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Research Article

Perturbation Analysis for the Matrix Equation $X - \sum_{i=1}^{m} A_i^* X A_i + \sum_{j=1}^{n} B_j^* X B_j = I$

Xue-Feng Duan[1,2] and Qing-Wen Wang[3]

[1] *School of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin 541004, China*
[2] *Research Center on Data Science and Social Computing, Guilin University of Electronic Technology, Guilin 541004, China*
[3] *Department of Mathematics, Shanghai University, Shanghai 200444, China*

Correspondence should be addressed to Qing-Wen Wang, wqw858@yahoo.com.cn

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We consider the perturbation analysis of the matrix equation $X - \sum_{i=1}^{m} A_i^* X A_i + \sum_{j=1}^{n} B_j^* X B_j = I$. Based on the matrix differentiation, we first give a precise perturbation bound for the positive definite solution. A numerical example is presented to illustrate the sharpness of the perturbation bound.

1. Introduction

In this paper, we consider the matrix equation

$$X - \sum_{i=1}^{m} A_i^* X A_i + \sum_{j=1}^{n} B_j^* X B_j = I, \tag{1.1}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_n$ are $n \times n$ complex matrices, I is an $n \times n$ identity matrix, m, n are nonnegative integers and the positive definite solution X is practical interest. Here, A_i^* and B_i^* denote the conjugate transpose of the matrices A_i and B_i, respectively. Equation (1.1) arises in solving some nonlinear matrix equations with Newton method. See, for example, the nonlinear matrix equation which appears in Sakhnovich [1]. Solving these nonlinear matrix equations gives rise to (1.1). On the other hand, (1.1) is the general case of the generalized Lyapunov equation

$$MYS^* + SYM^* + \sum_{k=1}^{t} N_k Y N_k^* + CC^* = 0, \tag{1.2}$$

whose positive definite solution is the controllability Gramian of the bilinear control system (see [2, 3] for more details)

$$M\dot{x}(t) = Sx(t) + \sum_{k=1}^{t} N_k x(t) u_k(t) + Cu(t). \tag{1.3}$$

Set

$$\begin{gathered} E = \frac{1}{\sqrt{2}}(M - S + I), \qquad F = \frac{1}{\sqrt{2}}(M + S + I), \qquad G = S - I, \qquad Q = CC^*, \\ X = Q^{-1/2} Y Q^{-1/2}, \qquad A_i = Q^{-1/2} N_i Q^{1/2}, \quad i = 1, 2, \ldots, t, \\ A_{t+1} = Q^{-1/2} F Q^{1/2}, \qquad A_{t+2} = Q^{-1/2} G Q^{1/2}, \\ B_1 = Q^{-1/2} E Q^{1/2}, \qquad B_2 = Q^{-1/2} C Q^{1/2}, \end{gathered} \tag{1.4}$$

then (1.2) can be equivalently written as (1.1) with $m = t + 2$ and $n = 2$.

Some special cases of (1.1) have been studied. Based on the kronecker product and fixed point theorem in partially ordered sets, Reurings [4] and Ran and Reurings [5, 6] gave some sufficient conditions for the existence of a unique positive definite solution of the linear matrix equations $X - \sum_{i=1}^{m} A_i^* X A_i = I$ and $X + \sum_{j=1}^{n} B_i^* X B_i = I$. And the expressions for these unique positive definite solutions were also derived under some constraint conditions. For the general linear matrix equation (1.1), Reurings [[4], Page 61] pointed out that it is hard to find sufficient conditions for the existence of a positive definite solution, because the map

$$G(X) = I + \sum_{i=1}^{m} A_i^* X A_i - \sum_{j=1}^{n} B_j^* X B_j \tag{1.5}$$

is not monotone and does not map the set of $n \times n$ positive definite matrices into itself. Recently, Berzig [7] overcame these difficulties by making use of Bhaskar-Lakshmikantham coupled fixed point theorem and gave a sufficient condition for (1.1) existing a unique positive definite solution. An iterative method was constructed to compute the unique positive definite solution, and the error estimation was given too.

Recently, the matrix equations of the form (1.1) have been studied by many authors (see [8–14]). Some numerical methods for solving the well-known Lyapunov equation $X + A^T X A = Q$, such as Bartels-Stewart method and Hessenberg-Schur method, have been proposed in [12]. Based on the fixed point theorem, the sufficient and necessary conditions for the existence of a positive definite solution of the matrix equation $X^s \pm A^* X^{-t} A = Q,\ s, t \in N$ have been given in [8, 9]. The fixed point iterative method and inversion-free iterative method were developed for solving the matrix equations $X \pm A^* X^{-\alpha} A = Q,\ \alpha > 0$ in [13, 14]. By making use of the fixed point theorem of mixed monotone operator, the matrix equation $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q,\ \ 0 < |\delta_i| < 1$ was studied in [10], and derived a sufficient condition for the existence of a positive definite solution. Assume that F maps positive definite matrices either into positive definite matrices or into negative definite matrices, the general nonlinear matrix equation $X + A^* F(X) A = Q$ was studied in [11], and the fixed point iterative method was constructed to compute the positive definite solution under some additional conditions.

Motivated by the works and applications in [2–6], we continue to study the matrix equation (1.1). Based on a new mathematical tool (i.e., the matrix differentiation), we firstly give a differential bound for the unique positive definite solution of (1.1), and then use it to derive a precise perturbation bound for the unique positive definite solution. A numerical example is used to show that the perturbation bound is very sharp.

Throughout this paper, we write $B > 0\ \ (B \geq 0)$ if the matrix B is positive definite (semidefinite). If $B - C$ is positive definite (semidefinite), then we write $B > C\ \ (B \geq C)$. If a positive definite matrix X satisfies $B \leq X \leq C$, we denote that $X \in [B, C]$. The symbols $\lambda_1(B)$ and $\lambda_n(B)$ denote the maximal and minimal eigenvalues of an $n \times n$ Hermitian matrix B, respectively. The symbol $H^{n\times n}$ stands for the set of $n \times n$ Hermitian matrices. The symbol $\|B\|$ denotes the spectral norm of the matrix B.

2. Perturbation Analysis for the Matrix Equation (1.1)

Based on the matrix differentiation, we firstly give a differential bound for the unique positive definite solution X_U of (1.1), and then use it to derive a precise perturbation bound for X_U in this section.

Definition 2.1 ([15], Definition 3.6). Let $F = (f_{ij})_{m\times n}$, then the matrix differentiation of F is $dF = (df_{ij})_{m\times n}$. For example, let

$$F = \begin{pmatrix} s+t & s^2-2t \\ 2s+t^3 & t^2 \end{pmatrix}. \tag{2.1}$$

Then

$$dF = \begin{pmatrix} ds+dt & 2sds-2dt \\ 2ds+3t^2dt & 2tdt \end{pmatrix}. \tag{2.2}$$

Lemma 2.2 ([15], Theorem 3.2). *The matrix differentiation has the following properties:*

(1) $d(F_1 \pm F_2) = dF_1 \pm dF_2$;

(2) $d(kF) = k(dF)$, *where k is a complex number;*

(3) $d(F^*) = (dF)^*$;

(4) $d(F_1F_2F_3) = (dF_1)F_2F_3 + F_1(dF_2)F_3 + F_1F_2(dF_3)$;

(5) $dF^{-1} = -F^{-1}(dF)F^{-1}$;

(6) $dF = 0$, *where F is a constant matrix.*

Lemma 2.3 ([7], Theorem 3.1). *If*

$$\sum_{i=1}^{m} A_i^* A_i < \frac{1}{2} I, \qquad \sum_{j=1}^{n} B_j^* B_j < \frac{1}{2} I, \tag{2.3}$$

then (1.1) *has a unique positive definite solution* X_U *and*

$$X_U \in \left[I - 2\sum_{j=1}^{n} B_j^* B_j,\ I + 2\sum_{i=1}^{m} A_i^* A_i \right]. \tag{2.4}$$

Theorem 2.4. *If*

$$\sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{2}, \qquad \sum_{j=1}^{n} \left\|B_j\right\|^2 < \frac{1}{2}, \tag{2.5}$$

then (1.1) *has a unique positive definite solution* X_U*, and it satisfies*

$$\|dX_U\| \le \frac{2\left(1 + 2\sum_{i=1}^{m} \|A_i\|^2\right)\left[\sum_{i=1}^{m} (\|A_i\| \|dA_i\|) + \sum_{j=1}^{n} \left(\left\|B_j\right\| \left\|dB_j\right\|\right)\right]}{1 - \sum_{i=1}^{m} \|A_i\|^2 - \sum_{j=1}^{n} \left\|B_j\right\|^2}. \tag{2.6}$$

Proof. Since

$$\begin{aligned} &\lambda_1 (A_i^* A_i) \le \left\|A_i^* A_i\right\| \le \|A_i\|^2, \quad i = 1, 2, \ldots, m, \\ &\lambda_1 \left(B_j^* B_j\right) \le \left\|B_j^* B_j\right\| \le \left\|B_j\right\|^2, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{2.7}$$

then

$$\begin{aligned} A_i^* A_i &\le \lambda_1 (A_i^* A_i) I \le \left\|A_i^* A_i\right\| I \le \|A_i\|^2 I, \quad i = 1, 2, \ldots, m, \\ B_j^* B_j &\le \lambda_1 \left(B_j^* B_j\right) I \le \left\|B_j^* B_j\right\| \le \left\|B_j\right\|^2 I, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{2.8}$$

consequently,

$$\begin{aligned} \sum_{i=1}^{m} A_i^* A_i &\le \sum_{i=1}^{m} \|A_i\|^2 I, \\ \sum_{j=1}^{n} B_j^* B_j &\le \sum_{j=1}^{n} \left\|B_j\right\|^2 I. \end{aligned} \tag{2.9}$$

Combining (2.5)–(2.9) we have

$$\sum_{i=1}^{m} A_i^* A_i \leq \sum_{i=1}^{m} \|A_i\|^2 I < \frac{1}{2} I,$$
$$\sum_{j=1}^{n} B_j^* B_j \leq \sum_{j=1}^{n} \left\|B_j\right\|^2 I < \frac{1}{2} I. \tag{2.10}$$

Then by Lemma 2.3 we obtain that (1.1) has a unique positive definite solution X_U, which satisfies

$$X_U \in \left[I - 2\sum_{j=1}^{n} B_j^* B_j,\ I + 2\sum_{i=1}^{m} A_i^* A_i \right]. \tag{2.11}$$

Noting that X_U is the unique positive definite solution of (1.1), then

$$X_U - \sum_{i=1}^{m} A_i^* X_U A_i + \sum_{j=1}^{n} B_j^* X_U B_j = I. \tag{2.12}$$

It is known that the elements of X_U are differentiable functions of the elements of A_i and B_i. Differentiating (2.12), and by Lemma 2.2, we have

$$\begin{aligned} dX_U &- \sum_{i=1}^{m} \left[(dA_i^*) X_U A_i + A_i^* (dX_U) A_i + A_i^* X_U (dA_i) \right] \\ &+ \sum_{j=1}^{n} \left[\left(dB_j^* \right) X_U B_j + B_j^* (dX_U) B_j + B_j^* X_U \left(dB_j \right) \right] = 0, \end{aligned} \tag{2.13}$$

which implies that

$$\begin{aligned} dX_U &- \sum_{i=1}^{m} A_i^* (dX_U) A_i + \sum_{j=1}^{n} B_j^* (dX_U) B_j \\ &= \sum_{i=1}^{m} (dA_i^*) X_U A_i + \sum_{i=1}^{m} A_i^* X_U (dA_i) - \sum_{j=1}^{n} \left(dB_j^* \right) X_U B_j - \sum_{j=1}^{n} B_j^* X_U \left(dB_j \right). \end{aligned} \tag{2.14}$$

By taking spectral norm for both sides of (2.14), we obtain that

$$\begin{aligned}
&\left\| dX_U - \sum_{i=1}^{m} A_i^*(dX_U)A_i + \sum_{j=1}^{n} B_j^*(dX_U)B_j \right\| \\
&= \left\| \sum_{i=1}^{m} (dA_i^*)X_U A_i + \sum_{i=1}^{m} A_i^* X_U (dA_i) - \sum_{j=1}^{n} \left(dB_j^*\right) X_U B_j - \sum_{j=1}^{n} B_j^* X_U (dB_j) \right\| \\
&\leq \left\| \sum_{i=1}^{m} (dA_i^*)X_U A_i \right\| + \left\| \sum_{i=1}^{m} A_i^* X_U (dA_i) \right\| + \left\| \sum_{j=1}^{n} \left(dB_j^*\right) X_U B_j \right\| + \left\| \sum_{j=1}^{n} B_j^* X_U (dB_j) \right\| \\
&\leq \sum_{i=1}^{m} \left\| (dA_i^*)X_U A_i \right\| + \sum_{i=1}^{m} \left\| A_i^* X_U (dA_i) \right\| + \sum_{j=1}^{n} \left\| \left(dB_j^*\right) X_U B_j \right\| + \sum_{j=1}^{n} \left\| B_j^* X_U (dB_j) \right\| \\
&\leq \sum_{i=1}^{m} \|dA_i^*\| \|X_U\| \|A_i\| + \sum_{i=1}^{m} \|A_i^*\| \|X_U\| \|dA_i\| + \sum_{j=1}^{n} \left\|dB_j^*\right\| \|X_U\| \|B_j\| + \sum_{j=1}^{n} \left\|B_j^*\right\| \|X_U\| \|dB_j\| \\
&= 2\sum_{i=1}^{m} (\|A_i\| \|X_U\| \|dA_i\|) + 2\sum_{j=1}^{n} \left(\|B_j\| \|X_U\| \|dB_j\|\right) \\
&= 2\left[\sum_{i=1}^{m} (\|A_i\| \|dA_i\|) + \sum_{j=1}^{n} \left(\|B_j\| \|dB_j\|\right) \right] \|X_U\|,
\end{aligned} \tag{2.15}$$

and noting (2.11) we obtain that

$$\|X_U\| \leq \left\| I + 2\sum_{i=1}^{m} A_i^* A_i \right\| \leq 1 + 2\sum_{i=1}^{m} \|A_i\|^2. \tag{2.16}$$

Then

$$\begin{aligned}
&\left\| dX_U - \sum_{i=1}^{m} A_i^*(dX_U)A_i + \sum_{j=1}^{n} B_j^*(dX_U)B_j \right\| \\
&\leq 2\left[\sum_{i=1}^{m} (\|A_i\| \|dA_i\|) + \sum_{j=1}^{n} \left(\|B_j\| \|dB_j\|\right) \right] \|X_U\| \\
&\leq 2\left(1 + 2\sum_{i=1}^{m} \|A_i\|^2 \right) \left[\sum_{i=1}^{m} (\|A_i\| \|dA_i\|) + \sum_{j=1}^{n} \left(\|B_j\| \|dB_j\|\right) \right],
\end{aligned} \tag{2.17}$$

$$
\begin{aligned}
&\left\| dX_U - \sum_{i=1}^{m} A_i^*(dX_U)A_i + \sum_{j=1}^{n} B_j^*(dX_U)B_j \right\| \\
&\quad \geq \|dX_U\| - \left\| \sum_{i=1}^{m} A_i^*(dX_U)A_i \right\| - \left\| \sum_{j=1}^{n} B_j^*(dX_U)B_j \right\| \\
&\quad \geq \|dX_U\| - \sum_{i=1}^{m} \left\| A_i^*(dX_U)A_i \right\| - \sum_{j=1}^{n} \left\| B_j^*(dX_U)B_j \right\| \\
&\quad \geq \|dX_U\| - \sum_{i=1}^{m} \left\| A_i^* \right\| \|dX_U\| \|A_i\| - \sum_{j=1}^{n} \left\| B_j^* \right\| \|dX_U\| \left\| B_j \right\| \\
&\quad = \left(1 - \sum_{i=1}^{m} \|A_i\|^2 - \sum_{j=1}^{n} \left\| B_j \right\|^2 \right) \|dX_U\|.
\end{aligned}
\tag{2.18}
$$

Due to (2.5) we have

$$
1 - \sum_{i=1}^{m} \|A_i\|^2 - \sum_{j=1}^{n} \left\| B_j \right\|^2 > 0. \tag{2.19}
$$

Combining (2.17), (2.18) and noting (2.19), we have

$$
\begin{aligned}
&\left(1 - \sum_{i=1}^{m} \|A_i\|^2 - \sum_{j=1}^{n} \left\| B_j \right\|^2 \right) \|dX_U\| \\
&\quad \leq \left\| dX_U - \sum_{i=1}^{m} A_i^*(dX_U)A_i + \sum_{j=1}^{n} B_j^*(dX_U)B_j \right\| \\
&\quad \leq 2\left(1 + 2\sum_{i=1}^{m} \|A_i\|^2 \right) \left[\sum_{i=1}^{m} (\|A_i\| \|dA_i\|) + \sum_{j=1}^{n} \left(\left\| B_j \right\| \left\| dB_j \right\| \right) \right],
\end{aligned}
\tag{2.20}
$$

which implies that

$$
\|dX_U\| \leq \frac{2\left(1 + 2\sum_{i=1}^{m} \|A_i\|^2 \right) \left[\sum_{i=1}^{m} (\|A_i\| \|dA_i\|) + \sum_{j=1}^{n} \left(\left\| B_j \right\| \left\| dB_j \right\| \right) \right]}{1 - \sum_{i=1}^{m} \|A_i\|^2 - \sum_{j=1}^{n} \left\| B_j \right\|^2}. \tag{2.21}
$$

□

Theorem 2.5. *Let* $\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_m, \widetilde{B}_1, \widetilde{B}_2, \dots, \widetilde{B}_n$ *be perturbed matrices of* $A_1, A_2, \dots, A_m$, $B_1, B_2, \dots, B_n$ *in* (1.1) *and* $\Delta_i = \widetilde{A}_i - A_i$, $i = 1, 2, \dots, m$, $\Delta_j = \widetilde{B}_j - B_j$, $j = 1, 2, \dots, n$. *If*

$$\sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{2}, \qquad \sum_{j=1}^{n} \|B_j\|^2 < \frac{1}{2}, \tag{2.22}$$

$$2\sum_{i=1}^{m} (\|A_i\|\|\Delta_i\|) + \sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{2} - \sum_{i=1}^{m} \|A_i\|^2, \tag{2.23}$$

$$2\sum_{j=1}^{n} (\|B_j\|\|\Delta_j\|) + \sum_{j=1}^{n} \|\Delta_j\|^2 < \frac{1}{2} - \sum_{j=1}^{n} \|B_j\|^2, \tag{2.24}$$

then (1.1) *and its perturbed equation*

$$\widetilde{X} - \sum_{i=1}^{m} \widetilde{A}_i^* \widetilde{X} \widetilde{A}_i + \sum_{j=1}^{n} \widetilde{B}_j^* \widetilde{X} \widetilde{B}_j = I \tag{2.25}$$

have unique positive definite solutions X_U *and* $\widetilde{X}_U$, *respectively, which satisfy*

$$\left\| \widetilde{X}_U - X_U \right\| \le S_{\text{err}}, \tag{2.26}$$

where

$$S_{\text{err}} = \frac{2\left[1 + 2\sum_{i=1}^{m} (\|A_i\| + \|\Delta_i\|)^2\right]\left[\sum_{i=1}^{m} (\|A_i\| + \|\Delta_i\|)^2 \|\Delta_i\| + \sum_{j=1}^{n} \left(\|B_j\| + \|\Delta_j\|\right)^2 \|\Delta_j\|\right]}{1 - \sum_{i=1}^{m} (\|A_i\| + \|\Delta_i\|)^2 - \sum_{j=1}^{n} \left(\|B_j\| + \|\Delta_j\|\right)^2}. \tag{2.27}$$

Proof. By (2.22) and Theorem 2.4, we know that (1.1) has a unique positive definite solution X_U. And by (2.23) we have

$$\begin{aligned}
\sum_{i=1}^{m} \left\| \widetilde{A}_i \right\|^2 &= \sum_{i=1}^{m} \|A_i + \Delta_i\|^2 \le \sum_{i=1}^{m} (\|A_i\| + \|\Delta_i\|)^2 \\
&= \sum_{i=1}^{m} \left(\|A_i\|^2 + 2\|A_i\|\|\Delta_i\| + \|\Delta_i\|^2 \right) \\
&= \sum_{i=1}^{m} \|A_i\|^2 + 2\sum_{i=1}^{m} (\|A_i\|\|\Delta_i\|) + \sum_{i=1}^{m} \|\Delta_i\|^2 \\
&< \sum_{i=1}^{m} \|A_i\|^2 + \frac{1}{2} - \sum_{i=1}^{m} \|A_i\|^2 = \frac{1}{2},
\end{aligned} \tag{2.28}$$

similarly, by (2.24) we have

$$\sum_{j=1}^{n} \left\| \widetilde{B}_j \right\|^2 < \frac{1}{2}. \tag{2.29}$$

By (2.28), (2.29), and Theorem 2.4 we obtain that the perturbed equation (2.25) has a unique positive definite solution $\widetilde{X}_U$. □

Set

$$A_i(t) = A_i + t\Delta_i, \quad B_j(t) = B_j + t\Delta_j, \quad t \in [0,1], \tag{2.30}$$

then by (2.23) we have

$$\begin{aligned}
\sum_{i=1}^{m} \|A_i(t)\|^2 &= \sum_{i=1}^{m} \|A_i + t\Delta_i\|^2 \le \sum_{i=1}^{m} (\|A_i\| + t\|\Delta_i\|)^2 \\
&= \sum_{i=1}^{m} \left(\|A_i\|^2 + 2t\|A_i\|\|\Delta_i\| + t^2\|\Delta_i\|^2 \right) \\
&\le \sum_{i=1}^{m} \left(\|A_i\|^2 + 2\|A_i\|\|\Delta_i\| + \|\Delta_i\|^2 \right) \\
&= \sum_{i=1}^{m} \|A_i\|^2 + 2\sum_{i=1}^{m} (\|A_i\|\|\Delta_i\|) + \sum_{i=1}^{m} \|\Delta_i\|^2 \\
&< \sum_{i=1}^{m} \|A_i\|^2 + \frac{1}{2} - \sum_{i=1}^{m} \|A_i\|^2 = \frac{1}{2},
\end{aligned} \tag{2.31}$$

similarly, by (2.24) we have

$$\sum_{j=1}^{n} \left\| B_j(t) \right\|^2 = \sum_{i=1}^{m} \left\| B_j + t\Delta_j \right\|^2 < \frac{1}{2}. \tag{2.32}$$

Therefore, by (2.31), (2.32), and Theorem 2.4 we derive that for arbitrary $t \in [0,1]$, the matrix equation

$$X - \sum_{i=1}^{m} A_i^*(t) X A_i(t) + \sum_{j=1}^{n} B_j^*(t) X B_j(t) = I \tag{2.33}$$

has a unique positive definite solution $X_U(t)$, especially,

$$X_U(0) = X_U, \qquad X_U(1) = \widetilde{X}_U. \tag{2.34}$$

From Theorem 2.4 it follows that

$$\begin{aligned}
\left\|\tilde{X}_U - X_U\right\| &= \|X_U(1) - X_U(0)\| = \left\|\int_0^1 dX_U(t)\right\| \le \int_0^1 \|dX_U(t)\| \\
&\le \int_0^1 \frac{2\left(1 + 2\sum_{i=1}^m \|A_i(t)\|^2\right)\left[\sum_{i=1}^m (\|A_i(t)\|\|dA_i(t)\|) + \sum_{j=1}^n \left(\left\|B_j(t)\right\|\left\|dB_j(t)\right\|\right)\right]}{1 - \sum_{i=1}^m \|A_i(t)\|^2 - \sum_{j=1}^n \left\|B_j(t)\right\|^2} \\
&\le \int_0^1 \frac{2\left(1 + 2\sum_{i=1}^m \|A_i(t)\|^2\right)\left[\sum_{i=1}^m (\|A_i(t)\|\|\Delta_i\|dt) + \sum_{j=1}^n \left(\left\|B_j(t)\right\|\left\|\Delta_j\right\|dt\right)\right]}{1 - \sum_{i=1}^m \|A_i(t)\|^2 - \sum_{j=1}^n \left\|B_j(t)\right\|^2} \\
&= \int_0^1 \frac{2\left(1 + 2\sum_{i=1}^m \|A_i(t)\|^2\right)\left[\sum_{i=1}^m (\|A_i(t)\|\|\Delta_i\|) + \sum_{j=1}^n \left(\left\|B_j(t)\right\|\left\|\Delta_j\right\|\right)\right]}{1 - \sum_{i=1}^m \|A_i(t)\|^2 - \sum_{j=1}^n \left\|B_j(t)\right\|^2} dt.
\end{aligned} \tag{2.35}$$

Noting that

$$\begin{aligned}
\|A_i(t)\| &= \|A_i + t\Delta_i\| \le \|A_i\| + t\|\Delta_i\|, \quad i = 1, 2, \ldots, m, \\
\left\|B_j(t)\right\| &= \left\|B_j + t\Delta_i\right\| \le \left\|B_j\right\| + t\|\Delta_i\|, \quad j = 1, 2, \ldots, n,
\end{aligned} \tag{2.36}$$

and combining Mean Value Theorem of Integration, we have

$$\begin{aligned}
&\left\|\tilde{X}_U - X_U\right\| \\
&\le \int_0^1 \frac{2\left(1 + 2\sum_{i=1}^m \|A_i(t)\|^2\right)\left[\sum_{i=1}^m (\|A_i(t)\|\|\Delta_i\|) + \sum_{j=1}^n \left(\left\|B_j(t)\right\|\left\|\Delta_j\right\|\right)\right]}{1 - \sum_{i=1}^m \|A_i(t)\|^2 - \sum_{j=1}^n \left\|B_j(t)\right\|^2} dt \\
&\le \int_0^1 \frac{2\left[1 + 2\sum_{i=1}^m (\|A_i\| + t\|\Delta_i\|)^2\right]\left[\sum_{i=1}^m (\|A_i\| + t\|\Delta_i\|)^2\|\Delta_i\| + \sum_{j=1}^n \left(\left\|B_j\right\| + t\left\|\Delta_j\right\|\right)^2\left\|\Delta_j\right\|\right]}{1 - \sum_{i=1}^m (\|A_i\| + t\|\Delta_i\|)^2 - \sum_{j=1}^n \left(\left\|B_j\right\| + t\left\|\Delta_j\right\|\right)^2} dt \\
&= \frac{2\left[1 + 2\sum_{i=1}^m (\|A_i\| + \xi\|\Delta_i\|)^2\right]\left[\sum_{i=1}^m (\|A_i\| + \xi\|\Delta_i\|)^2\|\Delta_i\| + \sum_{j=1}^n \left(\left\|B_j\right\| + \xi\left\|\Delta_j\right\|\right)^2\left\|\Delta_j\right\|\right]}{1 - \sum_{i=1}^m (\|A_i\| + \xi\|\Delta_i\|)^2 - \sum_{j=1}^n \left(\left\|B_j\right\| + \xi\left\|\Delta_j\right\|\right)^2} \\
&\quad \times (1 - 0), \quad \xi \in [0, 1] \\
&\le \frac{2\left[1 + 2\sum_{i=1}^m (\|A_i\| + \|\Delta_i\|)^2\right]\left[\sum_{i=1}^m (\|A_i\| + \|\Delta_i\|)^2\|\Delta_i\| + \sum_{j=1}^n \left(\left\|B_j\right\| + \left\|\Delta_j\right\|\right)^2\left\|\Delta_j\right\|\right]}{1 - \sum_{i=1}^m (\|A_i\| + \|\Delta_i\|)^2 - \sum_{j=1}^n \left(\left\|B_j\right\| + \left\|\Delta_j\right\|\right)^2} = S_{\text{err}}.
\end{aligned} \tag{2.37}$$

3. Numerical Experiments

In this section, we use a numerical example to confirm the correctness of Theorem 2.5 and the precision of the perturbation bound for the unique positive definite solution X_U of (1.1).

Example 3.1. Consider the symmetric linear matrix equation

$$X - A_1^* X A_1 - A_2^* X A_2 + B_1^* X B_1 + B_2^* X B_2 = I, \tag{3.1}$$

and its perturbed equation

$$\widetilde{X} - \widetilde{A}_1^* \widetilde{X} \widetilde{A}_1 - \widetilde{A}_2^* \widetilde{X} \widetilde{A}_2 + \widetilde{B}_1^* \widetilde{X} \widetilde{B}_1 + \widetilde{B}_2^* \widetilde{X} \widetilde{B}_2 = I, \tag{3.2}$$

where

$$A_1 = \begin{pmatrix} 0.02 & -0.10 & -0.02 \\ 0.08 & -0.10 & 0.02 \\ -0.06 & -0.12 & 0.14 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.08 & -0.10 & -0.02 \\ 0.08 & -0.10 & 0.02 \\ -0.06 & -0.12 & 0.14 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.47 & 0.02 & 0.04 \\ -0.10 & 0.36 & -0.02 \\ -0.04 & 0.01 & 0.47 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0.10 & 0.10 & 0.05 \\ 0.15 & 0.275 & 0.075 \\ 0.05 & 0.05 & 0.175 \end{pmatrix},$$

$$\widetilde{A}_1 = A_1 + \begin{pmatrix} 0.5 & 0.1 & -0.2 \\ -0.4 & 0.2 & 0.6 \\ -0.2 & 0.1 & -0.1 \end{pmatrix} \times 10^{-j}, \qquad \widetilde{A}_2 = A_2 + \begin{pmatrix} -0.4 & 0.1 & -0.2 \\ 0.5 & 0.7 & -1.3 \\ 1.1 & 0.9 & 0.6 \end{pmatrix} \times 10^{-j},$$

$$\widetilde{B}_1 = B_1 + \begin{pmatrix} 0.8 & 0.2 & 0.05 \\ -0.2 & 0.12 & 0.14 \\ -0.25 & -0.2 & 0.26 \end{pmatrix} \times 10^{-j}, \quad \widetilde{B}_2 = B_2 + \begin{pmatrix} 0.2 & 0.2 & 0.1 \\ -0.3 & 0.15 & -0.15 \\ 0.1 & -0.1 & 0.25 \end{pmatrix} \times 10^{-j}, \quad j \in N. \tag{3.3}$$

It is easy to verify that the conditions (2.22)–(2.24) are satisfied, then (3.1) and its perturbed equation (3.2) have unique positive definite solutions X_U and $\widetilde{X}_U$, respectively. From Berzig [7] it follows that the sequences $\{X_k\}$ and $\{Y_k\}$ generated by the iterative method

$$\begin{gathered} X_0 = 0, \qquad Y_0 = 2I, \\ X_{k+1} = I + A_1^* X_k A_1 + A_2^* X_k A_2 - B_1^* Y_k B_1 - B_2^* Y_k B_2, \\ Y_{k+1} = I + A_1^* Y_k A_1 + A_2^* Y_k A_2 - B_1^* X_k B_1 - B_2^* X_k B_2, \quad k = 0, 1, 2, \ldots \end{gathered} \tag{3.4}$$

both converge to X_U. Choose $\tau = 1.0 \times 10^{-15}$ as the termination scalar, that is,

$$R(X) = \left\| X - A_1^* X A_1 - A_2^* X A_2 + B_1^* X B_1 + B_2^* X B_2 - I \right\| \leq \tau = 1.0 \times 10^{-15}. \tag{3.5}$$

By using the iterative method (3.4) we can get the computed solution X_k of (3.1). Since $R(X_k) < 1.0 \times 10^{-15}$, then the computed solution X_k has a very high precision. For simplicity,

Table 1: Numerical results for the different values of j.

j	2	3	4	5	6
$\\|\widetilde{X}_U - X_U\\|/\\|X_U\\|$	2.13×10^{-4}	6.16×10^{-6}	5.23×10^{-8}	4.37×10^{-10}	8.45×10^{-12}
$S_{\mathrm{err}}/\\|X_U\\|$	3.42×10^{-4}	7.13×10^{-6}	6.63×10^{-8}	5.12×10^{-10}	9.77×10^{-12}

we write the computed solution as the unique positive definite solution X_U. Similarly, we can also get the unique positive definite solution $\widetilde{X}_U$ of the perturbed equation (3.2).

Some numerical results on the perturbation bounds for the unique positive definite solution X_U are listed in Table 1.

From Table 1, we see that Theorem 2.5 gives a precise perturbation bound for the unique positive definite solution of (3.1).

4. Conclusion

In this paper, we study the matrix equation (1.1) which arises in solving some nonlinear matrix equations and the bilinear control system. A new method of perturbation analysis is developed for the matrix equation (1.1). By making use of the matrix differentiation and its elegant properties, we derive a precise perturbation bound for the unique positive definite solution of (1.1). A numerical example is presented to illustrate the sharpness of the perturbation bound.

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