $x = y = 1$.

An equivalence ρ on A^* is called *a right congruence* if $x\rho y$ implies that $xz\rho yz$ for any $x, y, z \in A^*$. A left congruences can be defined dually. An equivalence is *a congruence* if it is a right congruence and also a left congruence.

A *fuzzy subset* α of a set X is a mapping $\alpha : X \to [0,1]$. By $\wedge$ and $\vee$ infimum and supremum in the unit segment $[0,1]$ will be denoted, respectively. Every element y of X can be considered as the following fuzzy subset of X:

$$y(x) = 1 \quad \text{for } x = y, \qquad y(x) = 0 \quad \text{for } x \neq y. \tag{2.1}$$

A *fuzzy language* over A is a fuzzy subset of A^*. A fuzzy language is *regular* if it is recognizable by a fuzzy automaton from the book [3]. For a fuzzy language λ over A, the relations defined on A^* by the following:

$$\begin{gathered} xP_\lambda^{(r)}y \text{ if } \lambda(xu) = \lambda(yu) \quad \text{for every } u \text{ in } A^*, \\ xP_\lambda^{(l)}y \text{ if } \lambda(ux) = \lambda(uy) \quad \text{for every } u \text{ in } A^*, \\ xP_\lambda y \text{ if } \lambda(uxv) = \lambda(uyv) \quad \text{for every } u, v \in A^*, \end{gathered} \tag{2.2}$$

are called the *principal right congruence* (resp., *principal left congruence, principal congruence*) determined by λ, respectively.

Now, we state the well-known Myhill-Nerode theorem for fuzzy languages which gives some algebraic characterizations for regular fuzzy languages. Recall that *the index* of an equivalence ρ on A^* is the number of ρ-classes of A^*.

Theorem 2.1 (see [3, 6, 8], Myhill-Nerode theorem). *For a fuzzy language λ over A, the following statements are equivalent:*

(1) *λ is regular.*

(2) *P_λ is of finite index.*

(3) $P_\lambda^{(r)}$ *is of finite index.*

(4) $P_\lambda^{(l)}$ *is of finite index.*

In the sequel, we recall some operations of fuzzy languages. For two fuzzy languages λ_1 and λ_2 over A, the *union, intersection, product, and left and right quotients* of λ_1 and λ_2 are defined, respectively, by the following:

$$\begin{aligned}
(\lambda_1 \cup \lambda_2)(w) &= \lambda_1(w) \vee \lambda_2(w), \\
(\lambda_1 \cap \lambda_2)(w) &= \lambda_1(w) \wedge \lambda_2(w), \\
(\lambda_1\lambda_2)(w) &= \bigvee_{xy=w} (\lambda_1(x) \wedge \lambda_2(y)) \\
\left(\lambda_1^{-1}\lambda_2\right)(w) &= \bigvee_{u\in A^*} (\lambda_2(uw) \wedge \lambda_1(u)), \\
\left(\lambda_2\lambda_1^{-1}\right)(w) &= \bigvee_{u\in A^*} (\lambda_2(wu) \wedge \lambda_1(u)).
\end{aligned} \tag{2.3}$$

Further, we also define *left-right quotient* of three fuzzy languages λ_1, λ_2 and λ over A by the following:

$$\left(\lambda_1^{-1}\lambda\lambda_2^{-1}\right)(w) = \left[\left(\lambda_1^{-1}\lambda\right)\lambda_2^{-1}\right](w). \tag{2.4}$$

Observe that $(s^{-1}\lambda t^{-1})(w) = \lambda(swt)$ for any $s, t, w \in A^*$ with the above notations.

On regular fuzzy languages, we have the following.

Lemma 2.2 (see [6]). *Finite unions, intersections, products, and left-right quotients of regular fuzzy languages over A are regular.*

3. Main Result

In this section, we shall introduce some kinds of generalized principal (resp., right, left) congruences determined by fuzzy languages by using prefix-suffix-free subsets (resp., prefix-free languages, suffix-free languages) and give an extended version of Myhill-Nerode theorem for fuzzy languages.

Now, let P be a prefix-free language, S be a suffix-free language over A, Δ be a prefix-suffix-free subset of $A^* \times A^*$, and λ be a fuzzy language over A, respectively. For a prefix-suffix-free subset Δ, denote

$$\Omega_\Delta = \left\{(sx, yt) \mid (s,t) \in \Delta, x, y \in A^*\right\}, \qquad N(\Delta) = \bigcup_{(s,t)\in\Delta} sA^*t. \tag{3.1}$$

Define the following relations on A^*:

$$\begin{gathered}
xP^{(l)}_{P,\lambda}y \text{ if } \lambda(ux)=\lambda(uy) \quad \text{for every } u \text{ in } PA^*,\\
xP^{(r)}_{S,\lambda}y \text{ if } \lambda(xu)=\lambda(yu) \quad \text{for every } u \text{ in } A^*S,\\
xP_{\Delta,\lambda}y \text{ if } \lambda(uxv)=\lambda(uyv) \quad \text{for every } (u,v) \text{ in } \Omega_{\Delta},\\
xP^{(r)}_{\mathcal{F},S,\lambda}y \text{ if there exists some finite subset } F \text{ of } A^* \text{ such that}\\
\lambda(xu)=\lambda(yu) \quad \text{for every } u \text{ in } F\overline{\overline{A^*S}},\\
xP^{(l)}_{\mathcal{F},P,\lambda}y \text{ if there exists some finite subset } F \text{ of } A^* \text{ such that}\\
\lambda(ux)=\lambda(uy) \quad \text{for every } u \text{ in } \overline{\overline{PA^*}}F.
\end{gathered} \tag{3.2}$$

Then we have the following observations.

Proposition 3.1. *The above $P^{(r)}_{S,\lambda}, P^{(r)}_{\mathcal{F},S,\lambda}$ (resp., $P^{(l)}_{P,\lambda}, P^{(l)}_{\mathcal{F},P,\lambda}$; $P_{\Delta,\lambda}$) are right congruences (resp., left congruences; congruence) on A^*. Furthermore,*

$$P^{(r)}_{\lambda} \subseteq P^{(r)}_{S,\lambda} \subseteq P^{(r)}_{\mathcal{F},S,\lambda}, \qquad P^{(l)}_{\lambda} \subseteq P^{(l)}_{P,\lambda} \subseteq P^{(l)}_{\mathcal{F},P,\lambda}, \qquad P_{\lambda} \subseteq P_{\Delta,\lambda}. \tag{3.3}$$

Proof. It is easy to check that $P^{(r)}_{S,\lambda}$ (resp., $P^{(l)}_{P,\lambda}$) is a right (resp., left) congruence, $P_{\Delta,\lambda}$ is a congruence, and

$$P^{(r)}_{\lambda} \subseteq P^{(r)}_{S,\lambda} \subseteq P^{(r)}_{\mathcal{F},S,\lambda}, \qquad P^{(l)}_{\lambda} \subseteq P^{(l)}_{P,\lambda} \subseteq P^{(l)}_{\mathcal{F},P,\lambda}, \qquad P_{\lambda} \subseteq P_{\Delta,\lambda} \tag{3.4}$$

by their definitions. In the sequel, we show that $P^{(r)}_{\mathcal{F},S,\lambda}$ is a right congruence and $P^{(l)}_{\mathcal{F},P,\lambda}$ is a left congruence. Clearly, both $P^{(r)}_{\mathcal{F},S,\lambda}$ and $P^{(l)}_{\mathcal{F},P,\lambda}$ are equivalences. Now, let x, y be two words in A^* and $xP^{(r)}_{\mathcal{F},S,\lambda}y$. Then there exists a finite subset F of A^* such that $\lambda(xu) = \lambda(yu)$ for any u in $F\overline{\overline{A^*S}}$. Now, let z be a word in A^* and F' be the union of $\{w \in A^* \mid zw \in F\}$ and $\{1\}$. Then zu is in $F\overline{\overline{A^*S}}$ for any u in $F'\overline{\overline{A^*S}}$. This implies that $\lambda(xzu) = \lambda(yzu)$ for any u in $F'\overline{\overline{A^*S}}$ whence $xzP^{(r)}_{\mathcal{F},S,\lambda}yz$ since F' is finite. Thus, $P^{(r)}_{\mathcal{F},S,\lambda}$ is a right congruence. Dually, $P^{(l)}_{\mathcal{F},P,\lambda}$ is a left congruence. □

Remark 3.2. Note that the above inclusions are all proper in general. For example, let $A = \{a\}$, $S = \{a^2\}$ and $F = \{1, a, a^2, a^3\}$. Then $F\overline{\overline{A^*S}} = A^*a^5$. Define a fuzzy language over A as follows:

$$\lambda(w) = \alpha \quad \text{for } w \in \left\{a^2, a^3\right\}, \qquad \lambda(w) = \beta \quad \text{for } w \in A^* \setminus \left\{a^2, a^3\right\}, \tag{3.5}$$

where α, β are in $[0,1]$ and $\alpha \neq \beta$. Then we have

$$\left(a^3, a^4\right) \notin P_{\lambda}^{(r)}, \qquad \left(a^3, a^4\right) \in P_{S,\lambda}^{(r)}, \qquad \left(1, a^2\right) \notin P_{S,\lambda}^{(r)}, \qquad \left(1, a^2\right) \in P_{\mathcal{F},S,\lambda}^{(r)}. \tag{3.6}$$

Similarly, we can show that the remainder inclusions are all proper.

To obtain our main result, we need a series of lemmas. First, we recall the following *alphabetic order* "$\leq$" on A^*: For two words u and v in A^* with different lengths, $u < v$ if $|u| < |v|$, for two words with same length, the order is the lexicographic order. Observe that the alphabetic order is a well order on A^*. We have the following result.

Lemma 3.3. *Let L be an infinite prefix-closed language over A. Then there exists an infinite subset $\{1, a_1, a_1a_2, \ldots, a_1a_2, \ldots, a_n, \ldots\}$ of L, where $a_i \in A$.*

Proof. Denote

$$\mathrm{Pre}f_A(L) = \left\{a \in A \mid \left(\exists y \in A^*\right)\ ay \in L\right\}. \tag{3.7}$$

Observe that A is finite and L is infinite, there exists $L_1 \subseteq L$ and $a_1 \in A$ such that L_1 is infinite and $\mathrm{Pre}f_A(L_1) = \{a_1\}$. Denote

$$a_1^{-1}L_1 = \{w \in A^* \mid a_1w \in L_1\}. \tag{3.8}$$

Then $a_1^{-1}L_1$ is infinite. Hence, there also exists $L_2 \subseteq a_1^{-1}L_1$ and $a_2 \in A$ such that L_2 is infinite and $\mathrm{Pre}f_A(L_2) = \{a_2\}$. In general, for any positive integer n, there exists $L_{n+1} \subseteq a_n^{-1}L_n$ and $a_{n+1} \in A$ such that L_{n+1} is infinite and $\mathrm{Pre}f_A(L_{n+1}) = \{a_{n+1}\}$. Let

$$C = \{1, a_1, a_1a_2, a_1a_2a_3, \ldots, a_1a_2a_3\cdots a_n, \ldots\}. \tag{3.9}$$

Clearly, C is infinite. We claim that $C \subseteq L$. Let $a_1a_2a_3\cdots a_n \in C$. Observe that

$$\begin{gathered} L_n \subseteq a_{n-1}^{-1}L_{n-1} \subseteq a_{n-1}^{-1}a_{n-2}^{-1}L_{n-2} \subseteq \cdots \subseteq a_{n-1}^{-1}a_{n-2}^{-1}\cdots a_1^{-1}L_1 = (a_1a_2\cdots a_{n-1})^{-1}L_1, \\ \{a_n\} = \mathrm{Pre}f_A(L_n) \subseteq \mathrm{Pre}f_A\Big((a_1a_2\cdots a_{n-1})^{-1}L_1\Big). \end{gathered} \tag{3.10}$$

Therefore, there exists $y \in A^*$ such that $a_ny \in (a_1a_2\cdots a_{n-1})^{-1}L_1$. And hence, $a_1a_2\cdots a_{n-1}a_ny \in L_1 \subseteq L$. Since L is prefix-closed, $a_1a_2\cdots a_{n-1}a_n \in L$. This implies that $C \subseteq L$. □

Lemma 3.4. *Let ρ be a right congruence on A^* and $\{L_i \mid i \in I\}$ be the set of all ρ-classes of A^*. Then,*

$$L_\rho = \{s_i \mid s_i \textit{ is the least element in } L_i \textit{ with respect to } \text{“} \leq \text{”},\ i \in I\} \tag{3.11}$$

is prefix-closed.

Proof. Clearly, 1 is in L_ρ. Let s_j be in L_ρ and $s_j = a_1 a_2 \cdots a_t$ for some positive integer $t > 1$ and $a_1, a_2, \ldots, a_t$ in A. Then, $a_1 a_2 \cdots a_{t-1}$ is not in L_j. Suppose that $a_1 a_2 \cdots a_{t-1}$ is in L_k. Then, $s_k \leq a_1 a_2 \cdots a_{t-1}$. This implies that $s_k a_t \leq a_1 a_2 \cdots a_{t-1} a_t = s_j$. On the other hand, since $s_k \rho a_1 a_2 \cdots a_{t-1}$ and ρ is a right congruence, we have $s_k a_t \rho s_j$. Hence, $s_k a_t$ is in L_j and so $s_k a_t \geq s_j$. Thus, $s_k a_t = s_j = a_1 a_2 a_3 \cdots a_{t-1} a_t$. This implies that $s_k = a_1 a_2 a_3 \cdots a_{t-1}$ whence $a_1 a_2 a_3 \cdots a_{t-1}$ is in L_ρ. □

Lemma 3.5. *Let S be a suffix-free language and λ be a fuzzy language over A.*

(1) *$P_{\{1\}\times S,\lambda}$ is of finite index if and only if $P^{(r)}_{S,\lambda}$ is of finite index.*

(2) *$P^{(r)}_{S,\lambda}$ is of finite index if and only if $P^{(r)}_{\mathcal{F},S,\lambda}$ is of finite index.*

Proof. (1) Similar to the proof of Proposition 3.11 in [10].

(2) Observe that $P^{(r)}_{S,\lambda} \subseteq P^{(r)}_{\mathcal{F},S,\lambda}$, the necessity holds. Conversely, if $P^{(r)}_{\mathcal{F},S,\lambda}$ is of finite index and $P^{(r)}_{S,\lambda}$ is of infinite index, then by Lemma 3.4, $L_{P^{(r)}_{S,\lambda}}$ is infinite and prefix-closed. By Lemma 3.3, there exists an infinite subset

$$C = \{1, a_1, a_1 a_2, \cdots, a_1 a_2 \cdots a_n, \ldots\}, \tag{3.12}$$

of $L_{P^{(r)}_{S,\lambda}}$, where $a_i \in A$. Since $P^{(r)}_{\mathcal{F},S,\lambda}$ is of finite index, there exist two distinct elements $x, y \in C$ such that $x P^{(r)}_{\mathcal{F},S,\lambda} y$. Therefore, there exists a finite subset F of A^* such that $\lambda(xu) = \lambda(yu)$ for every u in $\overline{F\overline{A^*S}}$. Denote $T = \max\{|f| \mid f \in F\}$ and take u in A^* satisfying $|u| > T$. We assert that uv is in $\overline{F\overline{A^*S}}$ for any v in A^*S. In fact, if $uv = fw$ for some f in F and w in $\overline{A^*S}$, then by the choice of u, f is a prefix of u and so v is a suffix of w whence w is in A^*S. A contradiction. Therefore, for any v in A^*S, we have $\lambda(xuv) = \lambda(yuv)$. This implies that $xu P^{(r)}_{S,\lambda} yu$.

Without loss of generality, we let $x < y$ with respect to the alphabetic order, $y = a_1 a_2 \cdots a_t$ and $u = a_{t+1} \cdots a_{t+T+1}$. Then, by the above discussions, $xu P^{(r)}_{S,\lambda} yu$ and yu is in C. Observe that C is a subset of $L_{P^{(r)}_{S,\lambda}}$, in view of the definition of $L_{P^{(r)}_{S,\lambda}}$, $xu \geq yu$. This implies that $x \geq y$. A contradiction. □

Lemma 3.6. *Let S be a finite suffix-free language over A. Then A^*S is cofinite if and only if S is maximal.*

Proof. It follows from Lemma 3.14 in [10]. □

Lemma 3.7. *Let Δ be a finite prefix-suffix-free subset of $A^* \times A^*$ and λ be a fuzzy language over A. Then the following are equivalent:*

(1) *$P_{\Delta,\lambda}$ is of finite index.*

(2) *The following fuzzy language λ_Δ over A defined by*

$$\lambda_\Delta(w) = \lambda(w) \quad \textit{for } w \in N(\Delta), \qquad \lambda_\Delta(w) = 0 \quad \textit{for } w \notin N(\Delta) \tag{3.13}$$

is regular.

(3) *$\lambda = \lambda_1 \cup \lambda_2$, where λ_1 is regular and $\lambda_2(w) = 0$ for any w in $N(\Delta)$.*

Proof. (1) *implies* (2). Let x, y be in A^*, (s, t) be in Δ and $xP_{\Delta,\lambda}y$. Then for any u, v in A^*, (su, vt) is in Ω_Δ. Therefore,

$$s^{-1}\lambda t^{-1}(uxv) = \lambda(suxvt) = \lambda(suyvt) = s^{-1}\lambda t^{-1}(uyv), \tag{3.14}$$

whence $xP_{s^{-1}\lambda t^{-1}}y$. Thus,

$$P_{\Delta,\lambda} \subseteq \bigcap_{(s,t)\in\Delta} P_{s^{-1}\lambda t^{-1}}. \tag{3.15}$$

Now, if $P_{\Delta,\lambda}$ is of finite index, then $s^{-1}\lambda t^{-1}$ is regular for any (s, t) in Δ. Observe that

$$\lambda_\Delta = \bigcup_{(s,t)\in\Delta} \left[s\left(s^{-1}\lambda t^{-1}\right)\right]t, \tag{3.16}$$

it follows that λ_Δ is regular from Lemma 2.2.

(2) *implies* (3). By (2), λ_Δ is regular. Let λ_2 be the following fuzzy language over A defined by

$$\lambda_2(w) = 0 \quad \text{for } w \in N(\Delta), \qquad \lambda_2(w) = \lambda(w) \quad \text{for } w \notin N(\Delta). \tag{3.17}$$

Then $\lambda = \lambda_\Delta \cup \lambda_2$, as required.

(3) *implies* (1). If $\lambda = \lambda_1 \cup \lambda_2$ for some regular fuzzy language λ_1 and a fuzzy language λ_2 such that $\lambda_2(w) = 0$ for any w in $N(\Delta)$, then P_{λ_1} is of finite index and $P_{\Delta,\lambda_2} = A^* \times A^*$. Observe that

$$P_{\lambda_1} \subseteq P_{\Delta,\lambda_1} \subseteq P_{\Delta,\lambda_1\cup\lambda_2} = P_{\Delta,\lambda}, \tag{3.18}$$

$P_{\Delta,\lambda}$ is of finite index. □

Remark 3.8. In general, for a given finite prefix-suffix-free subset of $A^* \times A^*$ and a fuzzy language λ over A, λ may be nonregular even if $P_{\Delta,\lambda}$ is of finite index. For example, let $A = \{a, b\}$ and $\Delta = \{(a, b)\}$. Define the following fuzzy language λ over A as follows:

$$\lambda(w) = 0 \quad \text{for } w \in N(\Delta), \qquad \lambda(w) = \frac{1}{|w| + 1} \quad \text{for } w \notin N(\Delta). \tag{3.19}$$

Clearly, $\lambda_\Delta(w) = 0$ for every w in A^* and so λ_Δ is trivially regular. By Lemma 3.7, $P_{\Delta,\lambda}$ is of finite index. However, for any pair w_1, w_2 in $\overline{N(\Delta)}$ with different lengths, we have $\lambda(w_1) \neq \lambda(w_2)$ whence w_1 is not P_λ related to w_2. Observe that $\overline{N(\Delta)}$ is infinite, there are infinite P_λ-classes of A^* and so P_λ is of infinite index. This implies that λ is nonregular by Theorem 2.1.

Now, we have our main theorem.

Theorem 3.9 (An extended version of Myhill-Nerode theorem). *For a fuzzy language λ over A, the following statements are equivalent:*

(1) *λ is regular.*

(2) *$P^{(r)}_{S,\lambda}$ is of finite index for some finite maximal suffix-free language S over A.*

(3) *$P^{(l)}_{P,\lambda}$ is of finite index for some finite maximal prefix-free language P over A.*

(4) *$P_{\Delta,\lambda}$ is of finite index for some finite prefix-suffix-free subset Δ of $A^* \times A^*$ such that $N(\Delta)$ is cofinite.*

(5) *$P^{(l)}_{\mathcal{F},P,\lambda}$ is of finite index for some finite maximal prefix-free language P over A.*

(6) *$P^{(r)}_{\mathcal{F},S,\lambda}$ is of finite index for some finite maximal suffix-free language S over A.*

Proof. (1) *implies* (2). Observe that $\{1\}$ is a maximal suffix-free language over A and $P^{(r)}_{\lambda} = P^{(r)}_{\{1\},\lambda}$, the result follows from Theorem 2.1.

(2) *implies* (4). Observe that $\{1\} \times S$ is a prefix-suffix-free subset of $A^* \times A^*$ and $N(\{1\} \times S) = A^*S$, the result follows from Lemma 3.5 (1) and Lemma 3.6.

(4) implies (1). By Lemma 3.7 (3), there exists a regular fuzzy language λ_1 and another fuzzy language λ_2 such that $\lambda = \lambda_1 \cup \lambda_2$ and $\lambda_2(w) = 0$ for any w in $N(\Delta)$. However, by (4), $\overline{N(\Delta)}$ is finite, which implies that λ_2 is also regular. In view of Lemma 2.2, λ is regular.

By symmetry, we can prove that the facts that (1) implies (3) and (3) implies (4). On the other hand, by Lemma 3.5 (2) and its dual, it follows that (3) is equivalent to (5) and (2) is equivalent to (6). □

4. Conclusions

In this short note, we have obtained an extended version of Myhill-Nerode theorem for fuzzy languages (Theorem 3.9) which provides some algebraic characterizations of regular fuzzy languages. On the other hand, for a given prefix-suffix-free subset Δ of $A^* \times A^*$, by Proposition 3.1 and Remark 3.8,

$$\mathbb{FR}_{\Delta}(A) = \{\lambda \mid \lambda \text{ is a fuzzy language over } A \text{ such that the index of } P_{\Delta,\lambda} \text{ is finite}\} \quad (4.1)$$

contains the class of regular fuzzy languages over A as a proper subclass. In fact, Lemma 3.7 gives some characterizations of members in $\mathbb{FR}_{\Delta}(A)$ for a given finite prefix-suffix-free subset Δ of $A^* \times A^*$. Thus the following questions could be considered as a future work. For a general prefix-suffix-free subset Δ of $A^* \times A^*$, what can be said about $\mathbb{FR}_{\Delta}(A)$? For example, can we obtain some results parallel to Theorems 3.5 and 3.17 in [10]?

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