

Research Article

Global Bifurcation in $2m$ -Order Generic Systems of Nonlinear Boundary Value Problems

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We consider the systems of $(-1)^m u^{(2m)} = \lambda u + \lambda v + u f(t, u, v)$, $t \in (0, 1)$, $u^{(2i)}(0) = u^{(2i)}(1) = 0$, and $0 \leq i \leq m-1$, $(-1)^m v^{(2m)} = \mu u + \mu v + v g(t, u, v)$, $t \in (0, 1)$, $v^{(2i)}(0) = v^{(2i)}(1) = 0$, $0 \leq i \leq m-1$, where $\lambda, \mu \in \mathbb{R}$ are real parameters. $f, g : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are $C^k, k \geq 3$ functions and $f(t, 0, 0) = g(t, 0, 0) = 0, t \in [0, 1]$. It will be shown that if the functions, f and g are "generic" then the solution set of the systems consists of a countable collection of 2-dimensional, C^k manifolds.

1. Introduction

Many scientific and technological problems that are modeled mathematically by systems of ODEs, for example, mathematical models of series circuits and mechanical systems involving several springs attached in series can lead to a system of differential equations. Furthermore, such systems are often encountered in chemical, ecological, biological, and engineering applications, thereby attracting constant interest of researchers in recent years, on several aspects of the problems; we focus on one aspect here, namely, the existence of solutions, from the point of view of bifurcations.

In [1], Rynne considered the global bifurcation in generic systems of coupled nonlinear Sturm-Liouville boundary value problem:

$$\begin{aligned} L_1 u &:= -(p_1 u')' + q_1 u = \mu u + u f(x, u, v), & x \in (0, 1), \\ a_{10} u(0) + b_{10} u'(0) &= 0, & a_{11} u(1) + b_{11} u'(1) = 0, \\ L_2 v &:= -(p_2 v')' + q_2 v = \nu v + v g(x, u, v), & x \in (0, 1), \\ a_{20} v(0) + b_{20} v'(0) &= 0, & a_{21} v(1) + b_{21} v'(1) = 0, \end{aligned} \tag{1.1}$$

where $p_i \in C^1[0, 1]$, $q_i \in C[0, 1]$, $i = 1, 2$, and q_i are positive on $[0, 1]$; $\mu, \nu \in R$ are parameter; $f, g : [0, 1] \times R^2 \rightarrow R$ are C^k , $k \geq 3$ functions. The interesting results are that if the functions f and g are "generic" then for all integers $m, n \geq 0$, there are smooth 2-dimensional manifolds $S_{m'}^1, S_{n'}^2$ of semitrivial solutions of the system which bifurcate from the eigenvalues $\lambda_{m'}^0, \mu_{n'}^0$ of L_1, L_2 , respectively. Furthermore, there are smooth curves $\mathcal{B}_{mn}^1 \subset S_{m'}^1, \mathcal{B}_{mn}^2 \subset S_{n'}^2$, along which secondary bifurcations take place, giving rise to smooth, 2-dimensional manifolds of nontrivial solutions. It is shown that there is a single such manifold, \mathcal{N}_{mn} , which "links" the curves $\mathcal{B}_{mn}^1, \mathcal{B}_{mn}^2$. Nodal properties of solutions on \mathcal{N}_{mn} and global properties of \mathcal{N}_{mn} are also discussed.

Inspired by [1], in this paper, we consider the $2m$ -order systems of coupled nonlinear boundary value problems:

$$\begin{aligned} (-1)^m u^{(2m)} &= \lambda u + \lambda v + u f(t, u, v), \quad t \in (0, 1), \\ u^{(2i)}(0) &= u^{(2i)}(1) = 0, \quad 0 \leq i \leq m-1, \\ (-1)^m v^{(2m)} &= \mu u + \mu v + v g(t, u, v), \quad t \in (0, 1), \\ v^{(2i)}(0) &= v^{(2i)}(1) = 0, \quad 0 \leq i \leq m-1, \end{aligned} \tag{1.2}$$

where $\lambda, \mu \in R$ are real parameters. $f, g : [0, 1] \times R^2 \rightarrow R$ are C^k , $k \geq 3$ functions and $f(t, 0, 0) = g(t, 0, 0) = 0$, $t \in [0, 1]$. It will be shown that if the functions f and g are "generic" then the solution set of the systems consists of a countable collection of 2-dimensional, C^k manifolds. The paper is organized as follows. In Section 2, by computing algebraic multiplicity of eigenvalue (λ, μ) , we get the set of bifurcation points of problem (1.2) and obtain the existence of nontrivial solution. In Section 3, we get some genericity result.

Note that problem (1.2) is different from the problem (1.1). In [1], the manifolds of nontrivial solution come from the secondary bifurcations which take place along the smooth curves in the manifolds of semitrivial solutions. In this paper, by computing algebraic multiplicity of eigenvalue (λ, μ) , we get the existence of nontrivial solution by first bifurcation.

2. Existence of Nontrivial Solution

Let $X = \{w \in C^{2m}[0, 1] : w^{(2i)}(0) = w^{(2i)}(1) = 0, 0 \leq i \leq m-1\}$, $X_0 = X \setminus \{0\}$, $Y = \{w \in C^{2m-1}[0, 1] : w^{(2i)}(0) = w^{(2i)}(1) = 0, 0 \leq i \leq m-1\}$, $Z = C[0, 1]$; these spaces are endowed with their usual supnorms. We also use the space $E = R^2 \times X^2$ with norm $\|(\lambda, \mu, u, v)\|^2 = |\lambda|^2 + |\mu|^2 + \|u\|^2 + \|v\|^2$. For $u, v \in C[0, 1]$, let $\langle u, v \rangle = \int_0^1 uv$.

Define $L : X \rightarrow Z$ by

$$Lu := (-1)^m u^{(2m)}, \quad u \in X, \tag{2.1}$$

then L is invertible and $\langle Lu, v \rangle = \langle u, Lv \rangle$.

Let $G_m(t, s)$ be the Green's function of the problem

$$\begin{aligned} (-1)^m w^{(2m)}(t) &= 0, \quad t \in (0, 1), \\ w^{(2i)}(0) &= w^{(2i)}(1) = 0, \quad 0 \leq i \leq m-1, \end{aligned} \tag{2.2}$$

then

$$G_i(t, s) = \int_0^1 G(t, \xi) G_{i-1}(\xi, s) d\xi, \quad 2 \leq i \leq m, \tag{2.3}$$

where

$$G_1(t, s) = G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.4}$$

So problem (1.2) is equivalent to

$$\begin{aligned} u(t) &= \lambda \int_0^1 G_m(t, s) u(s) ds + \lambda \int_0^1 G_m(t, s) v(s) ds + \int_0^1 G_m(t, s) u(s) f(s, u(s), v(s)) ds, \\ v(t) &= \mu \int_0^1 G_m(t, s) u(s) ds + \mu \int_0^1 G_m(t, s) v(s) ds + \int_0^1 G_m(t, s) v(s) g(s, u(s), v(s)) ds. \end{aligned} \tag{2.5}$$

In fact, problem (2.5) can be expressed in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \mu \begin{pmatrix} 0 & 0 \\ A & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} N(u, v) \\ M(u, v) \end{pmatrix}, \tag{2.6}$$

where $A : Y \rightarrow Y$ is given by

$$\begin{aligned} Ax(t) &= \int_0^1 G_m(t, s) x(s) ds, \\ N(u, v)(t) &:= \int_0^1 G_m(t, s) u(s) f(s, u(s), v(s)) ds, \\ M(u, v)(t) &:= \int_0^1 G_m(t, s) v(s) g(s, u(s), v(s)) ds, \end{aligned} \tag{2.7}$$

note that $N(u, v)/\|(u, v)\| \rightarrow 0, M(u, v)/\|(u, v)\| \rightarrow 0$ for (u, v) near $(0, 0)$ in Y^2 .

We discuss the bifurcation phenomena for problem (2.6).

From [2] we know that the set $\{(\lambda, \mu) \in R^2: (\lambda, \mu, 0, 0) \text{ is a bifurcation point for (2.6)}\}$ that is contained in the set

$$\Sigma_A \left\{ (\lambda, \mu) \in R^2 : \begin{pmatrix} u \\ v \end{pmatrix} = \left[\lambda \begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & 0 \\ A & A \end{pmatrix} \right] \begin{pmatrix} u \\ v \end{pmatrix} \text{ has a nontrivial solution } \begin{pmatrix} u \\ v \end{pmatrix} \text{ in } Y^2 \right\}. \tag{2.8}$$

Note that the equation defining Σ_A is equivalent to the system

$$\begin{aligned} Lu &= \lambda(u + v), \\ Lv &= \mu(u + v). \end{aligned} \tag{2.9}$$

If (2.9) has a nontrivial solution $\begin{pmatrix} u \\ v \end{pmatrix}$, then u and v can be shown to solve

$$L(L - (\lambda + \mu))x = 0. \tag{2.10}$$

It is known in [3] that L has a discrete sequence of simple eigenvalues

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \longrightarrow +\infty. \tag{2.11}$$

It can thus be shown that $\Sigma_A = \{(\lambda, \mu) : \lambda + \mu = \lambda_n, n = 1, 2, \dots\}$.

Theorem 2.1 of [2] guarantees that elements of Σ_A of odd algebraic multiplicity are bifurcation points. Suppose (λ, μ) satisfies $\lambda + \mu = \lambda_n$, and $Lx_n = \lambda_n x_n$. It is easy to show that (λ, μ) has geometric multiplicity 1. In fact, since u and v solve (2.10), it must be the case that $u = \alpha x_n$, $v = \beta x_n$. Then (2.9) yields

$$\begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} x_n = \begin{pmatrix} \lambda & \lambda \\ \mu & \mu \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} x_n. \tag{2.12}$$

Then (2.12) has a nontrivial solution with $\beta = (\mu/\lambda)\alpha$ if $\lambda \neq 0$; $\alpha = 0$ if $(\lambda, \mu) = (0, \lambda_n)$. So (λ, μ) has geometric multiplicity 1. We will show that the algebraic multiplicity at (λ, μ) is also 1.

Assume for the moment that $\lambda = 0$, $\mu = \lambda_n$. In this case, the equation

$$\begin{pmatrix} L - \lambda & -\lambda \\ -\mu & L - \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.13}$$

implies

$$\begin{pmatrix} L - \lambda & -\lambda \\ -\mu & L - \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \beta x_n \end{pmatrix}, \tag{2.14}$$

where $\beta \in R$. We have $u = 0$ from $Lu = 0$, so $Lv = \mu v + \beta x_n = \lambda_n v + \beta x_n$; then $\langle Lv, x_n \rangle = \langle \lambda_n v + \beta x_n, x_n \rangle$, so $\beta = 0$. It follows that the algebraic multiplicity at $(0, \lambda_n)$ is 1. By the homotopy invariance of Leray-Schauder degree, the algebraic multiplicity at (λ, μ) is odd for each (λ, μ) such that $\lambda + \mu = \lambda_n$.

We have established the following result.

Theorem 2.1. *If λ_n is an eigenvalue of L , then the points belonging to the set $\{(\lambda, \mu) : \lambda + \mu = \lambda_n\}$ are all bifurcation points.*

From Theorem 3.3 of [2] one has the following.

Theorem 2.2. *Suppose that $(\lambda_0, \mu_0) \in \Sigma_A$. Then there is a two-dimensional continuum \mathcal{C}_n emerging in $R^2 \times Y^2$ from $(\lambda_0, \mu_0, 0, 0)$. If $(\lambda_0, \mu_0) \neq (0, \lambda_n)$ or $(\lambda_n, 0)$, then at least locally, all nontrivial solutions (λ, μ, u, v) in \mathcal{C}_n are such that u and v have $n - 1$ simple zeros in $(0, 1)$.*

3. Genericity Result

We begin by stating the basic transversality theorems as given in [4] (see Theorems 1.1, 1.2 and Remark A1.1 in the appendix of [4]). Let X, Y, Z be real, separable Banach spaces. Let $V \subset X$, $W \subset Y$ be open sets, and let $F : V \times W \rightarrow Z$ be a C^k , $k \geq 1$ mapping such that for every $w_0 \in W$, $F(\cdot, w_0)$ is a Fredholm mapping of index $l < k$, that is, for all $(v^0, w^0) \in V \times W$; the linear operator $D_v F(v^0, w^0)$ is Fredholm with index $l < k$ (where $DF : X \times Y \rightarrow Z$, $D_v F : X \rightarrow Z$, $D_w F : Y \rightarrow Z$, will denote, resp., the Fréchet derivative of F). We say that 0 is a regular value of F if the operator $DF(v^0, w^0)$ is onto every point (v^0, w^0) such that $F(v^0, w^0) = 0$. Also, a subset of a topological space is said to be residual if it contains the intersection of a countable collection of open dense sets. Note that the intersection of a countable collection of residual sets is also residual.

Theorem 3.1 (see [4]). *If 0 is a regular value of F , then the set*

$$\Theta = \left\{ w^0 \in W : 0 \text{ is a regular value of } F(\cdot, w^0) \right\} \quad (3.1)$$

is a residual subset of W . For every $w^0 \in \Theta$, the set

$$\left\{ v \in V : F(v, w^0) = 0 \right\} \quad (3.2)$$

is the disjoint union of a finite, or countable, collection of connected C^k submanifolds of X of dimension l .

A property of the elements of a topological space is said to be generic if it holds for all elements in a residual subset of the space. Since we wish to discuss properties of the systems (1.2) which hold for “generic” functions f and g , we need an appropriate space of functions and a topology on this space. Let \mathcal{F} be the set of all real valued C^k functions defined on $[0, 1] \times R^2$. We define a topology on \mathcal{F} as follows. For any $f \in \mathcal{F}$ and any continuous, positive function $\epsilon : R^2 \rightarrow R_+$, we define an ϵ -neighbourhood of f by

$$U_\epsilon(f) = \left\{ h \in \mathcal{F} : \sum_{|\alpha| \leq k} |D^\alpha f(t, \eta, \zeta) - D^\alpha h(t, \eta, \zeta)| < \epsilon(\eta, \zeta), (t, \eta, \zeta) \in [0, 1] \times R^2 \right\} \quad (3.3)$$

(here D^α is the usual multi-index notation for partial derivatives); a subset $\mathcal{E} \subset \mathcal{F}$ is defined to be open if and only if for every $f \in \mathcal{E}$ there is a function ϵ such that the $U_\epsilon(f)$ of f lies in \mathcal{E} .

Let $\mathcal{F}_1 \subset \mathcal{F}$ denote the set of functions such that $f(\cdot, 0, 0) = 0$. Let $\mathcal{N}(f, g)$ denote the set of nontrivial solutions of (1.2), we get the following result.

Theorem 3.2. *There is a residual set $\mathcal{G} \subset \mathcal{F}_1^2$ such that if $(f, g) \in \mathcal{G}$, then the set \mathcal{N} consists of a countable collection of 2-dimensional, C^k manifolds.*

Proof. Let $E_0 = R^2 \times X_0^2$ and, for any $(f, g) \in \mathcal{F}_1^2$, define a C^k function $G(f, g) : E_0 \rightarrow Z^2$ by

$$G(f, g)(\lambda, \mu, u, v) = \begin{pmatrix} Lu - \lambda u - \lambda v - f(\cdot, u, v)u, \\ Lv - \mu u - \mu v - g(\cdot, u, v)v, \end{pmatrix} \quad (\lambda, \mu, u, v) \in E_0. \quad (3.4)$$

Clearly, $\mathcal{N} = G(f, g)^{-1}(0)$. For any $\rho > 1$, let

$$A_\rho = \{(\lambda, \mu, u, v) \in E_0 : \|(\lambda, \mu, u, v)\| < \rho, \|u\| > \rho^{-1}, \|v\| > \rho^{-1}\}, \quad (3.5)$$

and let \bar{A}_ρ be the closure of A_ρ . For any integer $r \geq 1$, let \mathcal{T}_r be the set of functions $(f, g) \in \mathcal{F}_1^2$ such that $G(f, g)$ is transverse on $A_{r+\delta}$ for some sufficiently small $\delta > 0$. \square

We first prove that each set \mathcal{T}_r is open in \mathcal{F}_1^2 . For any $(f, g) \in \mathcal{F}_1^2$, $\rho > 1$, the set $\mathcal{N}(f, g) \cap \bar{A}_\rho$ is compact. Thus $(f, g) \notin \mathcal{T}_r$ if and only if there exists $z \in \mathcal{N}(f, g) \cap \bar{A}_r$ such that the operator $DG(f, g)(z)$ is not surjective (here the derivative is with respect to (λ, μ, u, v)). Thus, in deciding whether $(f, g) \in \mathcal{T}_r$, the values of f and g on $|u| > r$, $|v| > r$ are irrelevant. Therefore, it suffices to show that if $(f^i, g^i) \in \mathcal{F}_1^2 \setminus \mathcal{T}_r$, $i = 1, 2, \dots$, is a sequence converging to $(f^0, g^0) \in \mathcal{F}_1^2$ with respect to the C^k topology on $[0, 1] \times [-r, r]^2$, then $(f^0, g^0) \notin \mathcal{T}_r$. Now, for each i , there exists $z^i \in \mathcal{N}(f^i, g^i) \cap \bar{A}_r$ such that $DG(f^i, g^i)(z^i)$ is not surjective. Furthermore, it can be shown that the sequence z^i converges (in E) to a point z^0 (using standard Sobolev embedding and regularity arguments; see for instance, the proof of Theorem 2.1 in [4] or the proof of Theorem 3.a.1 in [5]). Then, by continuity, $z^0 \in \mathcal{N}(f^0, g^0) \cap \bar{A}_r$, and the operator $DG(f^0, g^0)(z^0) = \lim_{i \rightarrow \infty} DG(f^i, g^i)(z^i)$ cannot be surjective since this would contradict standard perturbation results for Fredholm operators (see for instance, Theorem 13.6 in [6]). Thus $(f^0, g^0) \notin \mathcal{T}_r$, which completes the proof that \mathcal{T}_r is open in \mathcal{F}_1^2 .

We will now show that the sets \mathcal{T}_r are dense in \mathcal{F}_1^2 . Choose an arbitrary, fixed $(f, g) \in \mathcal{F}_1^2$ and $r \geq 1$. Let $\theta_r : R \rightarrow R$ be a decreasing C^∞ function such that $\theta_r(s) = 1$ if $s \leq r + 1$ and $\theta_r(s) = 0$ if $s \geq r + 2$. For any set of functions $\gamma = (\gamma_{ij})_{i,j=1}^2 \in \Gamma := C^k([0, 1])^4$ we define $f^\gamma : [0, 1] \times R^2 \rightarrow R$ by

$$f^\gamma(t, \eta, \zeta) = f(t, \eta, \zeta) + \theta_r\left(\left(\eta^2 + \zeta^2\right)^{1/2}\right)(\gamma_{11}(x)\eta + \gamma_{12}(x)\zeta), \quad (3.6)$$

for $(t, \eta, \zeta) \in [0, 1] \times R^2$; we define $g^\gamma : [0, 1] \times R^2 \rightarrow R$ similarly, using γ_{21}, γ_{22} . Clearly, for any given positive functions $\varepsilon_1, \varepsilon_2$ (see the definition of the topology on \mathcal{F}) we have $f^\gamma \in \mathcal{U}_{\varepsilon_1}(f)$, $g^\gamma \in \mathcal{U}_{\varepsilon_2}(g)$, for all sufficiently small $\gamma \in \Gamma$. We now define a function $H : A_{r+1} \times \Gamma \rightarrow Z \times Z$ by

$$H(\lambda, \mu, u, v, \gamma) = G(f^\gamma, g^\gamma)(\lambda, \mu, u, v), \quad (\lambda, \mu, u, v, \gamma) \in A_{r+1} \times \Gamma. \quad (3.7)$$

Since Γ is a Banach space (with norm $\|\gamma\|^2 = \sum_{i,j=1}^2 \|\gamma_{i,j}\|_k^2$), we can differentiate H with respect to γ and apply transversality results. Clearly, H is C^k .

Lemma 3.3. *The derivative $DH(z^0)$ is surjective at any point $z^0 \in H^{-1}(0)$.*

Proof. We must show that for any $\zeta = (\zeta_1, \zeta_2) \in Z^2$, the following equation can be solved for $(\bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v}, \bar{\gamma}) \in E \times \Gamma$:

$$H_{(u,v)}^0(\bar{u}, \bar{v}) + H_{(\lambda,\mu)}^0(\bar{\lambda}, \bar{\mu}) + H_{\gamma}^0 \bar{\gamma} = \zeta, \tag{3.8}$$

where $H_{(u,v)}^0, H_{(\lambda,\mu)}^0, H_{\gamma}^0$ denote the derivatives at $z^0 = (\lambda^0, \mu^0, u^0, v^0, \gamma^0)$ with respect to (u, v) , (λ, μ) and γ , respectively; these operators have the form

$$\begin{aligned} H_{(u,v)}^0(\bar{u}, \bar{v}) &= \begin{pmatrix} L\bar{u} \\ L\bar{v} \end{pmatrix} - \begin{pmatrix} (\lambda^0 + f^0 + u^0 f_u^0)\bar{u} + (\lambda^0 + u^0 f_v^0)\bar{v} \\ (\mu^0 + g^0 + v^0 g_v^0)\bar{v} + (\mu^0 + v^0 g_u^0)\bar{u} \end{pmatrix}, \\ H_{(\lambda,\mu)}^0(\bar{\lambda}, \bar{\mu}) &= \begin{pmatrix} -\bar{\lambda}(u^0 + v^0) \\ -\bar{\mu}(u^0 + v^0) \end{pmatrix}, \quad H_{\gamma}^0 \bar{\gamma} = \begin{pmatrix} \bar{\gamma}_{11} u^0 u^0 + \bar{\gamma}_{12} u^0 v^0 \\ \bar{\gamma}_{21} u^0 v^0 + \bar{\gamma}_{22} v^0 v^0 \end{pmatrix}, \end{aligned} \tag{3.9}$$

(where f^0 denotes $f^{i^0}(\cdot, u^0, v^0)$, etc.). From [4, Page 301] we know that the operator $H_{(u,v)}^0$ is Fredholm with index 0 and, if $\xi = (\xi_1, \xi_2) \in R(H_{(u,v)}^0)^\perp \subset Z^2$, then ξ is a solution of a homogeneous coupled pair of linear ordinary differential equations. Let $d^0 = \dim R(H_{(u,v)}^0)^\perp$. If $d^0 = 0$ there is nothing further to prove, so suppose that $d^0 \geq 1$ and let $\{\xi_i \in Z^2 : i = 1, \dots, d^0\}$ be a basis for $R(H_{(u,v)}^0)^\perp$. Since the functions u^0, v^0 and $\xi_i, i = 1, \dots, d^0$ are all nonzero and can be regarded as solutions of homogeneous linear ordinary differential equations (or systems), it follows from the uniqueness of the solution of the initial value problem for such equations that these functions cannot be identically zero on any open set. Thus there must exist an open interval $I^0 \subset (0, 1)$ and a number $\delta > 0$, such that for $x \in I^0$ we have $|u^0(x)| > \delta, |v^0(x)| > \delta$, and the functions $\xi_i, i = 1, \dots, d^0$ are linearly independent on I^0 .

Now, since u^0 and v^0 are bounded away from 0 on I^0 , for any $\psi = (\psi_1, \psi_2) \in C_0^\infty(I^0)^2$, the equation $H_{\gamma}^0 \bar{\gamma} = \psi$ has a solution $\bar{\gamma} \in C_0^k(I^0)^4$. Thus, we can choose $\gamma^j \in C_0^k(I^0)^4, j = 1, \dots, d^0$ such that $\langle H_{\gamma}^0 \gamma^j, \xi^i \rangle = \delta_{ij}, i, j = 1, \dots, d^0$ (δ_{ij} is the Kronecker delta); thus the set $\{H_{\gamma}^0 \gamma^j : j = 1, \dots, d^0\}$ spans a complement of $R(H_{(u,v)}^0)^\perp$ in Z^2 , and so the operator $(\bar{u}, \bar{v}, \bar{\gamma}) \rightarrow H_{(u,v)}^0(\bar{u}, \bar{v}) + H_{\gamma}^0 \bar{\gamma} \in Z^2$ is surjective. This proves Lemma 3.3.

It now follows from Lemma 3.3 and Theorem 3.1 (it can readily be verified that H satisfies the necessary Fredholm conditions) that there exists a residual set $\Theta \subset \Gamma$ such that if $\gamma \in \Theta$ then the mapping $H(\cdot, \gamma) : A_{r+1} \rightarrow Z^2$ is transverse, that is, $(f^\gamma, g^\gamma) \in \mathcal{T}_r$. Since $(f, g) \in \mathcal{F}_1^2$ was arbitrary, it follows from this that the set \mathcal{T}_r is dense in \mathcal{F}_1^2 . Now let $\mathcal{G} = \bigcap_{r=1}^\infty \mathcal{T}_r$. By construction, \mathcal{G} is a residual subset of \mathcal{F}_1^2 and, for any $(f, g) \in \mathcal{G}$, the mapping $G(f, g) : E_0 \rightarrow Z^2$ is transverse. It follows from $H_{(u,v)}^0$, and the results in [7], that $DG(f, g)(z)$ is Fredholm with index 2. So from Theorem 3.1 the zero set of this mapping is a 2-dimensional, C^k manifold. Thus we have proved Theorem 3.2. \square

Remark 3.4. For more information about generic result or nodal solutions, see [8–12].

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