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Research Article

Fixed Point Theorems via Auxiliary Functions

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We prove new fixed point theorems in the framework of partially ordered metric spaces. The main result is an extension and a generalization of many existing results in the literature. An example is also considered to illustrate the main result.

1. Introduction and Preliminaries

Fixed point theory is one of fundamental tools of nonlinear functional analysis. Since the fixed point theory has a wide application area in almost all quantitative sciences, many authors have been working on this field. One of the impressive initial results in this direction was given by Banach [1], known as Banach Contraction Mapping Principle. It states that each contraction in a complete metric space has a unique fixed point. Since then, a number of papers have been reported on various generalization of celebrated Banach Contraction Mapping Principle.

In 2008, Dutta and Choudhury proved the following theorem.

Theorem 1.1 (see [2]). *Let (X, d) be a complete metric space and $f : X \to X$ be such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad \forall x, y \in X, \tag{1.1}$$

where $\psi, \phi : [0, +\infty[\to [0, +\infty[$ are continuous, nondecreasing and $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then, f has a unique fixed point $x^* \in X$.

Remark 1.2. Notice that Theorem 1.1 remains true if the hypothesis on ϕ is replaced by ϕ is lower semicontinuous and $\phi(t) = 0$ if and only if $t = 0$ (see, e.g., [3, 4]).

Eslamian and Abkar [5] stated the following theorem as a generalization of Theorem 1.1.

Theorem 1.3. *Let (X, d) be a complete metric space and $f : X \to X$ be such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \quad \forall x, y \in X, \tag{1.2}$$

where $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, continuous,

$$\begin{gathered} \psi(t) = 0 \quad \text{iff } t = 0, \qquad \alpha(0) = \beta(0) = 0, \\ \psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0. \end{gathered} \tag{1.3}$$

Then, f has a unique fixed point $x^* \in X$.

Aydi et al. [6] proved that Theorem 1.3 is a consequence of Theorem 1.1

Harjani and Sadarangani [7] extended Theorem 1.1 in the framework of partially ordered metric spaces in the following way.

Theorem 1.4. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $f : X \to X$ be a continuous nondecreasing mapping such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad \forall x \leq y, \tag{1.4}$$

where $\psi, \phi : [0, +\infty[\to [0, +\infty[$ are continuous, nondecreasing and $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point $x^* \in X$.

Choudhury and Kundu [8] proved the following theorem as a generalization of Theorems 1.3 and 1.4.

Theorem 1.5. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreas-ing mapping such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \quad \forall x \leq y, \tag{1.5}$$

where $\psi, \alpha, \beta : [0, +\infty[\to [0, +\infty[$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, continuous,

$$\begin{gathered} \psi(t) = 0 \quad \text{iff } t = 0, \qquad \alpha(0) = \beta(0) = 0, \\ \psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0. \end{gathered} \tag{1.6}$$

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a unique fixed point $x^* \in X$.

Aydi et al. [6] proved that Theorem 1.5 is a consequence of Theorem 1.4.

2. Main Results

We define the following set of functions:

$$\Psi = \{\psi : [0,\infty) \longrightarrow [0,\infty) \text{ such that } \psi \text{ is nondecreasing and lower semicontinuous}\},$$
$$\Phi_\alpha = \{\alpha : [0,\infty) \longrightarrow [0,\infty) \text{ such that } \alpha \text{ is upper semicontinuous}\}, \quad (2.1)$$
$$\Phi_\beta = \{\beta : [0,\infty) \longrightarrow [0,\infty) \text{ such that } \beta \text{ is lower semicontinuous}\}.$$

Let $(X, \preceq)$ be ordered set. The pair (x, y) is said to be comparable if either $x \preceq y$ or $y \preceq x$ holds.

Theorem 2.1. *Let $(X, d, \preceq)$ be an ordered metric space such that (X, d) is complete and $f : X \to X$ be a nondecreasing self-mappings. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \quad \forall t > 0,\ s = t,\ \text{or } s = 0, \quad (2.2)$$
$$\psi(d(fx, fy)) \leq \alpha(d(x,y)) - \beta(d(x,y)), \quad (2.3)$$

for all comparable $x, y \in X$. Suppose that either

(a) *f is continuous, or*

(b) *if a nondecreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Let $x_0 \preceq fx_0$. We define an iterative sequence $\{x_n\}$ in the following way:

$$x_n = f^n x_0 = fx_{n-1} \quad \forall n \in \mathbb{N}. \quad (2.4)$$

Since f is nondecreasing and $x_0 \prec fx_0$, we have

$$x_0 \prec x_1 \preceq x_2 \preceq \cdots, \quad (2.5)$$

and hence $\{x_n\}$ is a nondecreasing sequence. If $x_{n_0} = x_{n_0+1} = fx_{n_0}$ for some $n_0 \in \mathbb{N}$, then the point x_0 is the desired fixed point of f which completes the proof.

Hence we suppose that $x_n \neq x_{n+1}$, that is, $d(x_{n-1}, x_n) > 0$ for all n. Hence, (2.5) turns in to

$$x_0 \prec x_1 \prec x_2 \prec \cdots. \quad (2.6)$$

We want to show that the sequence $\{d_n := d(x_n, x_{n+1})\}$ in nonincreasing. Suppose, to the contrary, that there exists some $n_0 \in \mathbb{N}$ such that

$$d(x_{n_0-1}, x_{n_0}) \leq d(x_{n_0}, x_{n_0+1}). \quad (2.7)$$

Since ψ is nondecreasing, we obtain that

$$\psi(d(x_{n_0-1}, x_{n_0})) \leq \psi(d(x_{n_0}, x_{n_0+1})). \tag{2.8}$$

By taking $x = x_{n_0-1}$ and $y = x_{n_0}$, the condition (2.3) together with (2.8) we derive that

$$\begin{aligned}\psi(d(x_{n_0-1}, x_{n_0})) &\leq \psi(d(x_{n_0}, x_{n_0+1})) = \psi\left(d\left(fx_{n_0-1}, fx_{n_0}\right)\right) \qquad (2.9)\\ &\leq \alpha(d(x_{n_0-1}, x_{n_0})) - \beta(d(x_{n_0-1}, x_{n_0})), \qquad (2.10)\end{aligned}$$

which contradicts (2.2). Therefore, we conclude that

$$d_n < d_{n-1}, \tag{2.11}$$

hold for all $n \in \mathbb{N}$. Hence $\{d_n\}$ is a nonincreasing sequence of positive real numbers. Thus, there exists $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. We will show that $r = 0$ by method of reductio ad absurdum. For this purpose, we assume that $r > 0$. By (2.9) together with the properties of α, β, ψ we have

$$\begin{aligned}\psi(r) &\leq \liminf_{n\to\infty} \psi(d(x_{n-1}, x_n)) \leq \limsup_{n\to\infty} \psi(d_{n-1})\\ &\leq \limsup_{n\to\infty}\left[\alpha(d_{n-1}) - \beta(d_{n-1})\right] \leq \alpha(r) - \beta(r),\end{aligned} \tag{2.12}$$

which is a contradiction. Hence

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.13}$$

We will show that the sequence $\{x_n\}$ is Cauchy. Suppose, to the contrary, that

$$\lim_{n\to\infty} d(x_n, x_m) > 0, \tag{2.14}$$

that is, there is $\varepsilon > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k with $n(k) > m(k) > k$

$$d\left(x_{n(k)}, x_{m(k)}\right) \geq \varepsilon. \tag{2.15}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.15) and $n(k) > m(k) \geq k$. Thus,

$$d\left(x_{n(k)}, x_{m(k)-1}\right) < \varepsilon. \tag{2.16}$$

Now for all $k \in \mathbb{N}$ we have

$$\varepsilon \leq d\left(x_{n(k)}, x_{m(k)}\right) \leq d\left(x_{n(k)}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{m(k)}\right) < \varepsilon + s_{m(k)-1}. \tag{2.17}$$

So

$$\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon. \tag{2.18}$$

Again, from

$$\begin{aligned}
d\left(x_{n(k)}, x_{m(k)}\right) &\le d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right),\\
d\left(x_{n(k)+1}, x_{m(k)+1}\right) &\le d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right).
\end{aligned} \tag{2.19}$$

Taking the limit as $k \to +\infty$, by (2.13) and (2.18), we deduce

$$\lim_{k\to\infty} d\left(x_{n(k)+1}, x_{m(k)+1}\right) = \varepsilon. \tag{2.20}$$

Now from (2.3) we have

$$\psi\left(d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right) = \psi\left(d\left(f x_{n(k)}, f x_{m(k)}\right)\right) \le \alpha\left(d\left(x_{n(k)}, x_{m(k)}\right)\right) - \beta\left(d\left(x_{n(k)}, x_{m(k)}\right)\right). \tag{2.21}$$

Taking the lim inf as $k \to +\infty$ in the inequality above, we have

$$\begin{aligned}
\psi(\varepsilon) &\le \liminf \psi\left(d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right) \le \limsup \psi\left(d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right)\\
&\le \limsup\left(\alpha\left(d\left(x_{n(k)}, x_{m(k)}\right)\right) - \beta\left(d\left(x_{n(k)}, x_{m(k)}\right)\right)\right)\\
&= \limsup \alpha\left(d\left(x_{n(k)}, x_{m(k)}\right)\right) - \liminf \beta\left(d\left(x_{n(k)}, x_{m(k)}\right)\right)\\
&\le \alpha(\varepsilon) - \beta(\varepsilon).
\end{aligned} \tag{2.22}$$

So we have

$$\psi(\varepsilon) \le \alpha(\varepsilon) - \beta(\varepsilon), \tag{2.23}$$

which contradicts the fact that $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_m) = 0, \tag{2.24}$$

that is, the sequence $\{x_n\}$ is Cauchy. Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Suppose that (a) holds. Then,

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_n) = f(x^*). \tag{2.25}$$

Hene, x^* is a fixed point of f.

Suppose that (b) holds, that is, $x_n \preceq x^*$ for all $n \geq 0$. We claim that x^* is a fixed point of f, that is, $\lim_{n\to\infty} d(x_{n+1}, fx^*) = 0$. Suppose, to the contrary, that $\lim_{n\to\infty} d(x_{n+1}, fx^*) = d(x^*, fx^*) > 0$. Due to condition (2.3), we have

$$\psi(d(x_{n+1}, fx^*)) = \psi(d(fx_n, fx^*)) \leq \alpha(d(x_n, x^*)) - \beta(d(x_n, x^*)). \tag{2.26}$$

Taking the lim inf as $n \to \infty$ in the inequality above, we obtain that

$$\begin{aligned}\psi(d(x^*, fx^*)) &\leq \liminf_{n\to\infty} \psi(d(x_{n+1}, fx^*)) \\ &= \liminf_{n\to\infty} \psi(d(fx_n, fx^*)) \leq \limsup_{n\to\infty} \psi(d(x_n, fx^*)) \\ &\leq \limsup_{n\to\infty} (\alpha(d(x_n, x^*)) - \beta(d(x_n, x^*))) \\ &\leq \alpha(0) - \beta(0),\end{aligned} \tag{2.27}$$

which is a contradiction. Hence $\lim_{n\to\infty} d(x_{n+1}, fx^*) = d(x^*, fx^*) = 0$ and hence, $x^* = fx^*$. □

In the following theorem we investigate the uniqueness of fixed points in the theorem above. In order to assure the uniqueness of fixed points we need the following notion on the partially ordered metric space $(X, \preceq)$ which is called the comparability condition:

(C) For every $x, y \in X$ there exists $z \in X$ such that either $x \preceq z$ and $y \preceq z$ or $z \preceq x$ and $z \preceq y$.

Theorem 2.2. *In addition to hypotheses of Theorem 2.1, suppose that X is the comparability condition (C). Then f has a unique fixed point.*

Proof. Due to Theorem 2.1, we guarantee that f has a fixed point. Suppose x and y are fixed points of f with $x \neq y$, that is, $d(x, y) > 0$.

We need to consider two different cases. First case: If x and y are comparable, then

$$\psi(d(x, y)) = \psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \tag{2.28}$$

which contradicts (2.2). Hence $x = y$.

Let us examine the second and last case: if x and y are not comparable, then by (C) there exists z such that $x \preceq z$ and $y \preceq z$. Notice that $x = fx \preceq fz$ and hence $x \preceq f^n z$ for each n. Analogously we have $y \preceq f^n z$ for each n. Then we have

$$\psi(d(x, fz)) = \psi(d(fx, fz)) \leq \alpha(d(x, z)) - \beta(d(x, z)), \tag{2.29}$$

which is true only if $x = fz$. On the other hand, we have

$$\psi(d(y, fz)) = \psi(d(fy, fz)) \leq \alpha(d(y, z)) - \beta(d(y, z)), \tag{2.30}$$

which yields that $y = fz$. Hence, $x = y$. □

Example 2.3. Let $X = [0,1]$. We define a partial order $\preceq$ by $x \preceq y$ if and only if $x \leq y$. Let $d(x,y) = |x-y|$ and $f : X \to X$ be defined by $fx = (1/2)x - (1/4)x^2$. Also define three functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\psi(t) = \begin{cases} t + \dfrac{3}{2}, & t > 0, \\ 1, & t = 0, \end{cases} \qquad \alpha(t) = t + \frac{5}{2}, \qquad \beta(t) = \frac{t}{2} + 1. \tag{2.31}$$

Clearly, ψ is lower semicontinuous, also $\psi(t) = t + (3/2) > \alpha(t) - \beta(t) = (t/2) + (3/2)$ and $\psi(t) = t + (3/2) > \alpha(0) - \beta(0) = 3/2$ for all $t > 0$. Let $y \geq x$. Then we have

$$\psi(d(fx,fy)) = \begin{cases} \dfrac{1}{2}\left[y - x - \dfrac{3}{2}(y^2 - x^2)\right] + \dfrac{3}{2}, & p(x,y) > 0, \\ 0, & p(x,y) = 0, \end{cases} \tag{2.32}$$

where $p(x,y) = y - x - (1/2)(y^2 - x^2)$. And

$$\alpha(d(x,y)) - \beta(d(x,y)) = \frac{y-x}{2} + \frac{3}{2}. \tag{2.33}$$

Then

$$\psi(d(fx,fy)) \leq \alpha(d(x,y)) - \beta(d(x,y)), \tag{2.34}$$

hold for all $y \geq x$. Hence condition of Theorem 2.1 is hold. That is, f has a fixed point. Theorem 1.5 cannot be applied to f in this example. Since ψ is lower semicontinuous and $\psi(0) \neq 0$, $\alpha(0) \neq 0$, and $\beta(0) \neq 0$. Further, the approach of Aydi et al. [6] cannot be modified for it.

Now, we use the following lemma, that is a consequence of the axiom of choice, to obtain common fixed point results for two self-mappings defined on a metric space.

Lemma 2.4 (see [9, Lemma 2.1]). *Let X be a nonempty set and $g : X \to X$ a mapping. Then there exists a subset $E \subset X$ such that $fE = gX$ and $g : E \to X$ is one-to-one.*

Theorem 2.5. *Let $(X, d, \preceq)$ be an ordered metric space such that (X, d) is complete and $f, g : X \to X$ be two self-mappings. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(d(fx,fy)) \leq \alpha(d(gx,gy)) - \beta(d(gx,gy)), \tag{2.35}$$

for all $x, y \in X$ with $gx \preceq gy$. If the following conditions hold:

(a) *f is g-nondecreasing, $g(X)$ is closed and $fX \subseteq gX$,*

(b) *there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$,*

(c) *if a nondecreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then f and g have a coincidence point.

Proof. By Lemma 2.4, there exists $E \subset X$ such that $gE = gX$ and $g : E \to X$ is one-to-one. Define

$$T : gE \longrightarrow gE \quad \text{by } Tgx = fx \quad \forall gx \in gE. \tag{2.36}$$

Since g is one-to-one on E and $fX \subset gX$, T is well defined. By condition (2.35), for all $gx, gy \in gE$, we have

$$\psi(d(Tgx, Tgy)) \leq \alpha(d(gx, gy)) - \beta(d(gx, gy)). \tag{2.37}$$

Since f is g-nondecreasing then T is nondecreasing. Indeed, $gx \preceq gy$ implies $fx \preceq fy$ and then $Tgx \preceq Tgy$. Also, since gE is complete, by Theorem 2.1, we have that T has a fixed point on gE, say gz. That is, $gz = Tgz = fz$. Then f and g have a coincidence point. □

Remark 2.6. One can easily conclude that Theorem 2.5 is still valid if we replace the condition (c) of Theorem 2.5 with

(c′) if a nonincreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \succcurlyeq$ for all $n \in N$.

We know state a new condition as follows:

(D) The set of points of coincidence of f and g, say, $PC(f, g)$ is totally ordered.

Theorem 2.7. *In addition to hypotheses of Theorem 2.5, suppose that X is the comparability condition (D). Then f and g have a unique coincidence point. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point.*

Proof. Assume that z and z^* are two coincidence points of f and g with $z \neq z^*$. By condition (2.35) we have

$$\psi(d(gz, gz^*)) = \psi(d(fz, fz^*)) \leq \alpha(d(gz, gz^*)) - \beta(d(gz, gz^*)), \tag{2.38}$$

which is a contradiction. Hence $z = z^*$. Let $fz = gz = w$. Since f and g are weakly compatible then $fw = fgz = gfz = gw$. Now since f and g have a unique coincidence point then $w = z$. That is, z is a unique common fixed point of f and g. □

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