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Research Article

New Generalized Mixed Equilibrium Problem with Respect to Relaxed Semi-Monotone Mappings in Banach Spaces

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We introduce the new generalized mixed equilibrium problem with respect to relaxed semi-monotone mappings. Using the KKM technique, we obtain the existence of solutions for the generalized mixed equilibrium problem in Banach spaces. Furthermore, we also introduce a hybrid projection algorithm for finding a common element in the solution set of a generalized mixed equilibrium problem and the fixed point set of an asymptotically nonexpansive mapping. The strong convergence theorem of the proposed sequence is obtained in a Banach space setting. The main results extend various results existing in the current literature.

1. Introduction

Let E be a Banach space with the dual E^* and let E^{**} denote the dual space of E^*. If $E = E^{**}$, then E is called reflexive. We denote by $\mathcal{N}$ and $\mathcal{R}$ the sets of positive integers and real numbers, respectively. Also, we denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Recall that if E is smooth, then J is single-valued, and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall still denote by J the single-valued duality mapping.

Let C be a nonempty subset of E^{**}, $\eta : C \times C \to E^{**}$ be a mapping and let $\xi : E^{**} \to \mathcal{R}$ a function with $\xi(tz) = t^p \xi(z)$ for all $t > 0$ and $z \in E^{**}$, where $p > 1$ is a constant. A mapping $A : C \times C \to E^*$ is said to be *relaxed η-ξ semimonotone* [1] if the following two conditions hold:

(i) for each fixed $u \in C$, $A(u, \cdot)$ is relaxed η-ξ monotone; that is,

$$\langle A(u,v) - A(u,w), \eta(v,w) \rangle \geq \xi(v-w), \quad \forall v, w \in C; \tag{1.2}$$

(ii) for each fixed $v \in C$, $A(\cdot, v)$ is completely continuous; that is, for any net $\{u_j\}$ in C, $u_j \to u_0$ in weak * topology of E^{**}, then $\{A(u_j, v)\}$ has a subsequence $\{A(u_{j_k}, v)\} \to A(u_0, v)$ in norm topology of E^*.

In case $\eta(x,y) = x - y$ for all $x, y \in C$ and $\xi \equiv 0$, A is called *semi-monotone* [2]. The following is an example of η-ξ semi-monotone mapping.

Example 1.1. Let $C = (-\infty, \infty)$, $A(x,y) = x + y$, and

$$\eta(x,y) = \begin{cases} -c(x-y), & x \geq y, \\ c(x-y), & x < y, \end{cases} \tag{1.3}$$

where $c > 0$ is a constant. Then, A is relaxed η-ξ semi-monotone with

$$\xi(z) = \begin{cases} -cz^2, & z \geq 0, \\ cz^2, & z < 0. \end{cases} \tag{1.4}$$

Let $f : C \times C \to \mathcal{R}$ be a bifunction, $\eta : C \times C \to E^{**}$ a mapping, and $\xi : E^{**} \to \mathcal{R}$, $\varphi : C \to \mathcal{R}$ two real-valued functions, and let $A : C \times C \to E^*$ be a η-ξ semi-monotone mapping. We consider the problem of finding $u \in C$ such that

$$f(u,v) + \langle A(u,u), \eta(v,u) \rangle + \varphi(v) \geq \varphi(u), \quad \forall\, v \in C, \tag{1.5}$$

which is called *the generalized mixed equilibrium problem with respect to relaxed η-ξ semi-monotone mapping* (GMEP(f, A, η, φ)). The set of such $u \in C$ is denoted by GMEP(f, A, η, φ), that is,

$$\mathrm{GMEP}(f, A, \eta, \varphi) = \{u \in C : f(u,v) + \langle A(u,u), \eta(v,u) \rangle + \varphi(v) \geq \varphi(u),\ \forall v \in C\}. \tag{1.6}$$

Now, let us consider some special cases of the problem (1.5).

(a) In the case of $f \equiv 0$, (1.5) is deduced to the following variational-like inequality problem:

$$\text{find } u \in C \text{ such that } \langle A(u,u), \eta(v,u) \rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in C. \tag{1.7}$$

The problem (1.7) was studied by Fang and Huang [1]. Using the KKM technique and η-ξ monotonicity of the mapping φ, they [1] obtained the existence of solutions of the variational-like inequality problem (1.7) in a real Banach space.

(b) In the case of $f \equiv 0$, $\varphi \equiv 0$ and $\eta(v,u) = v - u$ for all $v, u \in C$, the problem (1.5) is deduced to the following variational inequality problem:

$$\text{Find } u \in C \text{ such that } \langle A(u,u), v-u \rangle \geq 0, \quad \forall v \in C. \tag{1.8}$$

The problem (1.8) was studied by Chen [2]. They obtained the existence results of solutions in a real Banach space.

When E is a reflexive Banach space, we know $E^{**} = j(E)$, where $j : E \to E^{**}$ is the duality mapping defined by $\langle jx, f \rangle = \langle f, x \rangle$, for all $x \in E$, $f \in E^*$, which is an isometric mapping, so we may regard $E = E^{**}$ under an isometry. The following problems can be derived as special cases of the problem (1.5).

(c) In case E is reflexive (i.e., $E = E^{**}$), $f \equiv 0$ and $\eta(v,u) = v - u$ for all $v, u \in C$, the problem (1.5) is deduced to the following variational inequality problem:

$$\text{find } u \in C \text{ such that } \langle A(u,u), v-u \rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in C. \tag{1.9}$$

The problem (1.9) was studied by Chen [2].

(d) If E is reflexive (i.e., $E = E^{**}$) and $A \equiv 0$, (1.5) is deduced to the mixed equilibrium problem:

$$\text{find } u \in C \text{ such that } f(u,v) + \varphi(v) \geq \varphi(u), \quad \forall v \in C. \tag{1.10}$$

The problem (1.10) was considered and studied by Ceng and Yao [3]; Cholamjiak and Suantai [4].

(e) In the case of $A \equiv 0$ and $\varphi \equiv 0$, (1.5) is deduced to the following classical equilibrium problem:

$$\text{find } u \in C \text{ such that } f(u,v) \geq 0, \quad \forall v \in C. \tag{1.11}$$

The set of all solution of (1.11) is denoted by $\mathrm{EP}(f)$, that is,

$$\mathrm{EP}(f) = \{u \in C : f(u,v) \geq 0, \quad \forall v \in C\}. \tag{1.12}$$

Numerous problems in physics, optimization, and economics can be reduced to find a solution of the equilibrium problem, variational inequality problem, and related optimization problems; see, for instance, [5–11]. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; see, for instance, Blum and Oettli [12]; Combettes and Hirstoaga [13]; Moudafi [14].

Let C be a nonempty, closed convex subset of E. A mapping $S : C \to E$ is called *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Also a mapping $S : C \to C$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\|S^n x - S^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and for each $n \geq 1$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [15] as an important generalization of nonexpansive mappings. Denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. There are several methods for approximating fixed points of a nonexpansive mapping; see, for instance, [16–21]. Furthermore, since 1972, a host of authors have studied weak and strong convergence problems of the iterative processes for the class of asymptotically nonexpansive mappings; see, for instance, [22–25]. In 1953, Mann [16] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping S in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n, \quad \forall n \in \mathcal{N}, \tag{1.13}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in $[0, 1]$. However, we note that Mann's iteration process (1.13) has only weak convergence, in general; for instance, see [26–28]. In 2003, Nakajo and Takahashi [29] introduced the following iterative algorithm for the nonexpansive mapping S in the framework of Hilbert spaces:

$$\begin{gathered}
x_0 = x \in C, \\
y_n = \alpha_n x_n + (1 - \alpha_n) S x_n, \\
C_n = \left\{ z \in C : \left\| z - y_n \right\| \leq \| z - x_n \| \right\}, \\
Q_n = \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n} x, \quad n \geq 0,
\end{gathered} \tag{1.14}$$

where $\{\alpha_n\} \subset [0, \alpha]$, $\alpha \in [0, 1]$, and $P_{C_n \cap Q_n}$ is the metric projection from a Hilbert space H onto $C_n \cap Q_n$. They proved that $\{x_n\}$ generated by (1.14) converges strongly to a fixed point of S. Xu [30] extended Nakajo and Takahashi's theorem to Banach spaces by using the generalized projection.

Matsushita and Takahashi [17] introduced the following iterative algorithm in the framework of Banach spaces:

$$\begin{gathered}
x_0 = x \in C, \\
C_n = \overline{\text{co}} \{ z \in C : \| z - Sz \| \leq t_n \| x_n - S x_n \| \}, \\
D_n = \{ z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0 \}, \\
x_{n+1} = P_{C_n \cap D_n} x, \quad n \geq 0,
\end{gathered} \tag{1.15}$$

where $\overline{\text{co}} D$ denoted the convex closure of the set D, $\{t_n\}$ is a sequence in $(0, 1)$ with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$.

Very recently, Dehghan [24] introduced the following iterative algorithm for finding fixed points of an asymptotically nonexpansive mapping S in a uniformly convex and smooth Banach space:

$$\begin{aligned} x_0 &= x \in C, \quad C_0 = D_0 = C, \\ C_n &= \overline{\text{co}}\{z \in C_{n-1} : \|z - S^n z\| \le t_n \|x_n - S^n x_n\|\}, \\ D_n &= \{z \in D_{n-1} : \langle x_n - z, J(x - x_n)\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap D_n} x, \quad n \ge 0, \end{aligned} \tag{1.16}$$

where $\overline{\text{co}}D$ denotes the convex closure of the set D, J is the normalized duality mapping, $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$. The strong convergence theorem of the iterative sequence $\{x_n\}$ defined by (1.16) is obtained in a uniformly convex and smooth Banach space.

In this paper, motivated and inspired by the above results, we first suggest and analyze the new generalized mixed equilibrium problem with respect to relaxed η-ξ semi-monotone mapping. Using the KKM technique, we obtain the existence of solutions for such problem in a Banach space. Next, we also introduce a hybrid projection algorithm for finding a common element in the solution set of a generalized mixed equilibrium problem and the fixed point set of an asymptotically nonexpansive mapping. The strong convergence theorem of the proposed sequence is obtained in a Banach space setting. The main results extend various results existing in the current literature.

2. Preliminaries

Let E be a real Banach space, and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if for any $x, y \in U$,

$$x \ne y \text{ implies } \|x + y\| < 2. \tag{2.1}$$

It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \ge \varepsilon \text{ implies } \|x + y\| < 2(1 - \delta). \tag{2.2}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function $\delta : [0,2] \to [0,1]$ called the modulus of convexity of E as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : x, y \in E, \ \|x\| = \|y\| = 1, \ \|x - y\| \ge \varepsilon\right\}. \tag{2.3}$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.4}$$

exists for all $x, y \in U$. Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth Banach space E. Then for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| \le \min_{y\in C}\|y - x\|. \tag{2.5}$$

The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the *metric projection* from E onto C. The following theorem is wellknown.

Theorem 2.1 (see [31]). *Let C be a nonempty, closed convex subset of a smooth Banach space E and let $x \in E$, and $y \in C$. Then the following are equivalent:*

(a) *y is a best approximation to $x : y = P_C x$.*

(b) *y is a solution of the variational inequality:*

$$\langle y - z, J(x-y)\rangle \ge 0, \quad \forall z \in C, \tag{2.6}$$

where J is a duality mapping and P_C is the metric projection from E onto C.

It is wellknown that if P_C is a metric projection from a real Hilbert space H onto a nonempty, closed, and convex subset C, then P_C is nonexpansive. But, in a general Banach space, this fact is not true.

In the sequel, we will need the following lemmas.

Lemma 2.2 (see [32]). *Let E be a uniformly convex Banach space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < b \le \alpha_n \le c < 1$ for all $n \ge 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\limsup_{n\to\infty}\|x_n\| \le d$, $\limsup_{n\to\infty}\|y_n\| \le d$, and $\lim_{n\to\infty}\|\alpha_n x_n + (1-\alpha_n)y_n\| = d$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Theorem 2.3 (see [33]). *Let C be a bounded, closed, and convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing, convex, and continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\left\|S\left(\sum_{i=1}^{n}\lambda_i x_i\right) - \sum_{i=1}^{n}\lambda_i S x_i\right\|\right) \le \max_{1\le j\le k\le n}\left(\|x_j - x_k\| - \|Sx_j - Sx_k\|\right), \tag{2.7}$$

for all $n \in \mathcal{N}$, $\{x_1, x_2, \dots, x_n\} \subset C$, $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset [0,1]$ with $\sum_{i=1}^{n}\lambda_i = 1$ and nonexpansive mapping S of C into E.

Theorem 2.4 (see [24]). *Let C be a bounded, closed, and convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing, convex, and continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\frac{1}{k_m}\left\|S^m\left(\sum_{i=1}^{n}\lambda_i x_i\right) - \sum_{i=1}^{n}\lambda_i S^m x_i\right\|\right) \leq \max_{1\leq j\leq k\leq n}\left(\|x_j - x_k\| - \frac{1}{k_m}\|S^m x_j - S^m x_k\|\right), \tag{2.8}$$

for all $n \in \mathcal{N}$, $\{x_1, x_2, \ldots, x_n\} \subset C$; $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset [0,1]$ with $\sum_{i=1}^{n}\lambda_i = 1$ and an asymptotically nonexpansive mapping S of C into E with the sequence $\{k_m\}$.

Now, let us recall the following well-known concepts and results.

Definition 2.5. Let B be a subset of topological vector space X. A mapping $G : B \to 2^X$ is called a KKM mapping if $\mathrm{co}\{x_1, x_2, \ldots, x_m\} \subset \bigcup_{i=1}^{m} G(x_i)$ for $x_i \in B$ and $i = 1, 2, \ldots, m$, where $\mathrm{co}A$ denotes the convex hull of the set A.

Lemma 2.6 (see [34]). *Let B be a nonempty subset of a Hausdorff topological vector space X, and let $G : B \to 2^X$ be a KKM mapping. If $G(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x\in B} G(x) \neq \emptyset$.*

Theorem 2.7 (see [35] (Kakutani-Fan-Glicksberg Fixed Point Theorem)). *Let E be a locally convex Hausdorff topological vector space and C a nonempty, convex, and compact subset of E. Suppose $T : C \to 2^C$ is a upper semi-continuous mapping with nonempty, closed, and convex values. Then T has a fixed point in C.*

Definition 2.8 (see [36]). Let C be a nonempty, closed convex of a Banach space E. Let $T : C \to E^*$ and let $\eta : C \times C \to \mathcal{R}$ be two mappings. T is said to be *η-hemicontinuous* if, for any fixed $x, y \in C$, the mapping $f : [0,1] \to (-\infty, \infty)$ defined by $f(t) = \langle T(x + t(y - x)), \eta(y, x)\rangle$ is continuous at 0^+.

For solving the mixed equilibrium problem, let us assume the following conditions for a bifunction $f : C \times C \to \mathcal{R}$:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $y \in C$, $f(\cdot, y)$ is weakly upper semicontinuous;

(A4) for all $x \in C$, $f(x, \cdot)$ is convex.

The following lemmas can be found in [37].

Lemma 2.9 (see [37]). *Let C be a nonempty, bounded, closed, and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed η-ξ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1) *and* (A4)*, and let φ be a lower semicontinuous and convex function from C to $\mathcal{R}$. Let $r > 0$ and $z \in C$. Assume that*

(i) *$\eta(x, x) = 0$, for all $x \in C$;*

(ii) *for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x, u)\rangle$ is convex.*

Then the following problems (2.9) *and* (2.10) *are equivalent. Find* $x \in C$ *such that:*

$$f(x,y) + \varphi(y) + \langle Tx, \eta(y,x)\rangle + \frac{1}{r}\langle y - x, J(x-z)\rangle \geq \varphi(x), \quad \forall y \in C. \tag{2.9}$$

Find $x \in C$ *such that*

$$f(x,y) + \langle Ty, \eta(y,x)\rangle + \varphi(y) + \frac{1}{r}\langle y - x, J(x-z)\rangle \geq \varphi(x) + \xi(y-x), \quad \forall y \in C. \tag{2.10}$$

Lemma 2.10 (see [37]). *Let* C *be a nonempty, bounded, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space* E*, let* $T : C \to E^*$ *be an* η*-hemicontinuous and relaxed* η*-*ξ *monotone mapping. Let* f *be a bifunction from* $C \times C$ *to* $\mathcal{R}$ *satisfying* (A1), (A3), *and* (A4), *and let* φ *be a lower semicontinuous and convex function from* C *to* $\mathcal{R}$*. Let* $r > 0$ *and* $z \in C$*. Assume that*

(i) $\eta(x,y) + \eta(y,x) = 0$ *for all* $x, y \in C$*;*

(ii) *for any fixed* $u, v \in C$*, the mapping* $x \mapsto \langle Tv, \eta(x,u)\rangle$ *is convex and lower semicontinuous;*

(iii) $\xi : E \to \mathcal{R}$ *is weakly lower semicontinuous; that is, for any net* $\{x_\beta\}$*,* $\{x_\beta\}$ *converges to* x *in* $\sigma(E, E^*)$ *implies that* $\xi(x) \leq \liminf \xi(x_\beta)$*.*

Then, the solution set of the problem (2.9) *is nonempty, that is, there exists* $x_0 \in C$ *such that*

$$f(x_0,y) + \langle Tx_0, \eta(y,x_0)\rangle + \varphi(y) + \frac{1}{r}\langle y - x_0, J(x_0-z)\rangle \geq \varphi(x_0), \quad \forall y \in C. \tag{2.11}$$

3. Existence Results of Generalized Mixed Equilibrium Problem

In this section, we prove the following crucial lemma concerning the generalized mixed equilibrium problem with respect to relaxed η-ξ semi-monotone mapping (GMEP(f, A, η, φ)) in a real Banach space with the smooth and strictly convex second dual space.

Lemma 3.1. *Let* E *be a real Banach space with the smooth and strictly convex second dual space* E^{**}*, let* C *be a nonempty bounded closed convex subset of* E^{**}*, let* $A : C \times C \to E^*$ *be a relaxed* η*-*ξ *semi-monotone mapping. Let* $f : C \times C \to \mathcal{R}$ *be a bifunction satisfying* (A1), (A3), *and* (A4), *and let* $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* $r > 0$ *and* $z \in C$*. Assume that*

(i) $\eta(x,y) + \eta(y,x) = 0$ *for all* $x, y \in C$*;*

(ii) *for any fixed* $u, v, w \in C$*, the mapping* $x \mapsto \langle A(v,w), \eta(x,u)\rangle$ *is convex and lower semicontinuous;*

(iii) *for each* $x \in C$*,* $A(x,\cdot) : C \to E^*$ *is finite-dimensional continuous: that is, for any finite-dimensional subspace* $F \subset E^{**}$*,* $A(x,\cdot) : C \cap F \to E^*$ *is continuous;*

(iv) $\xi : E^{**} \to \mathcal{R}$ *is convex lower semicontinuous.*

Then there exists $u_0 \in C$ *such that*

$$f(u_0,v) + \langle A(u_0,u_0), \eta(v,u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0-z)\rangle \geq \varphi(u_0), \quad \forall v \in C. \tag{3.1}$$

Proof. Let $F \subseteq E^{**}$ be a finite-dimensional subspace with $C_F := F \cap C \neq \emptyset$. For each $w \in C$, consider the following problem: find $u_0 \in C_F$ such that

$$f(u_0, v) + \langle A(w, u_0), \eta(v, u_0) \rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z) \rangle - \varphi(u_0) \geq 0, \quad \forall v \in C_F. \tag{3.2}$$

Since $C_F \subseteq F$ is bounded closed and convex, $A(w, \cdot)$ is continuous on C_F and relaxed η-ξ monotone for each fixed $w \in C$, from Lemma 2.10, we know that problem (3.2) has a solution $u_0 \in C_F$.

Now, define a set-valued mapping $G : C_F \to 2^{C_F}$ as follows:

$$\text{Gw} = \left\{ u \in C_F : f(u, v) + \langle A(w, u), \eta(v, u) \rangle + \varphi(v) + \frac{1}{r}\langle v - u, J(u - z) \rangle - \varphi(u) \geq 0, \ \forall v \in C_F \right\}. \tag{3.3}$$

It follows from Lemma 2.9 that, for each fixed $w \in C_F$:

$$\begin{aligned} &\left\{ u \in C_F : f(u, v) + \langle A(w, u), \eta(v, u) \rangle + \varphi(v) + \frac{1}{r}\langle v - u, J(u - z) \rangle - \varphi(u) \geq 0, \ \forall v \in K_F \right\} \\ &\quad = \left\{ u \in C_F : f(u, v) + \langle A(w, v), \eta(v, u) \rangle + \varphi(v) + \frac{1}{r}\langle v - u, J(u - z) \rangle - \varphi(u) \geq \xi(v - u), \ \forall v \in K_F \right\}. \end{aligned} \tag{3.4}$$

Since every convex lower semicontinuous function in Banach spaces is weakly lower semicontinuous, the proper convex lower semicontinuity of φ and ξ, condition (ii), (A3) and (A4) implies that $G : C_F \to 2^{C_F}$ has nonempty bounded closed and convex values. Using (A3) and the complete continuity of $A(\cdot, u)$, we can conclude that G is upper semicontinuous. It follows from Theorem 2.7 that G has a fixed point $w^* \in C_F$, that is,

$$\langle f(w^*, v) + \langle A(w^*, w^*), \eta(v, w^*) \rangle + \varphi(v) + \frac{1}{r}\langle v - w^*, J(w^* - z) \rangle - \varphi(w^*) \geq 0, \quad \forall v \in C_F. \tag{3.5}$$

Let

$$\mathcal{F} = \{ F \subset E^{**} : F \text{ is finite dimensional with } F \cap C \neq \emptyset \}, \tag{3.6}$$

and let

$$\begin{aligned} W_F = \bigg\{ u \in C : {} & f(u, v) + \langle A(u, v), \eta(v, u) \rangle + \varphi(v) \\ & + \frac{1}{\text{r}}\langle v - u, J(u - z) \rangle - \varphi(u) \geq \xi(v - u), \ \forall v \in C_F \bigg\}, \quad \forall F \in \mathcal{F}. \end{aligned} \tag{3.7}$$

By (3.5) and Lemma 2.9, we know that W_F is nonempty and bounded. Denote by $\overline{W}_F^*$ the weak*-closure of W_F in E^{**}. Then, $\overline{W}_F^*$ is weak* compact in E^{**}.

For any $F_i \in \mathcal{F}$, $i = 1, 2, \ldots, N$, we know that $W_{\bigcap_{i=1}^N F_i} \subset \bigcap_{i=1}^N W_{F_i}$, so $\{\overline{W}_F^* : F \in \mathcal{F}\}$ has the finite intersection property. Therefore, it follows that

$$\bigcap_{F\in\mathcal{F}} \overline{W}_F^* \neq \emptyset. \tag{3.8}$$

Let $u_0 \in \bigcap_{F\in\mathcal{F}} \overline{W}_F^*$. We claim that

$$f(u_0, v) + \langle A(u_0, u_0), \eta(v, u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle - \varphi(u_0) \geq 0, \quad \forall v \in C. \tag{3.9}$$

Indeed, for each $v \in C$, let $F \in \mathcal{F}$ be such that $v \in C_F$ and $u_0 \in C_F$. Then, there exists $u_j \in W_F$ such that $u_j \rightharpoonup u_0$. The definition of W_F implies that

$$f(u_j, v) + \langle A(u_j, v), \eta(v, u_j)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_j, J(u_j - z)\rangle - \varphi(u_j) \geq \xi(v - u_j), \tag{3.10}$$

that is

$$f(u_j, v) + \langle A(u_j, v), \eta(v, u_j)\rangle + \varphi(v) + \frac{1}{r}\langle v - z, J(u_j - z)\rangle - \frac{1}{r}\|z - u_j\|^2 - \varphi(u_j) \geq \xi(v - u_j), \tag{3.11}$$

for all $j = 1, 2, \ldots$. Using the complete continuity of $A(\cdot, u)$, (A3), (ii), the continuity of J, the convex lower semicontinuity of φ, ξ, and $\|\cdot\|^2$, and letting $j \to \infty$, we get

$$f(u_0, v) + \langle A(u_0, v), \eta(v, u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle - \varphi(u_0) \geq \xi(v - u_0), \quad \forall v \in C. \tag{3.12}$$

From Lemma 2.9, we have

$$f(u_0, v) + \langle A(u_0, u_0), \eta(v, u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle - \varphi(u_0) \geq 0, \quad \forall v \in C. \tag{3.13}$$

Hence, we complete the proof. □

Setting $A \equiv 0$ and $\varphi \equiv 0$ in Lemma 3.1, we have the following result.

Corollary 3.2. *Let E be a real Banach space with the smooth and strictly convex second dual space E^{**}, let C be a nonempty bounded closed convex subset of E^{**}. Let $f : C \times C \to \mathcal{R}$ be a bifunction satisfying* (A1), (A3), *and* (A4). *Let $r > 0$ and $z \in C$. Then there exists $u_0 \in C$ such that*

$$f(u_0, v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle \geq 0, \quad \forall v \in C. \tag{3.14}$$

If E is reflexive (i.e., $E = E^{**}$) smooth and strictly convex real Banach space, then we have the following result.

Corollary 3.3. *Let E be a reflexive smooth and strictly convex Banach space, let C be a nonempty bounded closed convex subset of E, let $A : C \times C \to E^*$ be a relaxed η-ξ semi-monotone mapping. Let $f : C \times C \to \mathcal{R}$ be a bifunction satisfying* (A1), (A3), *and* (A4), *and let $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $r > 0$ and $z \in C$. Assume that*

(i) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$;*

(ii) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v, w), \eta(x, u)\rangle$ is convex and lower semicontinuous;*

(iii) *for each $x \in C$, $A(x, \cdot) : C \to E^*$ is finite-dimensional continuous.*

(iv) *$\xi : E \to \mathcal{R}$ is convex lower semicontinuous.*

Then, there exists $u_0 \in C$ such that

$$f(u_0, v) + \langle A(u_0, u_0), \eta(v, u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle \geq \varphi(u_0), \quad \forall v \in C. \tag{3.15}$$

If E is reflexive (i.e., $E = E^{**}$) smooth and strictly convex, A is semi-monotone, then we obtain the following result.

Corollary 3.4. *Let E be a reflexive smooth and strictly convex Banach space, let C be a nonempty bounded closed convex subset of E, let $A : C \times C \to E^*$ be a semi-monotone mapping. Let $f : C \times C \to \mathcal{R}$ be a bifunction satisfying* (A1), (A3), *and* (A4), *and let $\varphi : C \to \mathcal{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that, for any $r > 0$ and $z \in C$,*

(i) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v, w), x - u\rangle$ is convex and lower semicontinuous;*

(ii) *for each $x \in C$, $A(x, \cdot) : C \to E^*$ is finite-dimensional continuous.*

Then, there exists $u_0 \in C$ such that

$$f(u_0, v) + \langle A(u_0, u_0), v - u_0\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle \geq \varphi(u_0), \quad \forall v \in C. \tag{3.16}$$

Theorem 3.5. *Let E be a real Banach space with the smooth and strictly convex second dual space E^{**}, let C be a nonempty, bounded, closed, and convex subset of E^{**}, let $A : C \times C \to E^*$ be a relaxed η-ξ semi-monotone mapping. Let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1)–(A4) *and let*

φ be a lower semicontinuous and convex function from C to $\mathcal{R}$. For any $r > 0$, define a mapping $\Phi_r : E^{**} \to C$ as follows:

$$\Phi_r(x) = \left\{ u \in C : f(u,v) + \langle A(u,u), \eta(v,u) \rangle + \varphi(v) + \frac{1}{r}\langle v-u, J(u-x)\rangle \geq \varphi(u),\ \forall v \in C \right\}, \tag{3.17}$$

for all $x \in E$. Assume that

(i) *$\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in C$;*

(ii) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v,w), \eta(x,u)\rangle$ is convex and lower semicontinuous;*

(iii) *for each $x \in C$, $A(x,\cdot) : C \to E^*$ is finite-dimensional continuous: that is, for any finite-dimensional subspace $F \subset E^{**}$, $A(x,\cdot) : C \cap F \to E^*$ is continuous;*

(iv) *$\xi : E^{**} \to \mathcal{R}$ is convex lower semicontinuous;*

(v) *for any $x, y \in C$, $\xi(x-y) + \xi(y-x) \geq 0$;*

(vi) *for any $x, y \in C$, $A(x,y) = A(y,x)$.*

Then, the following holds:

(1) *Φ_r is single-valued;*

(2) *$\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle$ for all $x, y \in E$;*

(3) *$F(\Phi_r) = \mathrm{GMEP}(f, A, \eta, \varphi)$;*

(4) *$\mathrm{GMEP}(f, A, \eta, \varphi)$ is nonempty, closed, and convex.*

Proof. For each $x \in E^{**}$, by Lemma 2.10, we conclude that $\Phi_r(x)$ is nonempty.

(1) We prove that Φ_r is single-valued. Indeed, for $x \in E^{**}$ and $r > 0$, let $z_1, z_2 \in \Phi_r(x)$. Then,

$$\begin{aligned} &f(z_1,v) + \langle A(z_1,z_1), \eta(v,z_1)\rangle + \varphi(v) + \frac{1}{r}\langle v - z_1, J(z_1 - x)\rangle \geq \varphi(z_1), \quad \forall v \in C, \\ &f(z_2,v) + \langle A(z_2,z_2), \eta(v,z_2)\rangle + \varphi(v) + \frac{1}{r}\langle v - z_2, J(z_2 - x)\rangle \geq \varphi(z_2), \quad \forall v \in C. \end{aligned} \tag{3.18}$$

Hence,

$$\begin{aligned} &f(z_1,z_2) + \langle A(z_1,z_1), \eta(z_2,z_1)\rangle + \varphi(z_2) + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x)\rangle \geq \varphi(z_1), \\ &f(z_2,z_1) + \langle A(z_2,z_2), \eta(z_1,z_2)\rangle + \varphi(z_1) + \frac{1}{r}\langle z_1 - z_2, J(z_2 - x)\rangle \geq \varphi(z_2). \end{aligned} \tag{3.19}$$

Adding the two inequalities, from (i) we have

$$f(z_2,z_1) + f(z_1,z_2) + \langle A(z_1,z_1) - A(z_2,z_2), \eta(z_2,z_1)\rangle + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq 0. \tag{3.20}$$

From (A2), we have

$$\left\langle A(z_1, z_1) - A(z_2, z_2), \eta(z_2, z_1) \right\rangle + \frac{1}{r} \langle z_2 - z_1, J(z_1 - x) - J(z_2 - x) \rangle \geq 0. \tag{3.21}$$

That is,

$$\frac{1}{r} \langle z_2 - z_1, J(z_1 - x) - J(z_2 - x) \rangle \geq \left\langle A(z_2, z_2) - A(z_1, z_1), \eta(z_2, z_1) \right\rangle. \tag{3.22}$$

Calculating the right-hand side of (3.22), we have

$$\begin{aligned}
&\left\langle A(z_2, z_2) - A(z_1, z_1), \eta(z_2, z_1) \right\rangle \\
&\quad = \left\langle A(z_2, z_2) - A(z_2, z_1) + A(z_2, z_1) - A(z_1, z_2) + A(z_1, z_2) - A(z_1, z_1), \eta(z_2, z_1) \right\rangle \\
&\quad = \left\langle A(z_2, z_2) - A(z_2, z_1), \eta(z_2, z_1) \right\rangle + \left\langle A(z_2, z_1) - A(z_1, z_2), \eta(z_2, z_1) \right\rangle \\
&\qquad + \left\langle A(z_1, z_2) - A(z_1, z_1), \eta(z_2, z_1) \right\rangle \\
&\quad \geq 2\xi(z_2 - z_1) + \left\langle A(z_2, z_1) - A(z_1, z_2), \eta(z_2, z_1) \right\rangle,
\end{aligned} \tag{3.23}$$

and so,

$$\frac{1}{r} \langle z_2 - z_1, J(z_1 - x) - J(z_2 - x) \rangle \geq 2\xi(z_2 - z_1) + \left\langle A(z_2, z_1) - A(z_1, z_2), \eta(z_2, z_1) \right\rangle. \tag{3.24}$$

In (3.24) exchanging the position of z_1 and z_2, we get

$$\frac{1}{r} \langle z_1 - z_2, J(z_2 - x) - J(z_1 - x) \rangle \geq 2\xi(z_1 - z_2) + \left\langle A(z_1, z_2) - A(z_2, z_1), \eta(z_1, z_2) \right\rangle. \tag{3.25}$$

Adding the inequalities (3.24) and (3.25) and using (v) and (vi), we have

$$\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x) \rangle \geq r(\xi(z_2 - z_1) + \xi(z_1 - z_2)) \geq 0. \tag{3.26}$$

Hence,

$$0 \leq \langle z_2 - z_1, J(z_1 - x) - J(z_2 - x) \rangle = \langle (z_2 - x) - (z_1 - x), J(z_1 - x) - J(z_2 - x) \rangle. \tag{3.27}$$

Since J is monotone and E^{**} is strictly convex, we obtain that $z_1 - x = z_2 - x$ and hence $z_1 = z_2$. Therefore, Φ_r is single-valued.

(2) For $x, y \in C$, we have

$$f(\Phi_r x, \Phi_r y) + \langle A(\Phi_r x, \Phi_r x), \eta(\Phi_r y, \Phi_r x)\rangle + \varphi(\Phi_r y) - \varphi(\Phi_r x) + \frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x)\rangle \geq 0,$$
$$f(\Phi_r y, \Phi_r x) + \langle A(\Phi_r y, \Phi_r y), \eta(\Phi_r x, \Phi_r y)\rangle + \varphi(\Phi_r x) - \varphi(\Phi_r y) + \frac{1}{r}\langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle \geq 0. \tag{3.28}$$

Adding the above two inequalities and by (i) and (A2), we get

$$\langle A(\Phi_r x, \Phi_r x) - A(\Phi_r y, \Phi_r y), \eta(\Phi_r y, \Phi_r x)\rangle + \frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq 0, \tag{3.29}$$

that is

$$\frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq \langle A(\Phi_r y, \Phi_r y) - A(\Phi_r x, \Phi_r x), \eta(\Phi_r y, \Phi_r x)\rangle. \tag{3.30}$$

After calculating (3.30), we have

$$\frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq 2\xi(\Phi_r y, \Phi_r x) + \langle A(\Phi_r y, \Phi_r x) - A(\Phi_r x, \Phi_r y), \eta(\Phi_r y, \Phi_r x)\rangle. \tag{3.31}$$

In (3.30), exchanging the position of $\Phi_r x$ and $\Phi_r y$, we get

$$\frac{1}{r}\langle \Phi_r x - \Phi_r y, J(\Phi_r y - y) - J(\Phi_r x - x)\rangle \geq 2\xi(\Phi_r x, \Phi_r y) + \langle A(\Phi_r x, \Phi_r y) - A(\Phi_r y, \Phi_r x), \eta(\Phi_r x, \Phi_r y)\rangle. \tag{3.32}$$

Adding the inequalities (3.31) and (3.32), use (i) and (vi), we have

$$\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq r\left(\xi(\Phi_r x, \Phi_r y) + \xi(\Phi_r y, \Phi_r x)\right). \tag{3.33}$$

It follows from (iv) that

$$\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq 0. \tag{3.34}$$

Hence,

$$\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x) \rangle \le \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y) \rangle. \tag{3.35}$$

(3) Next, we show that $F(\Phi_r) = \text{GMEP}(f, A, \eta, \varphi)$. Indeed, we have the following:

$$\begin{aligned} u \in F(\Phi_r) &\Longleftrightarrow u = \Phi_r u \\ &\Longleftrightarrow f(u,v) + \langle A(u,u), \eta(v,u) \rangle + \varphi(v) + \frac{1}{r}\langle v - u, J(u-u) \rangle \ge \varphi(u), \quad \forall v \in C \\ &\Longleftrightarrow f(u,v) + \langle A(u,u), \eta(v,u) \rangle + \varphi(v) \ge \varphi(u), \quad \forall v \in C \\ &\Longleftrightarrow u \in \text{GMEP}(f, A, \eta, \varphi). \end{aligned} \tag{3.36}$$

Hence, $F(\Phi_r) = \text{GMEP}(f, A, \eta, \varphi)$.

(4) Finally, we prove that $\text{GMEP}(f, A, \eta, \varphi)$ is nonempty, closed, and convex. For each $v \in C$, we define the multivalued mapping $G : C \to 2^{E^{**}}$ by

$$G(v) = \{ u \in C : f(u,v) + \langle A(u,u), \eta(v,u) \rangle + \varphi(v) \ge \varphi(u) \}. \tag{3.37}$$

Since $v \in G(v)$, we have $G(v) \ne \emptyset$. We prove that G is a KKM mapping on C. Suppose that there exists a finite subset $\{z_1, z_2, \ldots, z_m\}$ of C, and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\widehat{z} = \sum_{i=1}^m \alpha_i z_i \notin G(z_i)$ for all $i = 1, 2, \ldots, m$. Then

$$f(\widehat{z}, z_i) + \langle A(\widehat{z}, \widehat{z}), \eta(z_i, \widehat{z}) \rangle + \varphi(z_i) - \varphi(\widehat{z}) < 0, \quad i = 1, 2, \ldots, m. \tag{3.38}$$

From (A1), (A4), (ii), and the convexity of φ, we have

$$\begin{aligned} 0 &= f(\widehat{z}, \widehat{z}) + \langle A(\widehat{z}, \widehat{z}), \eta(\widehat{z}, \widehat{z}) \rangle + \varphi(\widehat{z}) - \varphi(\widehat{z}) \\ &= f\left(\widehat{z}, \sum_{i=1}^m \alpha_i z_i \right) + \left\langle A(\widehat{z}, \widehat{z}), \eta\left(\sum_{i=1}^m \alpha_i z_i, \widehat{z} \right) \right\rangle + \varphi\left(\sum_{i=1}^m \alpha_i z_i \right) - \varphi(\widehat{z}) \\ &\le \sum_{i=1}^m \alpha_i \left(f(\widehat{z}, z_i) + \langle A(\widehat{z}, \widehat{z}), \eta(z_i, \widehat{z}) \rangle + \varphi(z_i) - \varphi(\widehat{z}) \right) \\ &< 0, \end{aligned} \tag{3.39}$$

which is a contradiction. Thus, G is a KKM mapping on C.

Next, we prove that $G(y)$ is closed for each $y \in C$. For any $y \in C$, let $\{x_n\}$ be any sequence in $G(y)$ such that $x_n \to x_0$. We claim that $x_0 \in G(y)$. Then, for each $y \in C$, we have

$$f(x_n, y) + \langle A(x_n, x_n), \eta(y, x_n) \rangle + \varphi(y) \ge \varphi(x_n). \tag{3.40}$$

By monotonicity of A, we obtain that

$$f(x_n, y) + \langle A(x_n, y), \eta(y, x_n)\rangle + \varphi(y) \geq \varphi(x_n) + \xi(y - x_n). \tag{3.41}$$

By (A3), (i), (ii), (iv), lower semicontinuity of φ, and the complete continuity of A, we obtain the following

$$\begin{aligned}
\varphi(x_0) + \langle A(x_0, y), \eta(x_0, y)\rangle &\leq \liminf_{n\to\infty} \varphi(x_n) + \liminf_{n\to\infty} \langle A(x_n, y), \eta(x_n, y)\rangle \\
&\leq \liminf_{n\to\infty} (\varphi(x_n) + \langle A(x_n, y), \eta(x_n, y)\rangle) \\
&= \liminf_{n\to\infty} (\varphi(x_n) - \langle A(x_n, y), \eta(y, x_n)\rangle) \\
&\leq \limsup_{n\to\infty} (\varphi(x_n) - \langle A(x_n, y), \eta(y, x_n)\rangle) \\
&\leq \limsup_{n\to\infty} (f(x_n, y) + \varphi(y) - \xi(y - x_n)) \\
&\leq f(x_0, y) + \varphi(y) - \xi(y - x_0).
\end{aligned} \tag{3.42}$$

Hence,

$$f(x_0, y) + \langle A(x_0, y), \eta(y, x_0)\rangle + \varphi(y) \geq \varphi(x_0) + \xi(y - x_0), \quad \forall y \in C. \tag{3.43}$$

From Lemma 2.9, we have

$$f(x_0, y) + \langle A(x_0, x_0), \eta(y, x_0)\rangle + \varphi(y) \geq \varphi(x_0), \quad \forall y \in C. \tag{3.44}$$

This shows that $x_0 \in G(y)$, and hence $G(y)$ is closed for each $y \in C$. Thus, $\mathrm{GMEP}(f, A, \eta, \varphi) = \bigcap_{y\in C} G(y)$ is also closed.

Next, we observe that $G(y)$ is weakly compact. In fact, since C is bounded, closed, and convex, we also have $G(y)$, which is weakly compact in the weak topology. By Lemma 2.6, we can conclude that $\bigcap_{y\in C} G(y) = \mathrm{GMEP}(f, A, \eta, \varphi) \neq \emptyset$.

Finally, we prove that $\mathrm{GMEP}(f, A, \eta, \varphi)$ is convex. In fact, let $u, v \in F(\Phi_r)$, and $z_t = tu + (1-t)v$ for $t \in (0,1)$. From (2), we know that

$$\langle \Phi_r u - \Phi_r z_t, J(\Phi_r z_t - z_t) - J(\Phi_r u - u)\rangle \geq 0. \tag{3.45}$$

This yields that

$$\langle u - \Phi_r z_t, J(\Phi_r z_t - z_t)\rangle \geq 0. \tag{3.46}$$

Similarly, we also have

$$\langle v - \Phi_r z_t, J(\Phi_r z_t - z_t)\rangle \geq 0. \tag{3.47}$$

It follows from (3.46) and (3.47) that

$$\begin{aligned}\|z_t - \Phi_r z_t\|^2 &= \langle z_t - \Phi_r z_t, J(z_t - \Phi_r z_t)\rangle \\ &= t\langle u - \Phi_r z_t, J(z_t - \Phi_r z_t)\rangle + (1-t)\langle v - \Phi_r z_t, J(z_t - \Phi_r z_t)\rangle \\ &\leq 0.\end{aligned} \tag{3.48}$$

Hence, $z_t \in F(\Phi_r) = \text{GMEP}(f, A, \eta, \varphi)$ and hence $\text{GMEP}(f, A, \eta, \varphi)$ is convex. This completes the proof. □

If E is reflexive (i.e., $E = E^{**}$) smooth and strictly convex, then the following result can be derived as a corollary of Theorem 3.5

Corollary 3.6. *Let E be a reflexive smooth and strictly convex Banach space, let C be a nonempty, bounded, closed, and convex subset of E, and let $A : C \times C \to E^*$ be a relaxed η-ξ semi-monotone mapping. Let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1)–(A4) *and let φ be a lower semicontinuous and convex function from C to $\mathcal{R}$. Let $r > 0$ and $z \in C$ and define a mapping $\Phi_r : E \to C$ as follows:*

$$\Phi_r(x) = \left\{u \in C : f(u, v) + \langle A(u, u), \eta(v, u)\rangle + \varphi(v) + \frac{1}{r}\langle v - u, J(u - x)\rangle \geq \varphi(u),\ \forall v \in C\right\}, \tag{3.49}$$

for all $x \in E$. Assume that

(i) $\eta(x, y) + \eta(y, x) = 0$ *for all $x, y \in C$;*

(ii) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v, w), \eta(x, u)\rangle$ is convex and lower semicontinuous;*

(iii) *for each $x \in C$, $A(x, \cdot) : C \to E^*$ is finite-dimensional continuous;*

(iv) *$\xi : E \to \mathcal{R}$ is convex lower semicontinuous;*

(v) *for any $x, y \in C$, $\xi(x - y) + \xi(y - x) \geq 0$;*

(vi) *for any $x, y \in C$, $A(x, y) = A(y, x)$.*

Then, the following holds:

(1) *Φ_r is single-valued;*

(2) *$\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle$ for all $x, y \in E$;*

(3) *$F(\Phi_r) = \text{GMEP}(f, A, \eta, \varphi)$;*

(4) *$\text{GMEP}(f, A, \eta, \varphi)$ is nonempty, closed, and convex.*

4. Strong Convergence Theorems

In this section, we prove a strong convergence theorem by using a hybrid projection algorithm for an asymptotically nonexpansive mapping in a uniformly convex and smooth Banach space.

Theorem 4.1. *Let E be a real Banach space with the smooth and uniformly convex second dual space E^{**}, let C be a nonempty, bounded, closed, and convex subset of E^{**}. Let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1)–(A4), *and let φ be a lower semicontinuous and convex function from C to $\mathcal{R}$. Let $A : C \times C \to E^*$ be a relaxed η-ξ semi-monotone and let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\Omega := F(S) \cap \mathrm{GMEP}(f, A, \eta, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{aligned}
&x_0 \in C, \quad D_0 = C_0 = C,\\
&C_n = \overline{\mathrm{co}}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \quad n \geq 1,\\
&u_n \in C \text{ such that}\\
&f(u_n, y) + \varphi(y) + \langle A(u_n, u_n), \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq \varphi(u_n), \quad \forall y \in C,\ n \geq 0,\\
&D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1,\\
&x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0,
\end{aligned} \tag{4.1}$$

where $\{t_n\}$ and $\{r_n\}$ are real sequences in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$, and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly, as $n \to \infty$, to $P_\Omega x_0$.

Proof. Firstly, we rewrite the (4.1) as follows:

$$\begin{aligned}
&x_0 \in C, \quad D_0 = C_0 = C,\\
&C_n = \overline{\mathrm{co}}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \quad n \geq 0,\\
&D_n = \{z \in D_{n-1} : \langle \Phi_{r_n} x_n - z, J(x_n - \Phi_{r_n} x_n)\rangle \geq 0\}, \quad n \geq 1,\\
&x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0,
\end{aligned} \tag{4.2}$$

where Φ_r is the mapping defined by

$$\Phi_r(x) = \left\{z \in C : f(z, y) + \langle A(z, z), \eta(y, z)\rangle + \varphi(y) + \frac{1}{r}\langle y - z, J(z - x)\rangle \geq \varphi(z),\ \forall y \in C\right\}. \tag{4.3}$$

We first show that the sequence $\{x_n\}$ is well defined. It is easy to verify that $C_n \cap D_n$ is closed and convex and $\Omega \subset C_n$ for all $n \geq 0$. Next, we prove that $\Omega \subset C_n \cap D_n$. Since $D_0 = C$, we also have $\Omega \subset C_0 \cap D_0$. Suppose that $\Omega \subset C_{k-1} \cap D_{k-1}$ for $k \geq 2$. It follows from Theorem 3.5 (2) that

$$\langle \Phi_{r_k} x_k - \Phi_{r_k} u, J(\Phi_{r_k} u - u) - J(\Phi_{r_k} x_k - x_k)\rangle \geq 0, \tag{4.4}$$

for all $u \in \Omega$. This implies that

$$\langle \Phi_{r_k} x_k - u, J(x_k - \Phi_{r_k} x_k) \rangle \geq 0, \tag{4.5}$$

for all $u \in \Omega$. Hence $\Omega \subset D_k$. By the mathematical induction, we get that $\Omega \subset C_n \cap D_n$ for each $n \geq 0$, and hence $\{x_n\}$ is welldefined. Put $w = P_\Omega x_0$. Since $\Omega \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n}$, we have

$$\|x_{n+1} - x_0\| \leq \|w - x_0\|, \quad n \geq 0. \tag{4.6}$$

Since $x_{n+2} \in D_{n+1} \subset D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|x_{n+2} - x_0\|. \tag{4.7}$$

Since $\{\|x_n - x_0\|\}$ is bounded, we have $\lim_{n\to\infty} \|x_n - x_0\| = d$ for some a constant d. Moreover, by the convexity of D_n, we also have $(1/2)(x_{n+1} + x_{n+2}) \in D_n$ and hence

$$\|x_0 - x_{n+1}\| \leq \left\| x_0 - \frac{x_{n+1} + x_{n+2}}{2} \right\| \leq \frac{1}{2}(\|x_0 - x_{n+1}\| + \|x_0 - x_{n+2}\|). \tag{4.8}$$

This implies that

$$\lim_{n\to\infty} \left\| \frac{1}{2}(x_0 - x_{n+1}) + \frac{1}{2}(x_0 - x_{n+2}) \right\| = \lim_{n\to\infty} \left\| x_0 - \frac{x_{n+1} + x_{n+2}}{2} \right\| = d. \tag{4.9}$$

By Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{4.10}$$

Next, we show that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{4.11}$$

To obtain (4.11), we need to show that $\lim_{n\to\infty} \|x_n - S^{n-k} x_n\| = 0$, for all $k \in \mathcal{N}$.

Fix $k \in \mathcal{N}$ and put $m = n - k$. Since $x_n = P_{C_{n-1} \cap D_{n-1}} x$, we have $x_n \in C_{n-1} \subseteq \cdots \subseteq C_m$. Since $t_m > 0$, there exist $y_1, \ldots, y_N \in C$ and nonnegative numbers $\lambda_1, \ldots, \lambda_N$ with $\lambda_1 + \cdots + \lambda_N = 1$ such that

$$\left\| x_n - \sum_{i=1}^{N} \lambda_i y_i \right\| < t_m, \tag{4.12}$$

and $\|y_i - S^m y_i\| \le t_m \|x_m - S^m x_m\|$ for all $i \in \{1,\ldots,N\}$. Put $M = \sup_{x\in C}\|x\|$, $u = P_{F(S)}x$ and $r_0 = \sup_{n\ge 1}(1+k_n)\|x_n - u\|$. Since C and $\{k_m\}$ are bounded, (4.12) implies

$$\left\|x_n - \frac{1}{k_m}\sum_{i=1}^{N}\lambda_i y_i\right\| \le \left(1-\frac{1}{k_m}\right)\|x\| + \frac{1}{k_m}\left\|x_n - \sum_{i=1}^{N}\lambda_i y_i\right\| \le \left(1-\frac{1}{k_m}\right)M + t_m, \tag{4.13}$$

and $\|y_i - S^m y_i\| \le t_m\|x_m - S^m x_m\| \le t_m(1+k_m)\|x_m - u\| \le r_0 t_m$ for all $i \in \{1,\ldots,N\}$. Therefore,

$$\left\|y_i - \frac{1}{k_m}S^m y_i\right\| \le \left(1-\frac{1}{k_m}\right)M + r_0 t_m, \tag{4.14}$$

for all $i \in \{1,\ldots,N\}$. Moreover, asymptotically nonexpansiveness of S and (4.6) give that

$$\left\|\frac{1}{k_m}S^m\left(\sum_{i=1}^{N}\lambda_i y_i\right) - S^m x_n\right\| \le \left(1-\frac{1}{k_m}\right)M + t_m. \tag{4.15}$$

It follows from Theorem 2.4, (4.13)–(4.15) that

$$\begin{aligned}
\|x_n - S^m x_n\| &\le \left\|x_n - \frac{1}{k_m}\sum_{i=1}^{N}\lambda_i y_i\right\| + \frac{1}{k_m}\left\|\sum_{i=1}^{N}\lambda_i(y_i - S^m y_i)\right\| \\
&\quad + \frac{1}{k_m}\left\|\sum_{i=1}^{N}\lambda_i S^m y_i - S^m\left(\sum_{i=1}^{N}\lambda_i y_i\right)\right\| + \left\|\frac{1}{k_m}S^m\left(\sum_{i=1}^{N}\lambda_i y_i\right) - S^m x_n\right\| \\
&\le 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m} + \gamma^{-1}\left(\max_{1\le i\le j\le N}\left(\left\|y_i - y_j\right\| - \frac{1}{k_m}\left\|S^m y_i - S^m y_j\right\|\right)\right) \\
&\le 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m} + \gamma^{-1}\left(\max_{1\le i\le j\le N}\left(\left\|y_i - \frac{1}{k_m}S^m y_i\right\| + \left\|y_j - \frac{1}{k_m}S^m y_j\right\|\right)\right) \\
&\le 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m} + \gamma^{-1}\left(2\left(1-\frac{1}{k_m}\right)M + 2r_0 t_m\right).
\end{aligned} \tag{4.16}$$

Since $\lim_{n\to\infty}k_n = 1$ and $\lim_{n\to\infty}t_n = 0$, it follows from the last inequality that $\lim_{n\to\infty}\|x_n - S^m x_n\| = 0$. We have that

$$\begin{aligned}
\|x_n - Sx_n\| &= \left\|x_n - S^{n-1}x_n\right\| + \left\|S^{n-1}x_n - Sx_n\right\| \\
&\le \left\|x_n - S^{n-1}x_n\right\| + k_1\left\|S^{n-2}x_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{4.17}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x} \in C$. Therefore, we obtain $\tilde{x} \in F(S)$. Next, we show that $\tilde{x} \in \mathrm{GMEP}(f, A, \eta, \varphi)$. By the construction of D_n, we see from Theorem 2.1 that $\Phi_{r_n}x_n = P_{D_n}x_n$. Since $x_{n+1} \in D_n$, we get

$$\|x_n - \Phi_{r_n}x_n\| \le \|x_n - x_{n+1}\| \longrightarrow 0. \tag{4.18}$$

From (C2), we also have

$$\frac{1}{r_n}\|J(x_n - \Phi_{r_n}x_n)\| = \frac{1}{r_n}\|x_n - \Phi_{r_n}x_n\| \longrightarrow 0, \tag{4.19}$$

as $n \to \infty$. By (4.19), we also have $\Phi_{r_{n_i}}x_{n_i} \rightharpoonup \tilde{x}$. By the definition of $\Phi_{r_{n_i}}$, for each $y \in C$, we obtain

$$\begin{aligned} &f\left(\Phi_{r_{n_i}}x_{n_i}, y\right) + \left\langle A\left(\Phi_{r_{n_i}}x_{n_i}, \Phi_{r_{n_i}}x_{n_i}\right), \eta\left(y, \Phi_{r_{n_i}}x_{n_i}\right)\right\rangle + \varphi(y) \\ &\quad + \frac{1}{r_{n_i}}\left\langle y - \Phi_{r_{n_i}}x_{n_i}, J\left(\Phi_{r_{n_i}}x_{n_i} - x_{n_i}\right)\right\rangle \geq \varphi\left(\Phi_{r_{n_i}}x_{n_i}\right). \end{aligned} \tag{4.20}$$

By (A3), (4.19), (ii), the weakly lower semicontinuity of φ and complete continuity of A we have

$$\begin{aligned} \varphi(\tilde{x}) &\leq \liminf_{i\to\infty} \varphi\left(\Phi_{r_{n_i}}x_{n_i}\right) \\ &\leq \liminf_{i\to\infty} f\left(\Phi_{r_{n_i}}x_{n_i}, y\right) + \liminf_{i\to\infty}\left\langle A\left(\Phi_{r_{n_i}}x_{n_i}, \Phi_{r_{n_i}}x_{n_i}\right), \eta\left(y, \Phi_{r_{n_i}}x_{n_i}\right)\right\rangle \\ &\quad + \varphi(y) + \liminf_{i\to\infty}\frac{1}{r_{n_i}}\left\langle y - \Phi_{r_{n_i}}x_{n_i}, J\left(\Phi_{r_{n_i}}x_{n_i} - x_{n_i}\right)\right\rangle \\ &\leq f(\tilde{x}, y) + \varphi(y) + \langle A(\tilde{x}, \tilde{x}), \eta(y, \tilde{x})\rangle. \end{aligned} \tag{4.21}$$

Hence,

$$f(\tilde{x}, y) + \varphi(y) + \langle A(\tilde{x}, \tilde{x}), \eta(y, \tilde{x})\rangle \geq \varphi(\tilde{x}). \tag{4.22}$$

This shows that $\tilde{x} \in \mathrm{GMEP}(f, A, \eta, \varphi)$, and hence $\tilde{x} \in \Omega := F(S) \cap \mathrm{GMEP}(f, A, \eta, \varphi)$.

Finally, we show that $x_n \to w$ as $n \to \infty$, where $w := P_\Omega x_0$. By the weakly lower semicontinuity of the norm, it follows from (4.6) that

$$\|x_0 - w\| \leq \|x_0 - \tilde{x}\| \leq \liminf_{i\to\infty}\|x_0 - x_{n_i}\| \leq \limsup_{i\to\infty}\|x_0 - x_{n_i}\| \leq \|x_0 - w\|. \tag{4.23}$$

This shows that

$$\lim_{i\to\infty}\|x_0 - x_{n_i}\| = \|x_0 - w\| = \|x_0 - \tilde{x}\|, \tag{4.24}$$

and $\tilde{x} = w$. Since E^{**} is uniformly convex, we obtain that $x_0 - x_{n_i} \to x_0 - w$. It follows that $x_{n_i} \to w$. So, we have $x_n \to w$ as $n \to \infty$. This completes the proof. □

If S is a nonexpansive mapping in Theorem 4.1, then we obtain the following result concerning the problem of finding a common element of $\mathrm{GMEP}(f, A, \eta, \varphi)$ and the fixed point set of a nonexpansive mapping in a Banach space setting.

Theorem 4.2. *Let E be a real Banach space with the smooth and uniformly convex second dual space E^{**}, let C be a nonempty, bounded, closed, and convex subset of E^{**}. Let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1)–(A4) *and let φ be a lower semicontinuous and convex function from C to $\mathcal{R}$. Let $A : C \times C \to E^*$ be a relaxed η-ξ semi-monotone and let $S : C \to C$ be a nonexpansive mapping such that $\Omega := F(S) \cap \mathrm{GMEP}(f, A, \eta, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$
\begin{aligned}
& x_0 \in C, \quad D_0 = C_0 = C, \\
& C_n = \overline{\mathrm{co}}\{z \in C_{n-1} : \|z - Sz\| \le t_n \|x_n - Sx_n\|\}, \quad n \ge 1, \\
& u_n \in C \text{ such that} \\
& f(u_n, y) + \varphi(y) + \langle A(u_n, u_n), \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \ge \varphi(u_n), \quad \forall y \in C,\ n \ge 0, \\
& D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \ge 0\}, \quad n \ge 1, \\
& x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \ge 0,
\end{aligned}
\tag{4.25}
$$

where $\{t_n\}$ and $\{r_n\}$ are real sequences in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$, and $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $P_\Omega x_0$.

Putting $A \equiv 0$ and $\varphi \equiv 0$ in Theorem 4.1, then we have the following result in a Banach space.

Theorem 4.3. *Let E be a real Banach space with the smooth and uniformly convex second dual space E^{**} and let C be a nonempty, bounded, closed, and convex subset of E^{**}. Let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying* (A1)–(A4). *Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\Omega := F(S) \cap \mathrm{EP}(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$
\begin{aligned}
& x_0 \in C, \quad D_0 = C_0 = C, \\
& C_n = \overline{\mathrm{co}}\{z \in C_{n-1} : \|z - S^n z\| \le t_n \|x_n - S^n x_n\|\}, \quad n \ge 1, \\
& u_n \in C \text{ such that} \\
& f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \ge 0, \quad \forall y \in C,\ n \ge 0, \\
& D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \ge 0\}, \quad n \ge 1, \\
& x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \ge 0,
\end{aligned}
\tag{4.26}
$$

where $\{t_n\}$ and $\{r_n\}$ are real sequences in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$, and $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $P_\Omega x_0$.

Putting $f \equiv 0$, $A \equiv 0$, $\varphi \equiv 0$, and $r_n \equiv 1$ in Theorem 4.1 and applying Theorem 2.1, we get $x_n = u_n$. Then, we have the following new approximation method concerning the problem of finding a fixed of an asymptotically nonexpansive mapping in a Banach space.

Theorem 4.4. *Let E be a real Banach space with the smooth and uniformly convex second dual space E^{**}, let C be a nonempty, bounded, closed, and convex subset of E^{**}. Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{gathered} x_0 \in C, \quad C_0 = C, \\ C_n = \overline{\text{co}}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n} x_0, \quad n \geq 0, \end{gathered} \tag{4.27}$$

where $\{t_n\}$ and $\{r_n\}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$. Then $\{x_n\}$ converges strongly, as $n \to \infty$, to $P_{F(S)} x_0$.

If E is reflexive (i.e., $E = E^{**}$) smooth and uniformly convex, then the following results can be derived as a corollary of Theorem 4.4.

Corollary 4.5. *Let E be a reflexive smooth and uniformly convex real Banach space, let C be a nonempty, bounded, closed, and convex subset of E. Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty]$ such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{gathered} x_0 \in C, \quad C_0 = C, \\ C_n = \overline{\text{co}}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n} x_0, \quad n \geq 0, \end{gathered} \tag{4.28}$$

where $\{t_n\}$ and $\{r_n\}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$. Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $P_{F(S)} x_0$.

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