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Research Article

Generalized α-ψ Contractive Type Mappings and Related Fixed Point Theorems with Applications

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We establish fixed point theorems for a new class of contractive mappings. As consequences of our main results, we obtain fixed point theorems on metric spaces endowed with a partial order and fixed point theorems for cyclic contractive mappings. Various examples are presented to illustrate our obtained results.

1. Introduction and Preliminaries

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;

(Ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$.

Very recently, Samet et al. [1] introduced the following concepts.

Definition 1.1. Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y)), \quad \forall x,y \in X. \tag{1.1}$$

Clearly, any contractive mapping, that is, a mapping satisfying Banach contraction, is an α-ψ contractive mapping with $\alpha(x,y) = 1$ for all $x,y \in X$ and $\psi(t) = kt$, $k \in (0,1)$.

Definition 1.2. Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that T is α-admissible if for all $x, y \in X$, and we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{1.2}$$

Various examples of such mappings are presented in [1].

The main results in [1] are the following fixed point theorems.

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ contractive mapping. Suppose that*

(i) *T is α admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Theorem 1.4. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ contractive mapping. Suppose that*

(i) *T is α admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then there exists $u \in X$ such that $Tu = u$.

Theorem 1.5. *Adding to the hypotheses of Theorem 1.3 (resp., Theorem 1.4) the condition, for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, and one obtains uniqueness of the fixed point.*

In the present work, we introduce the concept of generalized α-ψ contractive type mappings, and we study the existence and uniqueness of fixed points for such mappings. Presented theorems in this paper extend and generalize the above results derived by Samet et al. in [1]. Moreover, from our fixed point theorems, we will deduce various fixed point results on metric spaces endowed with a partial order and fixed point results for cyclic contractive mappings.

2. Main Results

We introduce the concept of generalized α-ψ contractive type mappings as follows.

Definition 2.1. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalized α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, and we have

$$\alpha(x, y) d(Tx, Ty) \leq \psi(M(x, y)), \tag{2.1}$$

where $M(x, y) = \max\{d(x, y), (d(x, Tx) + d(y, Ty))/2, (d(x, Ty) + d(y, Tx))/2\}$.

Remark 2.2. Clearly, since ψ is nondecreasing, every α-ψ contractive mapping is a generalized α-ψ contractive mapping.

Our first result is the following.

Theorem 2.3. *Let (X,d) be a complete metric space. Suppose that $T : X \to X$ is a generalized α-ψ contractive mapping and satisfies the following conditions:*

(i) *T is α admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ (such a point exists from condition (ii)). Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then $u = x_{n_0}$ is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is α admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. \tag{2.2}$$

Inductively, we have

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{2.3}$$

From (2.1) and (2.3), it follows that for all $n \geq 1$, we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1}) d(Tx_n, Tx_{n-1}) \leq \psi(M(x_n, x_{n-1})). \tag{2.4}$$

On the other hand, we have

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{ d(x_n, x_{n-1}), \frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{2}, \frac{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}{2} \right\} \\ &= \max\left\{ d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2}, \frac{d(x_{n-1}, x_{n+1})}{2} \right\} \\ &\leq \max\left\{ d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2} \right\} \\ &\leq \max\left\{ d(x_n, x_{n-1}), d(x_n, x_{n+1}) \right\}. \end{aligned} \tag{2.5}$$

From (2.4) and taking in consideration that ψ is a nondecreasing function, we get that

$$d(x_{n+1}, x_n) \leq \psi(\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}), \tag{2.6}$$

for all $n \geq 1$. If for some $n \geq 1$, we have $d(x_n, x_{n-1}) \leq d(x_n, x_{n+1})$, from (2.6), we obtain that

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}), \tag{2.7}$$

a contradiction. Thus, for all $n \geq 1$, we have

$$\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} = d(x_n, x_{n-1}). \tag{2.8}$$

Using (2.6) and (2.8), we get that

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})), \tag{2.9}$$

for all $n \geq 1$. By induction, we get

$$d(x_{n+1}, x_n) \leq \varphi^n(d(x_1, x_0)), \quad \forall n \geq 1. \tag{2.10}$$

From (2.10) and using the triangular inequality, for all $k \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} \varphi^n(d(x_1, x_0)) \\ &\leq \sum_{p=n}^{+\infty} \varphi^n(d(x_1, x_0)) \longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.11}$$

This implies that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{2.12}$$

Since T is continuous, we obtain from (2.12) that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0. \tag{2.13}$$

From (2.12), (2.13) and the uniqueness of the limit, we get immediately that u is a fixed point of T, that is, $Tu = u$. □

The next theorem does not require the continuity of T.

Theorem 2.4. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalized α-ψ contractive mapping and the following conditions hold:*

(i) *T is α admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 2.3, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (2.3) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (2.1), for all k, we get that

$$d(x_{n(k)+1}, Tu) = d(Tx_{n(k)}, Tu) \leq \alpha(x_{n(k)}, u) d(Tx_{n(k)}, Tu) \leq \psi(M(x_{n(k)}, u)). \tag{2.14}$$

On the other hand, we have

$$M(x_{n(k)}, u) = \max\left\{ d(x_{n(k)}, u), \frac{d(x_{n(k)}, x_{n(k)+1}) + d(u, Tu)}{2}, \frac{d(x_{n(k)}, Tu) + d(u, x_{n(k)+1})}{2} \right\}. \tag{2.15}$$

Letting $k \to \infty$ in the above equality, we get that

$$\lim_{k \to \infty} M(x_{n(k)}, u) = \frac{d(u, Tu)}{2}. \tag{2.16}$$

Suppose that $d(u, Tu) > 0$. From (2.16), for k large enough, we have $M(x_{n(k)}, u) > 0$, which implies that $\psi(M(x_{n(k)}, u)) < M(x_{n(k)}, u)$. Thus, from (2.14), we have

$$d(x_{n(k)+1}, Tu) < M(x_{n(k)}, u). \tag{2.17}$$

Letting $k \to \infty$ in the above inequality, using (2.16), we obtain that

$$d(u, Tu) \leq \frac{d(u, Tu)}{2}, \tag{2.18}$$

which is a contradiction. Thus we have $d(u, Tu) = 0$, that is, $u = Tu$. $\square$

With the following example, we will show that hypotheses in Theorems 2.3 and 2.4 do not guarantee uniqueness of the fixed point.

Example 2.5. Let $X = \{(1, 0), (0, 1)\} \subset \mathbb{R}^2$ be endowed with the Euclidean distance $d((x, y), (u, v)) = |x - u| + |y - v|$ for all $(x, y), (u, v) \in X$. Obviously, (X, d) is a complete metric space. The mapping $T(x, y) = (x, y)$ is trivially continuous and satisfies for any $\psi \in \Psi$

$$\alpha((x, y), (u, v)) d(T(x, y), T(u, v)) \leq \psi(M((x, y), (u, v))), \tag{2.19}$$

for all $(x,y),(u,v) \in X$, where

$$\alpha((x,y),(u,v)) = \begin{cases} 1 & \text{if } (x,y) = (u,v), \\ 0 & \text{if } (x,y) \neq (u,v). \end{cases} \tag{2.20}$$

Thus T is a generalized α-ψ contractive mapping. On the other hand, for all $(x,y),(u,v) \in X$, we have

$$\alpha((x,y),(u,v)) \geq 1 \longrightarrow (x,y) = (u,v) \longrightarrow T(x,y) = T(u,v) \longrightarrow \alpha(T(x,y),T(u,v)) \geq 1. \tag{2.21}$$

Thus T is α admissible. Moreover, for all $(x,y) \in X$, we have $\alpha((x,y),T(x,y)) \geq 1$. Then the assumptions of Theorem 2.3 are satisfied. Note that the assumptions of Theorem 2.4 are also satisfied; indeed if $\{(x_n,y_n)\}$ is a sequence in X that converges to some point $(x,y) \in X$ with $\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \geq 1$ for all n, then, from the definition of α, we have $(x_n,y_n) = (x,y)$ for all n, which implies that $\alpha((x_n,y_n),(x,y)) = 1$ for all n. However, in this case, T has two fixed points in X.

For the uniqueness of a fixed point of a generalized α-ψ contractive mapping, we will consider the following hypothesis.

(H) For all $x,y \in \operatorname{Fix}(T)$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

$\operatorname{Fix}(T)T$

Theorem 2.6. *Adding condition* (*H*) *to the hypotheses of Theorem 2.3 (resp., Theorem 2.4), one has obtains that u is the unique fixed point of T.*

Proof. Suppose that v is another fixed point of T. From (H), there exists $z \in X$ such that

$$\alpha(u,z) \geq 1, \qquad \alpha(v,z) \geq 1. \tag{2.22}$$

Since T is α admissible, from (2.22), we have

$$\alpha(u,T^n z) \geq 1, \quad \alpha(v,T^n z) \geq 1, \quad \forall n. \tag{2.23}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \geq 0$ and $z_0 = z$. From (2.23), for all n, we have

$$d(u,z_{n+1}) = d(Tu,Tz_n) \leq \alpha(u,z_n)d(Tu,Tz_n) \leq \psi(M(u,z_n)). \tag{2.24}$$

On the other hand, we have

$$\begin{aligned} M(u, z_n) &= \max\left\{ d(u, z_n), \frac{d(z_n, z_{n+1})}{2}, \frac{d(u, z_{n+1}) + d(z_n, u)}{2} \right\} \\ &\leq \max\left\{ d(u, z_n), \frac{d(z_n, u) + d(u, z_{n+1})}{2} \right\} \\ &\leq \max\{d(u, z_n), d(u, z_{n+1})\}. \end{aligned} \tag{2.25}$$

Using the above inequality, (2.24) and the monotone property of ψ, we get that

$$d(u, z_{n+1}) \leq \psi(\max\{d(u, z_n), d(u, z_{n+1})\}), \tag{2.26}$$

for all n. Without restriction to the generality, we can suppose that $d(u, z_n) > 0$ for all n. If $\max\{d(u, z_n), d(u, z_{n+1})\} = d(u, z_{n+1})$, we get from (2.26) that

$$d(u, z_{n+1}) \leq \psi(d(u, z_{n+1})) < d(u, z_{n+1}), \tag{2.27}$$

which is a contradiction. Thus we have $\max\{d(u, z_n), d(u, z_{n+1})\} = d(u, z_n)$, and

$$d(u, z_{n+1}) \leq \psi(d(u, z_n)), \tag{2.28}$$

for all n. This implies that

$$d(u, z_n) \leq \psi^n(d(u, z_0)), \quad \forall n \geq 1. \tag{2.29}$$

Letting $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n \to \infty} d(z_n, u) = 0. \tag{2.30}$$

Similarly, one can show that

$$\lim_{n \to \infty} d(z_n, v) = 0. \tag{2.31}$$

From (2.30) and (2.31), it follows that $u = v$. Thus we proved that u is the unique fixed point of T. □

Example 2.7. Let $X = [0, 1]$ be endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in X$. Obviously, (X, d) is a complete metric space. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \dfrac{1}{4} & \text{if } x \in [0, 1), \\ 0 & \text{if } x = 1. \end{cases} \tag{2.32}$$

In this case, T is not continuous. Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \left(\left[0,\frac{1}{4}\right] \times \left[\frac{1}{4},1\right]\right) \cup \left(\left[\frac{1}{4},1\right] \times \left[0,\frac{1}{4}\right]\right), \\ 0 & \text{otherwise.} \end{cases} \tag{2.33}$$

We will prove that

(A) $T : X \to X$ is a generalized α-ψ contractive mapping, where $\psi(t) = t/2$ for all $t \geq 0$;

(B) T is α-admissible;

(C) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(D) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k;

(E) condition (H) is satisfied.

Proof of (A). To show (A), we have to prove that (2.1) is satisfied for every $x, y \in X$. If $x \in [0,1/4]$ and $y = 1$, we have

$$\alpha(x,y)d(Tx,Ty) = d(Tx,Ty) = \left|\frac{1}{4} - 0\right| = \frac{1}{4}d(y,Ty) \leq \psi(M(x,y)). \tag{2.34}$$

Then (2.1) holds. If $x = 1$ and $y \in [0,1/4]$, we have

$$\alpha(x,y)d(Tx,Ty) = d(Tx,Ty) = \left|0 - \frac{1}{4}\right| = \frac{1}{4}d(x,Tx) \leq \psi(M(x,y)). \tag{2.35}$$

Then (2.1) holds also in this case. The other cases are trivial. Thus (2.1) is satisfied for every $x, y \in X$. □

Proof of (B). Let $(x,y) \in X \times X$ such that $\alpha(x,y) \geq 1$. From the definition of α, we have two cases.

Case 1 (if $(x,y) \in [0,1/4] \times [1/4,1]$). In this case, we have $(Tx,Ty) \in [1/4,1] \times [0,1/4]$, which implies that $\alpha(Tx,Ty) = 1$.

Case 2 (if $(x,y) \in [1/4,1] \times [0,1/4]$). In this case, we have $(Tx,Ty) \in [0,1/4] \times [1/4,1]$, which implies that $\alpha(Tx,Ty) = 1$.

So, in all cases, we have $\alpha(Tx,Ty) \geq 1$. Thus T is α admissible. □

Proof of (C). Taking $x_0 = 0$, we have $\alpha(x_0, Tx_0) = \alpha(0, 1/4) = 1$. □

Proof of (D). Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to +\infty$ for some $x \in X$. From the definition of α, for all n, we have

$$(x_n, x_{n+1}) \in \left(\left[0,\frac{1}{4}\right] \times \left[\frac{1}{4},1\right]\right) \cup \left(\left[\frac{1}{4},1\right] \times \left[0,\frac{1}{4}\right]\right). \tag{2.36}$$

Since $([0,1/4]\times[1/4,1])\cup([1/4,1]\times[0,1/4])$ is a closed set with respect to the Euclidean metric, we get that

$$(x,x)\in\left(\left[0,\frac{1}{4}\right]\times\left[\frac{1}{4},1\right]\right)\cup\left(\left[\frac{1}{4},1\right]\times\left[0,\frac{1}{4}\right]\right),\tag{2.37}$$

which implies that $x=1/4$. Thus we have $\alpha(x_n,x)\geq 1$ for all n. □

Proof of (E). Let $(x,y)\in X\times X$. It is easy to show that, for $z=1/4$, we have $\alpha(x,z)=\alpha(y,z)=1$. So, condition (H) is satisfied. □

Conclusion. Now, all the hypotheses of Theorem 2.6 are satisfied; thus T has a unique fixed point $u\in X$. In this case, we have $u=1/4$.

3. Consequences

Now, we will show that many existing results in the literature can be deduced easily from our Theorem 2.6.

3.1. Standard Fixed Point Theorems

Taking in Theorem 2.6, $\alpha(x,y)=1$ for all $x,y\in X$, we obtain immediately the following fixed point theorem.

Corollary 3.1. *Let (X,d) be a complete metric space and $T:X\to X$ be a given mapping. Suppose that there exists a function $\psi\in\Psi$ such that*

$$d(Tx,Ty)\leq\psi(M(x,y)),\tag{3.1}$$

for all $x,y\in X$. Then T has a unique fixed point.

The following fixed point theorems follow immediately from Corollary 3.1.

Corollary 3.2 (see Berinde [2]). *Let (X,d) be a complete metric space and $T:X\to X$ be a given mapping. Suppose that there exists a function $\psi\in\Psi$ such that*

$$d(Tx,Ty)\leq\psi(d(x,y)),\tag{3.2}$$

for all $x,y\in X$. Then T has a unique fixed point.

Corollary 3.3 (see Ćirić [3]). *Let (X,d) be a complete metric space and $T:X\to X$ be a given mapping. Suppose that there exists a constant $\lambda\in(0,1)$ such that*

$$d(Tx,Ty)\leq\lambda\max\left\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\right\},\tag{3.3}$$

for all $x,y\in X$. Then T has a unique fixed point.

Corollary 3.4 (see Hardy and Rogers [4]). *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist constants $A, B, C \geq 0$ with $(A + 2B + 2C) \in (0,1)$ such that*

$$d(Tx,Ty) \leq Ad(x,y) + B\left[d(x,Tx) + d(y,Ty)\right] + C\left[d(x,Ty) + d(y,Tx)\right], \tag{3.4}$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 3.5 (see Banach Contraction Principle [5]). *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda d(x,y), \tag{3.5}$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 3.6 (see Kannan [6]). *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0,1/2)$ such that*

$$d(Tx,Ty) \leq \lambda\left[d(x,Tx) + d(y,Ty)\right], \tag{3.6}$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 3.7 (see Chatterjea [7]). *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exists a constant $\lambda \in (0,1/2)$ such that*

$$d(Tx,Ty) \leq \lambda\left[d(x,Ty) + d(y,Tx)\right], \tag{3.7}$$

for all $x, y \in X$. Then T has a unique fixed point.

3.2. Fixed Point Theorems on Metric Spaces Endowed with a Partial Order

Recently there have been so many exciting developments in the field of existence of fixed point on metric spaces endowed with partial orders. This trend was started by Turinici [8] in 1986. Ran and Reurings in [9] extended the Banach contraction principle in partially ordered sets with some applications to matrix equations. The obtained result in [9] was further extended and refined by many authors (see, e.g., [10–15] and the references cited therein). In this section, from our Theorem 2.6, we will deduce very easily various fixed point results on a metric space endowed with a partial order. At first, we need to recall some concepts.

Definition 3.8. Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$x, y \in X, \quad x \preceq y \Longrightarrow Tx \preceq Ty. \tag{3.8}$$

Definition 3.9. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 3.10. Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preceq, d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

We have the following result.

Corollary 3.11. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(M(x, y)), \tag{3.9}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, one has uniqueness of the fixed point.

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \quad \text{or} \quad x \succeq y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Clearly, T is a generalized α-ψ contractive mapping, that is,

$$\alpha(x, y) d(Tx, Ty) \leq \psi(M(x, y)), \tag{3.11}$$

for all $x, y \in X$. From condition (i), we have $\alpha(x_0, Tx_0) \geq 1$. Moreover, for all $x, y \in X$, from the monotone property of T, we have

$$\alpha(x, y) \geq 1 \Longrightarrow x \succeq y \quad \text{or} \quad x \preceq y \Longrightarrow Tx \succeq Ty \quad \text{or} \quad Tx \preceq Ty \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{3.12}$$

Thus T is α admissible. Now, if T is continuous, the existence of a fixed point follows from Theorem 2.3. Suppose now that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. This implies from the definition of α that $\alpha(x_{n(k)}, x) \geq 1$ for all k. In this case, the existence of a fixed point follows from Theorem 2.4. To show the uniqueness, and let $x, y \in X$. By hypothesis, there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, which implies from the definition of α that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Thus we deduce the uniqueness of the fixed point by Theorem 2.6. □

The following results are immediate consequences of Corollary 3.11.

Corollary 3.12. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\varphi \in \Psi$ such that*

$$d(Tx, Ty) \leq \varphi(d(x, y)), \tag{3.13}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, one has uniqueness of the fixed point.

Corollary 3.13. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda \max\left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\}, \tag{3.14}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, one has uniqueness of the fixed point.

Corollary 3.14. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist constants $A, B, C \geq 0$ with $(A + 2B + 2C) \in (0, 1)$ such that*

$$d(Tx, Ty) \leq Ad(x, y) + B\left[d(x, Tx) + d(y, Ty)\right] + C\left[d(x, Ty) + d(y, Tx)\right], \tag{3.15}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, one has uniqueness of the fixed point.

Corollary 3.15 (see Ran and Reurings [9], Nieto and López [16]). *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y), \tag{3.16}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, one has uniqueness of the fixed point.

Corollary 3.16. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda\left[d(x, Tx) + d(y, Ty)\right], \tag{3.17}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, one has uniqueness of the fixed point.

Corollary 3.17. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a constant $\lambda \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda\left[d(x, Ty) + d(y, Tx)\right], \tag{3.18}$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) *T is continuous or $(X, \preceq, d)$ is regular.*

Then T has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, one has uniqueness of the fixed point.

3.3. Fixed Point Theorems for Cyclic Contractive Mappings

One of the remarkable generalizations of the Banach Contraction Mapping Principle was reported by Kirk et al. [17] via cyclic contraction. Following the paper [17], many fixed point theorems for cyclic contractive mappings have appeared (see, e.g., [18–23]). In this section, we will show that, from our Theorem 2.6, we can deduce some fixed point theorems for cyclic contractive mappings.

We have the following result.

Corollary 3.18. *Let $\{A_i\}_{i=1}^{2}$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) *$T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;*

(II) *there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi(M(x,y)), \quad \forall (x,y) \in A_1 \times A_2. \tag{3.19}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof. Since A_1 and A_2 are closed subsets of the complete metric space (X,d), then (Y,d) is complete. Define the mapping $\alpha : Y \times Y \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

From (II) and the definition of α, we can write

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)), \tag{3.21}$$

for all $x, y \in Y$. Thus T is a generalized α-ψ contractive mapping.

Let $(x,y) \in Y \times Y$ such that $\alpha(x,y) \geq 1$. If $(x,y) \in A_1 \times A_2$, from (I), $(Tx,Ty) \in A_2 \times A_1$, which implies that $\alpha(Tx,Ty) \geq 1$. If $(x,y) \in A_2 \times A_1$, from (I), $(Tx,Ty) \in A_1 \times A_2$, which implies that $\alpha(Tx,Ty) \geq 1$. Thus in all cases, we have $\alpha(Tx,Ty) \geq 1$. This implies that T is α-admissible.

Also, from (I), for any $a \in A_1$, we have $(a,Ta) \in A_1 \times A_2$, which implies that $\alpha(a,Ta) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. This implies from the definition of α that

$$(x_n,x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \quad \forall n. \tag{3.22}$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we get that

$$(x,x) \in (A_1 \times A_2) \cup (A_2 \times A_1), \tag{3.23}$$

which implies that $x \in A_1 \cap A_2$. Thus we get immediately from the definition of α that $\alpha(x_n,x) \geq 1$ for all n.

Finally, let $x, y \in \mathrm{Fix}(T)$. From (I), this implies that $x, y \in A_1 \cap A_2$. So, for any $z \in Y$, we have $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. Thus condition (H) is satisfied.

Now, all the hypotheses of Theorem 2.6 are satisfied, and we deduce that T has a unique fixed point that belongs to $A_1 \cap A_2$ (from (I)). □

The following results are immediate consequences of Corollary 3.18.

Corollary 3.19 (see Pacurar and Rus [21]). *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$;

(II) *there exists a function* $\varphi \in \Psi$ *such that*

$$d(Tx,Ty) \le \varphi(d(x,y)), \quad \forall (x,y) \in A_1 \times A_2. \tag{3.24}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 3.20. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$;

(II) *there exists a constant* $\lambda \in (0,1)$ *such that*

$$\begin{gathered} d(Tx,Ty) \le \lambda \max\left\{ d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2} \right\}, \\ \forall (x,y) \in A_1 \times A_2. \end{gathered} \tag{3.25}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 3.21. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$;

(II) *there exist constants* $A, B, C \ge 0$ *with* $(A + 2B + 2C) \in (0,1)$ *such that*

$$\begin{gathered} d(Tx,Ty) \le Ad(x,y) + B[d(x,Tx)+d(y,Ty)] + C[d(x,Ty)+d(y,Tx)], \\ \forall (x,y) \in A_1 \times A_2. \end{gathered} \tag{3.26}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 3.22 (see Kirk et al. [17]). *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X,d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$;

(II) *there exists a constant* $\lambda \in (0,1)$ *such that*

$$d(Tx,Ty) \le \lambda d(x,y), \quad \forall (x,y) \in A_1 \times A_2. \tag{3.27}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 3.23. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X, d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) *there exists a constant $\lambda \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)], \quad \forall(x, y) \in A_1 \times A_2. \tag{3.28}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Corollary 3.24. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric space (X, d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) *there exists a constant $\lambda \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda[d(x, Ty) + d(y, Tx)], \quad \forall(x, y) \in A_1 \times A_2. \tag{3.29}$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

References

[1] B. Samet, C. Vetro, and P. Vetro, “Fixed point theorems for α-ψ contractive type mappings,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 75, no. 4, pp. 2154–2165, 2012.

[2] V. Berinde, *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare, Romania, 2002.

[3] L. B. Ćirić, “Fixed points for generalized multi-valued contractions,” *Matematički Vesnik*, vol. 9, no. 24, pp. 265–272, 1972.

[4] G. E. Hardy and T. D. Rogers, “A generalization of a fixed point theorem of Reich,” *Canadian Mathematical Bulletin*, vol. 16, pp. 201–206, 1973.

[5] S. Banach, “Sur les operations dans les ensembles abstraits et leur application aux equations itegrales,” *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[6] R. Kannan, “Some results on fixed points,” *Bulletin of the Calcutta Mathematical Society*, vol. 10, pp. 71–76, 1968.

[7] S. K. Chatterjea, “Fixed-point theorems,” *Comptes Rendus de l’Académie Bulgare des Sciences*, vol. 25, pp. 727–730, 1972.

[8] M. Turinici, “Abstract comparison principles and multivariable Gronwall-Bellman inequalities,” *Journal of Mathematical Analysis and Applications*, vol. 117, no. 1, pp. 100–127, 1986.

[9] A. C. M. Ran and M. C. B. Reurings, “A fixed point theorem in partially ordered sets and some applications to matrix equations,” *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[10] R. P. Agarwal, M. A. El-Gebeily, and D. O’Regan, “Generalized contractions in partially ordered metric spaces,” *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.

[11] I. Altun and H. Simsek, “Some fixed point theorems on ordered metric spaces and application,” *Fixed Point Theory and Applications*, vol. 2010, Article ID 621469, 17 pages, 2010.

[12] L. Ćirić, N. Cakić, M. Rajović, and J. S. Ume, “Monotone generalized nonlinear contractions in partially ordered metric spaces,” *Fixed Point Theory and Applications*, vol. 2008, Article ID 131294, 11 pages, 2008.

[13] J. Harjani and K. Sadarangani, “Fixed point theorems for weakly contractive mappings in partially ordered sets,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 71, no. 7-8, pp. 3403–3410, 2009.

[14] A. Petruşel and I. A. Rus, “Fixed point theorems in ordered L-spaces,” *Proceedings of the American Mathematical Society*, vol. 134, no. 2, pp. 411–418, 2006.
[15] B. Samet, “Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 72, no. 12, pp. 4508–4517, 2010.
[16] J. J. Nieto and R. Rodríguez-López, “Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,” *Order*, vol. 22, no. 3, pp. 223–239, 2005.
[17] W. A. Kirk, P. S. Srinivasan, and P. Veeramani, “Fixed points for mappings satisfying cyclical contractive conditions,” *Fixed Point Theory*, vol. 4, no. 1, pp. 79–89, 2003.
[18] R. P. Agarwal, M. A. Alghamdi, and N. Shahzad, “Fixed point theory for cyclic generalized contractions in partial metric spaces,” *Fixed Point Theory and Applications*, vol. 2012, article 40, 2012.
[19] E. Karapınar, “Fixed point theory for cyclic weak ϕ-contraction,” *Applied Mathematics Letters*, vol. 24, no. 6, pp. 822–825, 2011.
[20] E. Karapınar and K. Sadaranagni, “Fixed point theory for cyclic (ϕ-ψ)-contractions,” *Fixed Point Theory and Applications*, vol. 2011, article 69, 2011.
[21] M. Păcurar and I. A. Rus, “Fixed point theory for cyclic φ-contractions,” *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 72, no. 3-4, pp. 1181–1187, 2010.
[22] M. A. Petric, “Some results concerning cyclical contractive mappings,” *General Mathematics*, vol. 18, no. 4, pp. 213–226, 2010.
[23] I. A. Rus, “Cyclic representations and fixed points,” *Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity*, vol. 3, pp. 171–178, 2005.