Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 782389, 7 pages
doi:10.1155/2012/782389

Research Article

Best Periodic Proximity Points for Cyclic Weaker Meir-Keeler Contractions

Chiming Chen and Chingju Lin

Department of Applied Mathematics, National Hsinchu University of Education, 521 Nan-Dah Road, Hsinchu City 30033, Taiwan

Correspondence should be addressed to Chiming Chen, ming@mail.nhcue.edu.tw

Received 30 January 2012; Revised 8 April 2012; Accepted 11 April 2012

Academic Editor: Song Cen

The purpose of this paper is to present the existence of the best period proximity point for cyclic weaker Meir-Keeler contractions and asymptotic cyclic weaker Meir-Keeler contractions in metric spaces.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$ we denote the set of all nonnegative numbers, while $\mathbb{N}$ is the set of all natural numbers. Let A and B be nonempty subsets of a metric space (X, d). Consider a mapping $f : A \cup B \to A \cup B$, f is called a cyclic map if $f(A) \subseteq B$ and $f(B) \subseteq A$. A point x in A is called a best proximity point of f in A if $d(x, fx) = d(A, B)$ is satisfied, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$, and $x \in A$ is called a best periodic proximity point of f in A if $d(x, f^{2\kappa+1}x) = d(A, B)$ is satisfied, for some $\kappa \in \mathbb{N} \cup \{0\}$. In 2005, Eldred et al. [1] proved the existence of a best proximity point for relatively nonexpansive mappings using the notion of proximal normal structure. In 2006, Eldred and Veeramani [2] proved the following existence theorem.

Theorem 1.1 (see Theorem 3.10 in [2]). *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $f : A \cup B \to A \cup B$ is a cyclic contraction, that is, $f(A) \subseteq B$ and $f(B) \subseteq A$, and there exists $k \in (0, 1)$ such that*

$$d(fx, fy) \le kd(x, y) + (1 - k)d(A, B) \quad \text{for every } x \in A, y \in B. \tag{1.1}$$

Then there exists a unique best proximity point in A. Further, for each $x \in A$, $\{f^{2n}x\}$ converges to the best proximity point.

In this paper, we also recall the notion of Meir-Keeler type mapping. A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler-type mapping (see [3]) if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in \mathbb{R}^+$ with $\eta \le t < \eta + \delta$, we have $\psi(t) < \eta$.

In the recent, Eldred et al. [1] introduced the below notion of cyclic Meir-Keeler contraction.

Definition 1.2 (see [1]). Let (X, d) be a metric space, and let A and B be nonempty subsets of X. Then $f : A \cup B \to A \cup B$ is called a cyclic Meir-Keeler contraction if the following are satisfied:

(i) $f(A) \subset B$ and $f(B) \subset A$;

(ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < d(A, B) + \varepsilon + \delta \quad \text{implies } d(fx, fy) < d(A, B) + \varepsilon \tag{1.2}$$

for all $x \in A$ and $y \in B$.

In the recent, Di Bari et al. [4] proved the following best proximity point theorem.

Theorem 1.3 (see [4]). *Let X be a uniformly convex Banach space, and let A and B be nonempty subsets of X. Suppose A is closed and convex and $f : A \cup B \to A \cup B$ is a cyclic Meir-Keeler contraction. Then there exists a unique best proximity point in A. Further, for each $x \in A$, $\{f^{2n}x\}$ converges to best proximity point.*

Later, many authors studied this subject, and many results on best proximity points are proved. (see, e.g., [5–10]). In this study, we will introduce the new concepts of cyclic weaker Meir-Keeler contractions and asymptotic cyclic weaker Meir-Keeler contractions in metric spaces, and the purpose of this paper is to present the existence of the best period proximity point for these contractions.

2. The Best Periodic Proximity Points for Cyclic Weaker Meir-Keeler Contractions

In this section, we first introduce the below notions of the weaker Meir-Keeler-type mapping, φ-mapping, and cyclic weaker Meir-Keeler contraction in metric spaces.

Definition 2.1. Let (X, d) be a metric space, and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Then φ is called a weaker Meir-Keeler-type mapping in X if for each $\eta > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \le d(x, y) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(d(x, y)) < \eta$.

The following provides an example of a weaker Meir-Keeler-type mapping that is not a Meir-Keeler-type mapping in a metric space (X, d).

Example 2.2. Let $X = \mathbb{R}^2$, and we define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad \forall x = (x_1, x_2),\ y = (y_1, y_2) \in X. \tag{2.1}$$

If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\varphi(t) = \begin{cases} 0 & \text{if } t \leq 1, \\ 2t & \text{if } 1 < t < 2, \\ 1 & \text{if } t \geq 2, \end{cases} \tag{2.2}$$

where $t = d(x,y)$, $x, y \in X$, then φ is a weaker Meir-Keeler-type mapping that is not a Meir-Keeler-type mapping in X.

Definition 2.3. Let (X, d) be a metric space. A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a φ-mapping in X if the mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions:

(φ_1) φ is a weaker Meir-Keeler-type mapping in X;

(φ_2) for all $t > 0$, $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ is nonincreasing;

(φ_3) for all $t > 0$, $\varphi(t) > 0$ and $\varphi(0) = 0$.

The following provides two examples of a φ-mapping.

Example 2.4. Let $X = \mathbb{R}^2$, and we define $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \forall x = (x_1, x_2),\ y = (y_1, y_2) \in X. \tag{2.3}$$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be

$$\varphi(t) = \frac{1}{2}t \quad \forall t \in \mathbb{R}^+. \tag{2.4}$$

Then $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a φ-mapping in X.

Example 2.5. Let $X = [0, 4]$, and we define $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = |x - y| \quad \forall x, y \in X. \tag{2.5}$$

If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\varphi(t) = \begin{cases} \frac{3}{4}t & \text{if } 0 \leq t \leq 1, \\ 2t & \text{if } 1 < t < 2, \\ 1 & \text{if } 2 \leq t \leq 4, \end{cases} \tag{2.6}$$

where $t = d(x,y)$, $x, y \in X$, then φ is a φ-mapping in X.

Definition 2.6. Let (X, d) be a metric space, and let A and B be nonempty subsets of X. Then $f : A \cup B \to A \cup B$ is called a cyclic weaker Meir-Keeler contraction if the following conditions hold:

(1) $f(A) \subset B$ and $f(B) \subset A$;

(2) there is a φ-mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ in X such that for all $n \in \mathbb{N}$ and $x \in A$, $y \in B$ with $d(x,y) - d(A,B) > 0$,

$$\begin{gathered} d(f^n x, f^n y) - d(A,B) < \varphi^n(d(x,y) - d(A,B)), \\ d(x,y) - d(A,B) = 0 \quad \text{implies } d(f^n x, f^n y) - d(A,B) = 0. \end{gathered} \tag{2.7}$$

The following provides an example of a cyclic weaker Meir-Keeler contraction.

Example 2.7. Let $A = [-2, 0]$ and $B = [0, 2]$ in the metric space $(\mathbb{R}, d)$, where $d(x,y) = |x - y|$. Define

$$f(x) = \frac{-x}{4} \quad \forall x \in A \cup B. \tag{2.8}$$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\varphi(t) = \begin{cases} \frac{3}{4}t & \text{if } 0 \le t \le 1, \\ 2t & \text{if } 1 < t < 2, \\ 1 & \text{if } 2 \le t \le 4, \end{cases} \tag{2.9}$$

where $t = d(x,y)$, $x \in A$, $y \in B$. Then all conditions (1) and (2) of Definition 2.6 and therefore f are a cyclic weaker Meir-Keeler contraction. Notice that $d(A,B) = 0$.

Now, we are in this position to state the following results.

Lemma 2.8. *Let (X,d) be a metric space, and let A, B be nonempty subsets of X. Suppose $f : A \cup B \to A \cup B$ is a cyclic weaker Meir-Keeler contraction. Then* $\lim_{n\to\infty} d(f^n x, f^{n+1} x) = d(A,B)$ *holds.*

Proof. Since $f : A \cup B \to A \cup B$ is a cyclic weaker Meir-Keeler contraction, there is a φ-mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ in X such that

$$d(f^n x, f^n y) - d(A,B) < \varphi^n(d(x,y) - d(A,B)), \tag{2.10}$$

for all $n \in \mathbb{N}$ and $x \in A$, $y \in B$.

Since $\{\varphi^n(d(x,y))\}_{n\in\mathbb{N}}$ is nonincreasing, hence we also conclude $\{\varphi^n(d(x,y) - d(A,B))\}_{n\in\mathbb{N}}$ is nonincreasing, and it must converge to some $\eta \ge 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. By the definition of the weaker Meir-Keeler-type mapping φ, corresponding to η use, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \le d(x,y) - d(A,B) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(d(x,y) - d(A,B)) < \eta$. Since $\lim_{n\to\infty} \varphi^n(d(x,y) - d(A,B)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \le \varphi^m(d(x,y) - d(A,B)) < \delta + \eta$, forall $m \ge m_0$. Thus, we conclude that $\varphi^{m_0+n_0}(d(x,y) - d(A,B)) < \eta$. So we get a contradiction. So $\lim_{n\to\infty} \varphi^n(d(x,y) - d(A,B)) = 0$, and so $\lim_{n\to\infty} d(f^n x, f^n y) - d(A,B) = 0$, that is, $\lim_{n\to\infty} d(f^n x, f^n y) = d(A,B)$. Thus, we also conclude that $\lim_{n\to\infty} d(f^n x, f^{n+1} x) = d(A,B)$. □

Applying above Lemma 2.8, it is easy to conclude the following theorem.

Theorem 2.9. *Let (X,d) be a metric space, and let A,B be nonempty subsets of X. Suppose $f : A\cup B \to A\cup B$ is a cyclic weaker Meir-Keeler contraction and if for some $x \in A$, the sequence $\{f^{2n+1}x\}$ converges to $\overline{x} \in A$, then $\overline{x}$ is a best periodic proximity point of f in A.*

Proof. By the definition of the weaker Meir-Keeler-type mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ in X, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(\eta) \le \eta$ for each $\eta > 0$. Since $\{f^{2n+1}x\}$ converges to$\overline{x} \in A$, corresponding to above n_0 use, we have

$$\begin{aligned}
d(A,B) &\le d\left(\overline{x}, f^{2n_0+1}\overline{x}\right)\\
&\le d\left(\overline{x}, f^{2n+1}x\right) + d\left(f^{2n+1}x, f^{2n_0+1}\overline{x}\right) - d(A,B) + d(A,B)\\
&\le d\left(\overline{x}, f^{2n+1}x\right) + \varphi^{2n_0+1}\left(d\left(f^{2(n-n_0)}x, \overline{x}\right) - d(A,B)\right) + d(A,B)\\
&\le d\left(\overline{x}, f^{2n+1}x\right) + \varphi^{2n_0}\left(d\left(f^{2(n-n_0)}x, \overline{x}\right) - d(A,B)\right) + d(A,B)\\
&\le d\left(\overline{x}, f^{2n+1}x\right) + d\left(f^{2(n-n_0)}x, \overline{x}\right) - d(A,B) + d(A,B)\\
&\le d\left(\overline{x}, f^{2n+1}x\right) + d\left(f^{2(n-n_0)}x, f^{2(n-n_0)+1}x\right) + d\left(f^{2(n-n_0)+1}x, \overline{x}\right),
\end{aligned} \tag{2.11}$$

Letting $n \to \infty$. Then $d(A,B) = d(\overline{x}, f^{2n_0+1}\overline{x})$. Thus $\overline{x}$ is a best period proximity point of f in A. □

3. The Best Periodic Proximity Points for Asymptotic Cyclic Weaker Meir-Keeler Contractions

In this section, we introduce the below notions of the asymptotic cyclic weaker Meir-Keeler-type sequence and asymptotic cyclic weaker Meir-Keeler contraction in a metric space (X,d).

Definition 3.1. Let (X,d) be a metric space. A sequence $\{\varphi_n \mid \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+\}_{n\in\mathbb{N}}$ in X is called an asymptotic weaker Meir-Keeler-type sequence if $\{\varphi_n \mid \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+\}_{n\in\mathbb{N}}$ satisfies the following conditions:

(C_1) for each $\eta > 0$, there exists $\delta > 0$ such that for $x,y \in X$ with $\eta \le d(x,y) < \delta + \eta$, there exists $2n_0 \in \mathbb{N}$ such that $\varphi_{2n_0}(d(x,y)) < \eta$;

(C_2) for all $n \in \mathbb{N}$ and $t > 0$, $\{\varphi_n(t)\}_{n\in\mathbb{N}}$ is nonincreasing;

(C_3) for all $n \in \mathbb{N}$, $\varphi_n(0) = 0$ and $\varphi_n(t) > 0$, $t > 0$.

Example 3.2. Let $X = \mathbb{R}^2$ and we define $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2| \quad \forall x = (x_1,x_2),\ y = (y_1,y_2) \in X. \tag{3.1}$$

Let $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ be

$$\varphi_n(t) = \frac{1}{2^n} t \quad \forall t \in \mathbb{R}^+,\ n \in \mathbb{N}, \tag{3.2}$$

where $t = d(x, y)$, $x, y \in X$. Then $\{\varphi_n \mid \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+\}_{n\in\mathbb{N}}$ is an asymptotic weaker Meir-Keeler-type sequence in a metric space (X, d).

Definition 3.3. Let (X, d) be a metric space, and let A and B be nonempty subsets of X. Then $f : A\cup B \to A\cup B$ is an asymptotic cyclic weaker Meir-Keeler contraction if the following conditions hold:

(1) $f(A) \subset B$ and $f(B) \subset A$;

(2) there is an asymptotic weaker Meir-Keeler-type sequence $\{\varphi_n \mid \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ and $x \in A$, $y \in B$ with $d(x, y) - d(A, B) > 0$,

$$\begin{gathered} d(f^n x, f^n y) - d(A, B) < \varphi_n(d(x, y) - d(A, B)), \\ d(x, y) - d(A, B) = 0 \quad \text{implies } d(f^n x, f^n y) - d(A, B) = 0. \end{gathered} \tag{3.3}$$

Now, we are in this position to state the following results.

Lemma 3.4. *Let (X, d) be a metric space and A, B nonempty subsets of X. Suppose $f : A\cup B \to A\cup B$ is an asymptotic cyclic weaker Meir-Keeler contraction. Then $\lim_{n\to\infty} d(f^n x, f^{n+1} x) = d(A, B)$ holds.*

Proof. Since $f : A \cup B \to A \cup B$ is an asymptotic cyclic weaker Meir-Keeler contraction, there is an asymptotic weaker Meir-Keeler-type sequence $\{\varphi_n \mid \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+\}_{n\in\mathbb{N}}$ such that

$$d(f^n x, f^n y) - d(A, B) < \varphi_n(d(x, y) - d(A, B)), \tag{3.4}$$

for all $n \in \mathbb{N}$ and $x \in A$, $y \in B$.

Since $\{\varphi_n(d(x, y))\}_{n\in\mathbb{N}}$ is nonincreasing, hence we also conclude $\{\varphi_n(d(x, y) - d(A, B))\}_{n\in\mathbb{N}}$ is nonincreasing, and it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. By the definition of asymptotic weaker Meir-Keeler-type sequence, corresponding to η use, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \leq d(x, y) - d(A, B) < \delta + \eta$, there exists $2n_0 \in \mathbb{N}$ such that $\varphi_{2n_0}(d(x, y) - d(A, B)) < \eta$. Since $\lim_{n\to\infty} \varphi_n(d(x, y) - d(A, B)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \leq \varphi_m(d(x, y) - d(A, B)) < \delta + \eta$, for all $m \geq m_0$. Thus, we conclude that $\varphi_{m_0+2n_0}(d(x, y) - d(A, B)) < \eta$. So we get a contradiction. Therefore, $\lim_{n\to\infty} \varphi_n(d(x, y) - d(A, B)) = 0$, and so $\lim_{n\to\infty} d(f^n x, f^n y) - d(A, B) = 0$, that is, $\lim_{n\to\infty} d(f^n x, f^n y) = d(A, B)$. Thus, we also conclude that $\lim_{n\to\infty} d(f^n x, f^{n+1} x) = d(A, B)$. □

Applying above Lemma 3.4, we are easy to conclude the following theorem.

Theorem 3.5. *Let (X, d) be a metric space and A, B nonempty subsets of X. Suppose $f : A \cup B \to A \cup B$ is an asymptotic cyclic weaker Meir-Keeler contraction, and if for some $x \in A$, the sequence $\{f^{2n+1} x\}$ converges to $\overline{x} \in A$, then $\overline{x}$ is a best periodic proximity point of f in A.*

Proof. By the definition of the asymptotic weaker Meir-Keeler-type sequence $\{\varphi_n \mid \varphi_n : \mathbb{R}^+ \to \mathbb{R}^+\}_{n\in\mathbb{N}}$, thus there exists $2n_0 \in \mathbb{N}$ such that $\varphi_{2n_0}(\eta) \le \eta$ for each $\eta > 0$. Since $\{f^{2n+1}x\}$ converges to $\overline{x} \in A$, corresponding to above $2n_0$ use, we have

$$\begin{aligned}
d(A,B) &\le d\left(\overline{x}, f^{2n_0+1}\overline{x}\right) \\
&\le d\left(\overline{x}, f^{2n+1}x\right) + d\left(f^{2n+1}x, f^{2n_0+1}\overline{x}\right) - d(A,B) + d(A,B) \\
&\le d\left(\overline{x}, f^{2n+1}x\right) + \varphi_{2n_0+1}\left(d\left(f^{2(n-n_0)}x, \overline{x}\right) - d(A,B)\right) + d(A,B) \\
&\le d\left(\overline{x}, f^{2n+1}x\right) + \varphi_{2n_0}\left(d\left(f^{2(n-n_0)}x, \overline{x}\right) - d(A,B)\right) + d(A,B) \\
&\le d\left(\overline{x}, f^{2n+1}x\right) + d\left(f^{2(n-n_0)}x, \overline{x}\right) - d(A,B) + d(A,B) \\
&\le d\left(\overline{x}, f^{2n+1}x\right) + d\left(f^{2(n-n_0)}x, f^{2(n-n_0)+1}x\right) + d\left(f^{2(n-n_0)+1}x, \overline{x}\right).
\end{aligned} \tag{3.5}$$

Letting $n \to \infty$. Then $d(A,B) = d(\overline{x}, f^{2n_0+1}\overline{x})$. Thus $\overline{x}$ is a best period proximity point of f in A. □

Acknowledgments

The authors would like to thank referee(s) for many useful comments and suggestions for the improvement of the paper. This paper was supported by the National Science Council of the Republic of China.

References

[1] A. A. Eldred, W. A. Kirk, and P. Veeramani, "Proximal normal structure and relatively nonexpansive mappings," *Studia Mathematica*, vol. 171, no. 3, pp. 283–293, 2005.

[2] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.

[3] A. Meir and E. Keeler, "A theorem on contraction mappings," *Journal of Mathematical Analysis and Applications*, vol. 28, pp. 326–329, 1969.

[4] C. Di Bari, T. Suzuki, and C. Vetro, "Best proximity points for cyclic Meir-Keeler contractions," *Nonlinear Analysis*, vol. 69, no. 11, pp. 3790–3794, 2008.

[5] S. Sadiq Basha, "Best proximity point theorems," *Journal of Approximation Theory*, vol. 163, no. 11, pp. 1772–1781, 2011.

[6] S. Karpagam and S. Agrawal, "Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps," *Nonlinear Analysis*, vol. 74, no. 4, pp. 1040–1046, 2011.

[7] W. A. Kirk, P. S. Srinivasan, and P. Veeramani, "Fixed points for mappings satisfying cyclical contractive conditions," *Fixed Point Theory*, vol. 4, no. 1, pp. 79–89, 2003.

[8] M. De la Sen, "Linking contractive self-mappings and cyclic Meir-Keeler contractions with Kannan self-mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 572057, 23 pages, 2010.

[9] S. L. Singh, S. N. Mishra, and R. Pant, "Fixed points of generalized asymptotic contractions," *Fixed Point Theory*, vol. 12, no. 2, pp. 475–484, 2011.

[10] C. Vetro, "Best proximity points: convergence and existence theorems for p-cyclic mappings," *Nonlinear Analysis*, vol. 73, no. 7, pp. 2283–2291, 2010.