Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 761242, 16 pages
doi:10.1155/2012/761242

Research Article

A Discontinuous Finite Volume Method for the Darcy-Stokes Equations

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Received 12 June 2012; Accepted 7 December 2012

Academic Editor: Claudio Padra

This paper proposes a discontinuous finite volume method for the Darcy-Stokes equations. An optimal error estimate for the approximation of velocity is obtained in a mesh-dependent norm. First-order L^2-error estimates are derived for the approximations of both velocity and pressure. Some numerical examples verifying the theoretical predictions are presented.

1. Introduction

The study of discontinuous Galerkin methods has been a very active research field since they were proposed by Reed and Hill [1] in 1973. Discontinuous Galerkin methods use discontinuous functions as finite element approximation and enforce the connections of the approximate solutions between elements by adding some penalty terms. The flexibility of discontinuous functions gives discontinuous Galerkin methods many advantages, such as high parallelizability and localizability. Arnold et al. [2] provided a framework for the analysis of a large class of discontinuous Galerkin methods for second-order elliptic problems.

Based on the advantages of using discontinuous functions for approximation in discontinuous Galerkin methods, it is natural to consider using discontinuous functions as trial functions in the finite volume method, which is called the discontinuous finite volume method. Such a method has the flexibility of the discontinuous Galerkin method and the simplicity and conservative properties of the finite volume method. Ye [3] developed a new discontinuous finite volume method and analyzed it for the second-order elliptic problem. Bi and Geng [4] proposed the semidiscrete and the backward Euler fully discrete discontinuous finite volume element methods for the second-order parabolic problems. Ye [5] considered the discontinuous finite volume method for solving the Stokes problems on both triangular and rectangular meshes and derived an optimal order error estimate for the approximation of velocity in a mesh-dependent norm and first-order L^2-error estimates for the approximations of both velocity and pressure.

The Darcy-Stokes problem is interesting for a variety of reasons. Apart from being a modeling tool in its own right, it also appears, less obviously, in time-stepping methods for Stokes and for high Reynolds number flows (where of course the convective term causes additional difficulties). In [6], the nonconforming Crouzeix-Raviart element is stabilized for the Darcy-Stokes problem with terms motivated by a discontinuous Galerkin approach. In [7], a new stabilized mixed finite element method is presented for the Darcy-Stokes equations.

In this paper, we will extend the discontinuous finite volume methods to solve the Darcy-Stokes equations. In our methods, velocity is approximated by discontinuous piecewise linear functions on triangular meshes and by discontinuous piecewise rotated bilinear functions on rectangular meshes. Piecewise constant functions are used as the test functions for velocity in the discontinuous finite volume methods. We obtained an optimal error estimate for the approximation of velocity in a mesh-dependent norm. First-order L^2-error estimates are derived for the approximations of both velocity and pressure. For the sake of simplicity and easy presentation of the main ideas of our method, we restrict ourselves to the model problem.

We consider the Darcy-Stokes equations

$$\sigma \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad x \in \Omega, \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = 0, \quad x \in \Omega, \tag{1.1b}$$

$$\mathbf{u} = \mathbf{0}, \quad x \in \partial\Omega, \tag{1.1c}$$

where Ω is a bounded polygonal domain in R^2 with boundary $\partial\Omega$. $\mathbf{u} = (u_1, u_2)$ is the velocity, p is the pressure, and $\mathbf{f}$ is a given force term. We assume $\sigma = 1,\ \mu = 1$.

2. Discontinuous Finite Volume Formulation

Let $\mathcal{R}_h$ be a triangular or rectangular partition of Ω. The triangles or rectangles in $\mathcal{R}_h$ are divided into three or four subtriangles by connecting the barycenter of the triangle or the center of the rectangles to their corner nodes, respectively. Then we define the dual partition $\mathcal{T}_h$ of the primal partition $\mathcal{R}_h$ to be the union of the triangles shown in Figures 1 and 2 for both triangular and rectangular meshes.

Let $P_k(T)$ consist of all the polynomials with degree less than or equal to k defined on T. We define the finite dimensional trial function space for velocity on a triangular partition by

$$V_h = \left\{ \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in P_1(K)^2,\ \forall\ K \in \mathcal{R}_h \right\} \tag{2.1}$$

and on rectangular partition by

$$V_h = \left\{ \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in \widehat{Q}_1(K)^2,\ \forall\ K \in \mathcal{R}_h \right\}, \tag{2.2}$$

where $\widehat{Q}_1$ denotes the space of functions of the form $a + bx_1 + cx_2 + d(x_1^2 - x_2^2)$ on K.

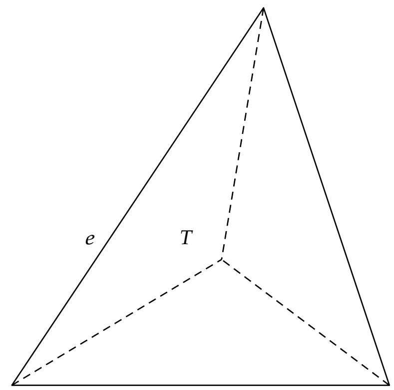

Figure 1: Element $T \in \mathcal{T}_h$ for triangular mesh.

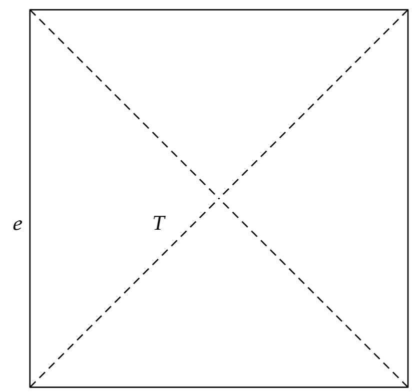

Figure 2: Element $T \in \mathcal{T}_h$ for rectangular mesh.

Let Q_h be the finite dimensional space for pressure

$$Q_h = \left\{ q \in L_0^2(\Omega) : q\big|_K \in P_0(K),\ \forall K \in \mathcal{R}_h \right\}, \tag{2.3}$$

where

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \right\}. \tag{2.4}$$

Define the finite dimensional test function space W_h for velocity associated with the dual partition $\mathcal{T}_h$ as

$$W_h = \left\{ \boldsymbol{\xi} \in L^2(\Omega)^2 : \boldsymbol{\xi}|_T \in P_0(T)^2, \forall\ T \in \mathcal{T}_h \right\}. \tag{2.5}$$

Multiplying (1.1a) and (1.1b) by $\xi \in W_h$ and $q \in Q_h$, respectively, we have

$$(\mathbf{u}, \boldsymbol{\xi}) - \sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\xi} ds + \sum_{T\in\mathcal{T}_h} \int_{\partial T} p\boldsymbol{\xi} \cdot \mathbf{n}\, ds = (\mathbf{f}, \boldsymbol{\xi}),$$
$$\sum_{K\in\mathcal{R}_h} \int_K \nabla \cdot \mathbf{u} q\, dx = 0, \tag{2.6}$$

where $\mathbf{n}$ is the unit outward normal vector on ∂T.

Let $T_j \in \mathcal{T}_h$ $(j = 1, \ldots, t)$ be the triangles in $K \in \mathcal{R}_h$, where $t = 3$ for triangular meshes and $t = 4$ for rectangular meshes, as shown as Figures 3 and 4. Then we have

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\xi}\, ds = \sum_{K\in\mathcal{R}_h} \sum_{j=1}^{t} \int_{A_{j+1}CA_j} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\xi}\, ds + \sum_{K\in\mathcal{R}_h} \int_{\partial K} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\xi}\, ds, \tag{2.7}$$

where $A_{t+1} = A_1$.

For vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{n} = (n_1, n_2)$, let $\mathbf{v} \otimes \mathbf{n}$ denote the matrix whose ijth component is $v_i \cdot n_j$ as in [5]. For two matrix valued variables σ and τ, we define $\sigma : \tau = \sum_{i,j=1}^{2} \sigma_{i,j}\tau_{i,j}$. Let $\Gamma = \sum_{K\in\mathcal{R}_h} \partial K$, $\Gamma_0 = \Gamma \setminus \partial\Omega$. Let e be an interior edge shared by two elements K_1 and K_2 in $\mathcal{R}_h$. We define the average $\{\cdot\}$ and jump $[\cdot]$ on e for scalar q, vector w, and matrix τ, respectively. If $e \in \Gamma_0$,

$$\{q\} = \frac{1}{2}(q|_{\partial K_1} + q|_{\partial K_2}), \qquad \{\mathbf{w}\} = \frac{1}{2}(\mathbf{w}|_{\partial K_1} + \mathbf{w}|_{\partial K_2}), \qquad \{\tau\} = \frac{1}{2}(\tau|_{\partial K_1} + \tau|_{\partial K_2}),$$
$$[q] = q|_{\partial K_1}\mathbf{n}_1 + q|_{\partial K_2}\mathbf{n}_2, \qquad [\mathbf{w}] = \mathbf{w}|_{\partial K_1} \cdot \mathbf{n}_1 + \mathbf{w}|_{\partial K_2} \cdot \mathbf{n}_2, \qquad [\tau] = \tau|_{\partial K_1} \cdot \mathbf{n}_1 + \tau|_{\partial K_2} \cdot \mathbf{n}_2, \tag{2.8}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit normal vectors on e pointing exterior to K_1 and K_2, respectively. We also define a matrix valued jump $[\![\cdot]\!]$ for a vector $\mathbf{w}$ as

$$[\![\mathbf{w}]\!] = \mathbf{w}|_{\partial K_1} \otimes \mathbf{n}_1 + \mathbf{w}|_{\partial K_2} \otimes \mathbf{n}_2. \tag{2.9}$$

If $e \in \partial\Omega$, define

$$\{q\} = q, \qquad [\mathbf{w}] = \mathbf{w} \cdot \mathbf{n}, \qquad \{\tau\} = \tau, \qquad [\![\mathbf{w}]\!] = \mathbf{w} \otimes \mathbf{n}. \tag{2.10}$$

A straightforward computation gives

$$\sum_{K\in\mathcal{R}_h} \int_{\partial K} q\mathbf{v} \cdot \mathbf{n}\, ds = \sum_{e\in\Gamma_0} \int_e [q] \cdot \{\mathbf{v}\} ds + \sum_{e\in\Gamma} \int_e \{q\}[\mathbf{v}] ds, \tag{2.11}$$

$$\sum_{K\in\mathcal{R}_h} \int_{\partial K} \mathbf{v} \cdot \tau\mathbf{n}\, ds = \sum_{e\in\Gamma_0} \int_e [\tau] \cdot \{\mathbf{v}\} ds + \sum_{e\in\Gamma} \int_e \{\tau\} : [\![\mathbf{v}]\!]\, ds. \tag{2.12}$$

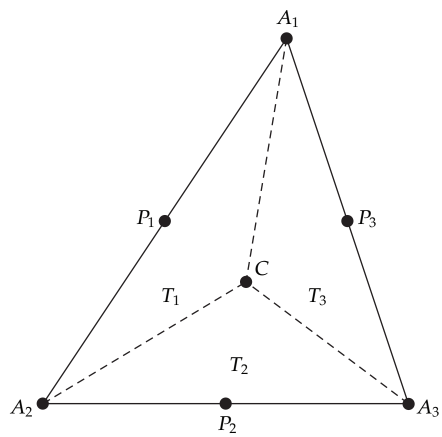

Figure 3: Triangular partition and its dual.

Let $\int_\Gamma q ds = \sum_{e\in\Gamma} \int_e q ds$. Using (2.7), (2.12), and the fact that $[\nabla \mathbf{u}] = 0$ for $\mathbf{u} \in (H_0^1(\Omega)\cap H^2(\Omega))^2$ on Γ_0, (2.7) becomes

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\xi} ds = \sum_{K\in\mathcal{R}_h} \sum_{j=1}^{t} \int_{A_{j+1}CA_j} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \boldsymbol{\xi}\, ds + \int_\Gamma [\![\boldsymbol{\xi}]\!] : \{\nabla \mathbf{u}\} ds. \tag{2.13}$$

Since $[p] = 0$ for $p \in H^1(\Omega)$ on Γ_0, we also have

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} p\boldsymbol{\xi} \cdot \mathbf{n}\, ds = \sum_{K\in\mathcal{R}_h} \sum_{j=1}^{t} \int_{A_{j+1}CA_j} p\boldsymbol{\xi} \cdot \mathbf{n}\, ds + \int_\Gamma \{p\} [\boldsymbol{\xi}] ds. \tag{2.14}$$

Let $V(h) = V_h + (H^2(\Omega) \cap H_0^1(\Omega))^2$. Define a mapping $\gamma : V(h) \rightarrow W_h$,

$$\gamma \mathbf{v}|_T = \frac{1}{h_e} \int_e \mathbf{v}|_T ds, \quad \forall T \in \mathcal{T}_h, \tag{2.15}$$

where h_e is the length of the edge e.

We define two norms for $V(h)$ as follows:

$$\begin{aligned} |||\mathbf{v}|||_1^2 &= \|\mathbf{v}\|_{1,h}^2 + \sum_{e\in\Gamma} [\![\gamma \mathbf{v}]\!]_e^2, \\ |||\mathbf{v}|||^2 &= |||\mathbf{v}|||_1^2 + \sum_{K\in\mathcal{R}_h} h_K^2 |\mathbf{v}|_{2,K}^2, \end{aligned} \tag{2.16}$$

where $\|\mathbf{v}\|_{1,h}^2 = |\mathbf{v}|_{0,h}^2 + |\mathbf{v}|_{1,h}^2$, $|\mathbf{v}|_{0,h}^2 = \sum_{K\in\mathcal{R}_h} |\mathbf{v}|_{0,K}^2$, $|\mathbf{v}|_{1,h}^2 = \sum_{K\in\mathcal{R}_h} |\mathbf{v}|_{1,K}^2$, and h_K = diameter of K.

As in [5], the standard inverse inequality implies that there is a constant C such that

$$|||\mathbf{v}||| \le C|||\mathbf{v}|||_1, \quad \forall \mathbf{v} \in V_h. \tag{2.17}$$

Lemma 2.1. *There exists a positive constant C independent of h such that*

$$h|||\mathbf{v}||| \le C\|\mathbf{v}\| , \quad \|\mathbf{v}\| \le C|||\mathbf{v}|||, \quad \forall \mathbf{v} \in V_h. \tag{2.18}$$

Proof. As in [4],

$$h|||\mathbf{v}|||_{1,h} \le C\|\mathbf{v}\|, \quad \|\mathbf{v}\| \le C|||\mathbf{v}|||_{1,h}, \quad \forall \mathbf{v} \in V_h, \tag{2.19}$$

where $|||\mathbf{v}|||_{1,h}^2 = |\mathbf{v}|_{1,h}^2 + \sum_{e\in\Gamma} [\![\gamma\mathbf{v}]\!]_e^2 + \sum_{K\in\mathcal{R}_h} h_K^2 |\mathbf{v}|_{2,K}^2$. Since $|||\mathbf{v}|||_{1,h} \le |||\mathbf{v}|||$, we have $\|\mathbf{v}\| \le C|||\mathbf{v}|||$. Note that $\mathbf{v} \in V_h$ is a piecewise linear function, and $h^2|||\mathbf{v}|||^2 = h^2|\mathbf{v}|_{0,h}^2 + h^2|\mathbf{v}|_{1,h}^2 + h^2 \sum_{e\in\Gamma} [\![\gamma\mathbf{v}]\!]_e^2 = I_1 + I_2 + I_3$. By Lemma 3.6 in [4], $I_2 \le C\|\mathbf{v}\|^2$, $I_3 \le C\|\mathbf{v}\|^2$, we have $h|||\mathbf{v}||| \le C\|\mathbf{v}\|$. □

Lemma 2.2 (see [4]). *The operator γ is self-adjoint with respect to the L^2-inner product, $(\mathbf{u}, \gamma\mathbf{v}) = (\mathbf{v}, \gamma\mathbf{u})$, $\forall \mathbf{u}, \mathbf{v} \in V_h$. Define $|||\mathbf{v}|||_0 = (\mathbf{v}, \gamma\mathbf{v})^{1/2}$. Then $|||\cdot|||_0$ and $\|\cdot\|$ are equivalent; here the equivalence constants are independent of h. And $\|\gamma\mathbf{v}\| = \|\mathbf{v}\|$, $\forall \mathbf{v} \in V_h$.*

Let

$$\begin{aligned}
a_0(\mathbf{v}, \xi) &= (\mathbf{v}, \xi) - \sum_{K\in\mathcal{R}_h} \sum_{j=1}^{t} \int_{A_{j+1}CA_j} \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \xi \, ds - \int_\Gamma [\![\boldsymbol{\xi}]\!] : \{\nabla \mathbf{v}\} ds, \\
c(\xi, q) &= \sum_{K\in\mathcal{R}_h} \sum_{j=1}^{t} \int_{A_{j+1}CA_j} q\xi \cdot \mathbf{n} \, ds + \int_\Gamma \{q\}[\xi] ds, \\
b_0(\mathbf{v}, q) &= \sum_{K\in\mathcal{R}_h} \int_K \nabla \cdot \mathbf{v} q \, dx.
\end{aligned} \tag{2.20}$$

It is clear that the solutions $(\mathbf{u}, p)$ of the Darcy-Stokes equations (1.1a)–(1.1c) satisfy the following:

$$\begin{aligned}
a_0(\mathbf{u}, \xi) + c(\xi, p) &= (\mathbf{f}, \xi), \quad \forall \xi \in W_h, \\
b_0(\mathbf{u}, q) &= 0, \quad \forall q \in Q_h.
\end{aligned} \tag{2.21}$$

Define the following bilinear forms:

$$\begin{aligned}
A_0(\mathbf{v}, \mathbf{w}) &= a_0(\mathbf{v}, \gamma\mathbf{w}), \quad \forall \mathbf{w}, \mathbf{v} \in V(h), \\
B_0(\mathbf{v}, q) &= b_0(\mathbf{v}, q), \quad \forall \mathbf{v} \in V(h),\ \forall q \in L_0^2(\Omega), \\
C(\mathbf{v}, q) &= c(\gamma\mathbf{v}, q), \quad \forall \mathbf{v} \in V(h),\ \forall q \in L_0^2(\Omega).
\end{aligned} \tag{2.22}$$

Then systems (2.21) are equivalent to

$$\begin{aligned}
A_0(\mathbf{u}, \mathbf{v}) + C(\mathbf{v}, p) &= (\mathbf{f}, \gamma\mathbf{v}), \quad \forall \mathbf{v} \in V_h, \\
B_0(\mathbf{u}, q) &= 0, \quad \forall q \in Q_h.
\end{aligned} \tag{2.23}$$

We propose two discontinuous finite volume formulations based on modification of the weak formulation (2.23) for Darcy-Stokes problem (1.1a)–(1.1c). Let us introduce the bilinear forms as follows:

$$A_1(\mathbf{v},\mathbf{w}) = A_0(\mathbf{v},\mathbf{w}) + \alpha \sum_{e\in\Gamma} [\![\gamma\mathbf{v}]\!]_e : [\![\gamma\mathbf{w}]\!]_e,$$
$$B(\mathbf{v},q) = B_0(\mathbf{v},q) - \int_\Gamma \{q\}[\gamma\mathbf{v}]\,ds, \tag{2.24}$$

where $\alpha > 0$ is a parameter to be determined later. For the exact solution $(\mathbf{u},p)$ of (1.1a)–(1.1c), we have

$$A_0(\mathbf{u},\mathbf{v}) = A_1(\mathbf{u},\mathbf{v}), \quad \forall\mathbf{v}\in V_h,$$
$$B_0(\mathbf{u},q) = B(\mathbf{u},q), \quad \forall q\in Q_h. \tag{2.25}$$

Therefore, it follows from (2.23) that

$$A_1(\mathbf{u},\mathbf{v}) + C(\mathbf{v},p) = (\mathbf{f},\gamma\mathbf{v}), \quad \forall\mathbf{v}\in V_h,$$
$$B(\mathbf{u},q) = 0, \quad \forall q\in Q_h. \tag{2.26}$$

The corresponding discontinuous finite volume scheme seeks $(\mathbf{u}_h,p_h)\in V_h\times Q_h$, such that

$$A_1(\mathbf{u}_h,\mathbf{v}) + C(\mathbf{v},p_h) = (\mathbf{f},\gamma\mathbf{v}), \quad \forall\mathbf{v}\in V_h,$$
$$B(\mathbf{u}_h,q) = 0, \quad \forall q\in Q_h. \tag{2.27}$$

Let e be an edge of element K. It is well known (see [2]) that there exists a constant C such that for any function $g\in H^2(K)$,

$$\|g\|_e^2 \le C\left(h_K^{-1}\|g\|_K^2 + h_K|g|_{1,K}^2\right), \tag{2.28}$$

$$\left\|\frac{\partial g}{\partial\mathbf{n}}\right\|_e^2 \le C\left(h_K^{-1}|g|_{1,K}^2 + h_K|g|_{2,K}^2\right), \tag{2.29}$$

where C depends only on the minimum angle of K.

Let $\nabla_h\mathbf{v}$ and $\nabla_h\cdot\mathbf{v}$ be the functions whose restriction to each element $\forall K\in\mathcal{R}_h$ is equal to $\nabla\mathbf{v}$ and $\nabla\cdot\mathbf{v}$, respectively.

Lemma 2.3. *For $\mathbf{v},\mathbf{w}\in V(h)$, there exists a positive constant C independent of h such that*

$$A_1(\mathbf{v},\mathbf{w}) \le C|||\mathbf{v}|||\,|||\mathbf{w}|||. \tag{2.30}$$

Proof. Let $A_{**}(\mathbf{v},\mathbf{w}) = (\mathbf{v},\gamma\mathbf{w}) + A_{*}(\mathbf{v},\mathbf{w})$,

$$A_{*}(\mathbf{v},\mathbf{w}) = -\sum_{K\in\mathcal{R}_h}\sum_{j=1}^{t}\int_{A_{j+1}CA_j}\frac{\partial \mathbf{v}}{\partial \mathbf{n}}\cdot\gamma\mathbf{w}\,ds. \tag{2.31}$$

By Lemma 3.1 in [5],

$$\begin{aligned}
A_{*}(\mathbf{v},\mathbf{w}) &= (\nabla_h\mathbf{v},\nabla_h\mathbf{w}) + \sum_{K\in\mathcal{R}_h}\int_{\partial K}(\gamma\mathbf{w}-\mathbf{w})\frac{\partial \mathbf{v}}{\partial \mathbf{n}}ds + \sum_{K\in\mathcal{R}_h}(\Delta\mathbf{v},\mathbf{w}-\gamma\mathbf{w})_K. \\
|A_{**}(\mathbf{v},\mathbf{w})| &\le \left|(\mathbf{v},\gamma\mathbf{w})\right| + |(\nabla_h\mathbf{v},\nabla_h\mathbf{w})| + \left|\sum_{K\in\mathcal{R}_h}\int_{\partial K}(\gamma\mathbf{w}-\mathbf{w})\frac{\partial \mathbf{v}}{\partial \mathbf{n}}ds\right| + \left|\sum_{K\in\mathcal{R}_h}(\Delta\mathbf{v},\mathbf{w}-\gamma\mathbf{w})_K\right| \\
&\le C\Bigg(|\mathbf{v}|_{0,h}|\mathbf{w}|_{0,h} + |\mathbf{v}|_{1,h}|\mathbf{w}|_{1,h} + \sum_{K\in\mathcal{R}_h}\left(h_K^{-1}\|\mathbf{w}-\gamma\mathbf{w}\|_K^2 + h_K\left|\mathbf{w}-\gamma\mathbf{w}\right|_{1,K}^2\right)^{1/2} \\
&\qquad \times\left(h_K^{-1}|\mathbf{v}|_{1,K}^2 + h_K|\mathbf{v}|_{2,K}^2\right)^{1/2} + \sum_{K\in\mathcal{R}_h}h_K|\mathbf{v}|_{2,K}|\mathbf{w}|_{1,K}\Bigg) \\
&\le C\Bigg(|\mathbf{v}|_{0,h}|\mathbf{w}|_{0,h} + |\mathbf{v}|_{1,h}|\mathbf{w}|_{1,h} + \left(\sum_{K\in\mathcal{R}_h}|\mathbf{w}|_{1,K}^2\right)^{1/2} \\
&\qquad \times\left(|\mathbf{v}|_{1,h} + \left(\sum_{K\in\mathcal{R}_h}h_K^2|\mathbf{v}|_{2,K}^2\right)^{1/2}\right) + \left(\sum_{K\in\mathcal{R}_h}h_K^2|\mathbf{v}|_{2,K}^2\right)^{1/2}|\mathbf{w}|_{1,h}\Bigg) \\
&\le C|||\mathbf{v}|||\;|||\mathbf{w}|||, \\
A_1(\mathbf{v},\mathbf{w}) &= A_{**}(\mathbf{v},\mathbf{w}) - \int_\Gamma [\![\gamma\mathbf{w}]\!] : \{\nabla\mathbf{v}\}ds + \alpha\sum_{e\in\Gamma}[\![\gamma\mathbf{v}_e]\!] : [\![\gamma\mathbf{w}_e]\!], \\
&\le C\Bigg(|||\mathbf{v}|||\;|||\mathbf{w}||| + \left(\sum_{K\in\mathcal{R}_h}\left(|\mathbf{v}|_{1,K}^2 + h_K^2|\mathbf{v}|_{2,K}^2\right)^{1/2}\right)\left(\sum_{e\in\Gamma}[\![\gamma\mathbf{w}_e^2]\!]\right)^{1/2} \\
&\qquad +\alpha\left(\sum_{e\in\Gamma}[\![\gamma\mathbf{v}_e^2]\!]\right)^{1/2}\left(\sum_{e\in\Gamma}[\![\gamma\mathbf{w}_e^2]\!]\right)^{1/2}\Bigg) \\
&\le C|||\mathbf{v}|||\;|||\mathbf{w}|||.
\end{aligned} \tag{2.32}$$

□

Lemma 2.4 (see [5]). *For any* $(\mathbf{v},q)\in V(h)\times Q_h$*, one has*

$$C(\mathbf{v},q) = -B(\mathbf{v},q). \tag{2.33}$$

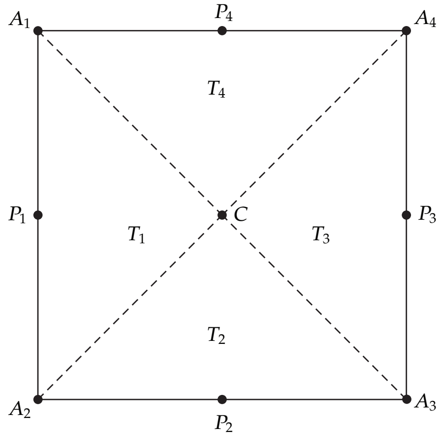

Figure 4: Rectangular partition and its dual.

Lemma 2.5 (see [5]). *For $(\mathbf{v}, q) \in V(h) \times L_0^2(\Omega)$, there exists a positive constant M independent of h such that*

$$C(\mathbf{v}, q) \leq M|||\mathbf{v}|||\left(\|q\| + \left(\sum_{K\in\mathcal{R}_h} h_K^2 |q|_{1,K}^2 \right)^{1/2} \right). \tag{2.34}$$

If $(\mathbf{v}, q) \in V_h \times Q_h$, then

$$C(\mathbf{v}, q) \leq M|||\mathbf{v}|||\, \|q\|. \tag{2.35}$$

Lemma 2.6. *For any $\mathbf{v} \in V_h$, there is a constant C_0 independent of h such that for α large enough*

$$A_1(\mathbf{v}, \mathbf{v}) \geq C_0|||\mathbf{v}|||^2. \tag{2.36}$$

Proof. Using the proof of Lemmas 3.1 and 3.5 in [5], for $\mathbf{v} \in V_h$,

$$\begin{gathered} \int_\Gamma \gamma\mathbf{v} : [\![\nabla\mathbf{v}]\!] ds \leq C|||\mathbf{v}|||_1 \left(\sum_{e\in\Gamma} [\![\gamma\mathbf{v}]\!]_e^2 \right)^{1/2}, \\ A_*(\mathbf{v}, \mathbf{w}) = (\nabla_h\mathbf{v}, \nabla_h\mathbf{w}), \quad \forall \mathbf{v}, \mathbf{w} \in V_h, \end{gathered} \tag{2.37}$$

we have

$$\begin{aligned} A_1(\mathbf{v},\mathbf{v}) &= (\mathbf{v},\gamma\mathbf{v}) + (\nabla_h\mathbf{v},\nabla_h\mathbf{v}) + \alpha\sum_{e\in\Gamma}[\![\gamma\mathbf{v}]\!]_e^2 - \int_\Gamma [\![\gamma\mathbf{v}]\!] : \{\nabla\mathbf{v}\}ds, \\ &\geq |\mathbf{v}|_{0,h}^2 + |\mathbf{v}|_{1,h}^2 + \alpha\sum_{e\in\Gamma}[\![\gamma\mathbf{v}]\!]_e^2 - C|||\mathbf{v}|||_1\left(\sum_{e\in\Gamma}[\![\gamma\mathbf{v}]\!]_e^2\right)^{1/2} \\ &\geq C|||\mathbf{v}|||_1^2 \geq C_0|||\mathbf{v}|||^2, \end{aligned} \tag{2.38}$$

when α is large enough. □

The value of α depends on the constant in the inverse inequality. Therefore, the value of α for which $A_1(\cdot,\cdot)$ is coercive is mesh dependent. We introduce a second discontinuous finite volume scheme which is parameter insensitive. Define a bilinear form as follows:

$$A_2(\mathbf{v},\mathbf{w}) = A_1(\mathbf{v},\mathbf{w}) + \int_\Gamma [\![\gamma\mathbf{v}]\!] : \{\nabla\mathbf{w}\}ds. \tag{2.39}$$

Similar to the bilinear form $A_1(\cdot,\cdot)$, for the exact solution $(\mathbf{u},p)$ of the Darcy-Stokes problem we have

$$A_2(\mathbf{u},\mathbf{v}) = A_0(\mathbf{u},\mathbf{v}), \quad \forall\mathbf{v}\in V_h. \tag{2.40}$$

Consequently, the solution of the Darcy-Stokes problem satisfies the following variational equations:

$$\begin{gathered} A_2(\mathbf{u},\mathbf{v}) + C(\mathbf{v},p) = (\mathbf{f},\gamma\mathbf{v}), \quad \forall\mathbf{v}\in V_h, \\ B(\mathbf{u},q) = 0, \quad \forall q\in Q_h. \end{gathered} \tag{2.41}$$

Our second discontinuous finite volume scheme for (1.1a)–(1.1c) seeks $(\mathbf{u}_h,p_h)\in V_h\times Q_h$, such that

$$\begin{gathered} A_2(\mathbf{u}_h,\mathbf{v}) + C(\mathbf{v},p_h) = (\mathbf{f},\gamma\mathbf{v}), \quad \forall\mathbf{v}\in V_h, \\ B(\mathbf{u}_h,q) = 0, \quad \forall q\in Q_h. \end{gathered} \tag{2.42}$$

For any value of $\alpha > 0$, we have

$$A_2(\mathbf{v},\mathbf{v}) = (\mathbf{v},\gamma\mathbf{v}) + (\nabla_h\mathbf{v},\nabla_h\mathbf{v}) + \alpha\sum_{e\in\Gamma}[\![\gamma\mathbf{v}]\!]_e^2 \geq C|||\mathbf{v}|||_1^2 \geq C_0|||\mathbf{v}|||^2, \quad \forall\mathbf{v}\in V_h. \tag{2.43}$$

Similarly, we can prove that

$$A_2(\mathbf{v},\mathbf{w}) \leq C|||\mathbf{w}|||\,|||\mathbf{v}|||, \quad \forall\mathbf{v},\mathbf{w}\in V(h). \tag{2.44}$$

Let $A(\mathbf{v},\mathbf{w}) = A_1(\mathbf{v},\mathbf{w})$ or $A(\mathbf{v},\mathbf{w}) = A_2(\mathbf{v},\mathbf{w})$. In the rest of the paper, we assume that the following is true:

$$A(\mathbf{v},\mathbf{v}) \geq C_0 |||\mathbf{v}|||^2. \tag{2.45}$$

If $A(\mathbf{v},\mathbf{w}) = A_2(\mathbf{v},\mathbf{w})$, (2.45) holds for any $\alpha > 0$. If $A(\mathbf{v},\mathbf{w}) = A_1(\mathbf{v},\mathbf{w})$, (2.45) holds for only α large enough.

3. Error Estimates

We will derive optimal error estimates for velocity in the norm $|||\cdot|||$ and for pressure in the L^2-norm. A first-order error estimate for velocity in L^2-norm will be obtained.

Let e be an interior edge shared by two elements K_1 and K_2 in $\mathcal{R}_h$. If $\int_e \mathbf{v}|_{K_1} ds = \int_e \mathbf{v}|_{K_2} ds$, we say that $\mathbf{v}$ is continuous on e. We say that $\mathbf{v}$ is zero at $e \in \partial\Omega$ if $\int_e \mathbf{v}\, ds = 0$. Define a subspace $\widehat{V}_h$ of V_h by

$$\widehat{V}_h = \left\{ \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in \widehat{Q}_1(K)^2 \ \forall K \in \mathcal{R}_h \text{ is continuous at } e \in \Gamma_0 \text{ and is zero at } e \in \partial\Omega \right\} \tag{3.1}$$

for rectangular meshes and by

$$\widehat{V}_h = \left\{ \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in P_1(K)^2 \ \forall K \in \mathcal{R}_h \text{ is continuous at } e \in \Gamma_0 \text{ and is zero at } e \in \partial\Omega \right\} \tag{3.2}$$

for triangular mesh.

It has been proven in [8, 9] that the following discrete inf-sup condition is satisfied; that is, there exists a positive constant β_0 such that

$$\sup_{\mathbf{v}\in\widehat{V}_h} \frac{(\nabla_h \cdot \mathbf{v}, q)}{|\mathbf{v}|_{1,h}} \geq \beta_0 \|q\|, \quad \forall q \in Q_h. \tag{3.3}$$

Lemma 3.1. *The bilinear form $B(\cdot,\cdot)$ satisfies the discrete inf-sup condition*

$$\sup_{\mathbf{v}\in V_h} \frac{B(\mathbf{v}, q)}{|||\mathbf{v}|||} \geq \beta \|q\|, \quad \forall q \in Q_h, \tag{3.4}$$

where β is a positive constant independent of the mesh size h.

Proof. For $\mathbf{v} \in \widehat{V}_h \subset V_h$ and $q \in Q_h$, we have $B(\mathbf{v},q) = (\nabla_h \cdot \mathbf{v}, q)$, and $|||\mathbf{v}|||_1 = \|\mathbf{v}\|_{1,h}$. By Poincare-Friedrichs $\|\mathbf{v}\|_{1,h} \leq C|\mathbf{v}|_{1,h}$, with (3.3), and (2.17) we get for any $q \in Q_h$

$$\beta_0 \|q\| \leq \sup_{\mathbf{v}\in\widehat{V}_h} \frac{(\nabla \cdot \mathbf{v}, q)}{|\mathbf{v}|_{1,h}} \leq C \sup_{\mathbf{v}\in\widehat{V}_h} \frac{B(\mathbf{v},q)}{|||\mathbf{v}|||_1} \leq C_1 \sup_{\mathbf{v}\in V_h} \frac{B(\mathbf{v},q)}{|||\mathbf{v}|||}. \tag{3.5}$$

With $\beta = \beta_0/C_1$, we have proven (3.4). □

Define an operator $\pi_K : H^1(K) \rightarrow P_1(K)$ or $\widehat{Q}_1(K)$. For all $v \in H^1(K)$,

$$\int_{e_i} \pi_K v\, ds = \int_{e_i} v\, ds, \quad i = 1, \dots, t, \tag{3.6}$$

where $e_i,\ i = 1, \dots, t$, are the t sides of the element K. $t = 3$ if K is a triangle and $t = 4$ if K is a rectangle. It was proven in [8] that

$$|\pi_K v - v|_{s,K} \le Ch^{2-s}|v|_{2,K}, \quad s = 0, 1, 2. \tag{3.7}$$

For all $\mathbf{v} = (v_1, v_2) \in H_0^1(\Omega)^2$, define $\Pi_1 \mathbf{v} = (\Pi_1 \mathbf{v}_1, \Pi_1 \mathbf{v}_2) \in V_h$ by

$$\Pi_1 v_i|_K = \pi_K v_i, \quad \forall K \in \mathcal{R}_h,\ i = 1, 2. \tag{3.8}$$

Using the definition of Π_1 and integration by parts, we can show that

$$B(\mathbf{v} - \Pi_1 \mathbf{v}, q) = 0, \quad \forall q \in Q_h. \tag{3.9}$$

The Cauchy-Schwarz inequality implies

$$[\![\gamma \mathbf{v}]\!]_e^2 = \left(\frac{1}{h_e}\int_e [\![\mathbf{v}]\!] ds\right)^2 \le \left(\frac{1}{h_e}\right)^2 \int_e [\![\mathbf{v}]\!]^2 ds \int_e ds = \int_e \frac{1}{h_e}[\![\mathbf{v}]\!]^2 ds. \tag{3.10}$$

Equations (2.28) and (3.8) imply that

$$\sum_{e\in\Gamma}[\![\gamma(\mathbf{u} - \Pi_1\mathbf{u})]\!]_e^2 \le C\left(|\mathbf{u} - \Pi_1\mathbf{u}|_{1,h}^2 + \sum_{K\in\mathcal{R}_h} h^{-2}\|\mathbf{u} - \Pi_1\mathbf{u}\|_K^2\right) \le Ch^2\|\mathbf{u}\|_2^2. \tag{3.11}$$

The definitions of the norm $|||\cdot|||$, (3.7), and (3.11) give

$$\begin{aligned} |||\mathbf{u} - \Pi_1\mathbf{u}|||^2 &= |\mathbf{u} - \Pi_1\mathbf{u}|_{0,h}^2 + |\mathbf{u} - \Pi_1\mathbf{u}|_{1,h}^2 + \sum_{e\in\Gamma}[\![\gamma(\mathbf{u} - \Pi_1\mathbf{u})]\!]_e^2 + \sum_{K\in\mathcal{R}_h} h^2|\mathbf{u} - \Pi_1\mathbf{u}|_{2,K}^2 \\ &\le Ch^2\|\mathbf{u}\|_2^2. \end{aligned} \tag{3.12}$$

Theorem 3.2. *Let* $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ *be the solution of* (2.27), *and let* $(\mathbf{u}, p) \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (L_0^2(\Omega) \cap H^1(\Omega))$ *be the solution of* (1.1a)–(1.1c). *Then there exists a constant* C *independent of* h *such that*

$$|||\mathbf{u} - \mathbf{u}_h||| + \|p - p_h\| \le Ch(\|\mathbf{u}\|_2 + \|p\|_1), \tag{3.13}$$

$$\|\mathbf{u} - \mathbf{u}_h\| \le Ch(\|\mathbf{u}\|_2 + \|p\|_1). \tag{3.14}$$

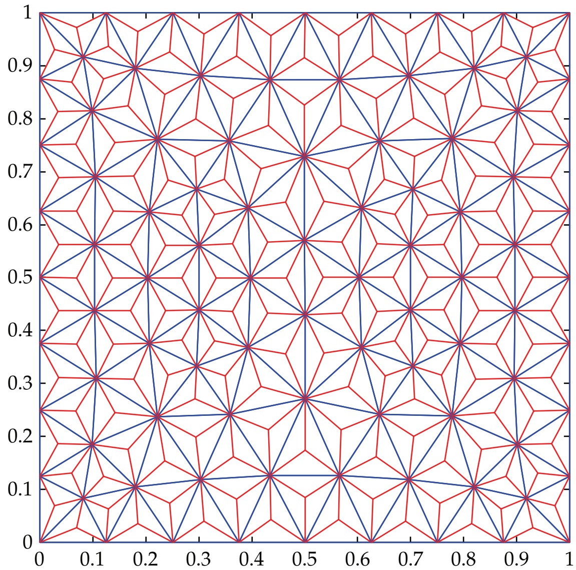

Figure 5: Triangular and its dual partition of $(0,1)\times(0,1)$.

Proof. Let $\varepsilon = \mathbf{u} - \Pi_1\mathbf{u}$, $\varepsilon_h = \mathbf{u}_h - \Pi_1\mathbf{u}$, $\eta = p - \Pi_2 p$, $\eta_h = p_h - \Pi_2 p$, where Π_2 is L^2 projection from $L_0^2(\Omega) \to Q_h$. Then $\mathbf{u} - \mathbf{u}_h = \varepsilon - \varepsilon_h$, $p - p_h = \eta - \eta_h$. Subtracting (2.26) from (2.27) and using Lemma 2.4, we get error equations

$$A(\varepsilon_h, \mathbf{v}) - B(\mathbf{v}, \eta_h) = A(\boldsymbol{\varepsilon}, \mathbf{v}) + C(\mathbf{v}, \eta), \quad \forall \mathbf{v} \in V_h, \tag{3.15a}$$

$$B(\boldsymbol{\varepsilon}_h, q) = B(\boldsymbol{\varepsilon}, q) = 0, \quad \forall q \in Q_h. \tag{3.15b}$$

By letting $\mathbf{v} = \varepsilon_h$ in (3.15a) and $q = \eta_h$ in (3.15b), the sum of (3.15a) and (3.15b) gives

$$A(\boldsymbol{\varepsilon}_h, \boldsymbol{\varepsilon}_h) = A(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_h) + C(\boldsymbol{\varepsilon}_h, \eta). \tag{3.16}$$

Thus, it follows from the coercivity (2.45), the boundedness (2.30), (2.44), and (2.34) that

$$|||\boldsymbol{\varepsilon}_h|||^2 \le C\left(|||\boldsymbol{\varepsilon}|||\,|||\boldsymbol{\varepsilon}_h||| + \left(\|\eta\| + \left(\sum_{K\in\mathcal{R}_h} h_K^2 |\eta|_{1,K}^2 \right)^{1/2} \right) |||\boldsymbol{\varepsilon}_h||| \right), \tag{3.17}$$

which implies the following:

$$|||\boldsymbol{\varepsilon}_h||| \le C\left(|||\boldsymbol{\varepsilon}||| + \|\eta\| + \left(\sum_{K\in\mathcal{R}_h} h_K^2 |\eta|_{1,K}^2 \right)^{1/2} \right). \tag{3.18}$$

The previous estimate can be rewritten as

$$|||\mathbf{u}_h - \Pi_1\mathbf{u}||| \le C\left(|||\mathbf{u} - \Pi_1\mathbf{u}||| + \|p - \Pi_2 p\| + \left(\sum_{K\in\mathcal{R}_h} h_K^2 |p - \Pi_2 p|_{1,K}^2 \right)^{1/2} \right). \tag{3.19}$$

Now using the triangle inequality, (3.7), the definition of Π_2, and the inequality mentioned previously, we get

$$|||\mathbf{u} - \mathbf{u}_h||| \le C(|||\mathbf{u} - \Pi_1\mathbf{u}||| + |||\mathbf{u}_h - \Pi_1\mathbf{u}|||) \le Ch(\|\mathbf{u}\|_2 + \|p\|_1), \tag{3.20}$$

which completes the estimate for the velocity approximation.

Discrete inf-sup condition (3.4), (3.15a), (3.15b), Lemmas 2.5, 2.4, and inverse inequality give

$$\begin{aligned}
\|p_h - \Pi_2 p\| &\le \frac{1}{\beta} \sup_{\mathbf{v}\in V_h} \frac{B(\mathbf{v}, \Pi_2 p - p_h)}{|||\mathbf{v}|||_{1,h}} = \frac{1}{\beta} \sup_{\mathbf{v}\in V_h} \frac{C(\mathbf{v}, p_h - \Pi_2 p)}{|||\mathbf{v}|||_{1,h}} \\
&= \frac{1}{\beta} \sup_{\mathbf{v}\in V_h} \frac{C(\mathbf{v}, p_h - p) + C(\mathbf{v}, p - \Pi_2 p)}{|||\mathbf{v}|||} \\
&= \frac{1}{\beta} \sup_{\mathbf{v}\in V_h} \frac{A(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + C(\mathbf{v}, p - \Pi_2 p)}{|||\mathbf{v}|||} \\
&\le C\left(|||\mathbf{u} - \mathbf{u}_h||| + \|p - \Pi_2 p\| + \left(\sum_{K\in\mathcal{R}_h} h_K^2 |p - \Pi_2 p|_{1,K}^2 \right)^{1/2} \right) \\
&\le Ch(\|\mathbf{u}\|_2 + \|p\|_1).
\end{aligned} \tag{3.21}$$

Using the previous inequality and the triangle inequality, we have completed the proof of (3.13).

Using Lemma 2.1, (3.12), and (3.13), we have

$$\|\mathbf{u}_h - \Pi_1\mathbf{u}_h\| \le C|||\mathbf{u}_h - \Pi_1\mathbf{u}_h||| \le C(|||\mathbf{u} - \mathbf{u}_h||| + |||\mathbf{u} - \Pi_1\mathbf{u}_h|||) \le Ch(\|\mathbf{u}\|_2 + \|p\|_1). \tag{3.22}$$

Equations (3.22) and (3.7) and the triangle inequality imply (3.14). We have completed the proof. □

4. Numerical Experiments

In this section, we present a numerical example for solving the problems (1.1a)–(1.1c) by using the discontinuous finite volume element method presented with (2.27) and (2.42). Let $\Omega = (0,1) \times (0,1)$, $\mathcal{R}_h$ be the Delaunay triangulation generated by EasyMesh [10] over Ω with mesh size h as shown in Figure 5. We consider the case of $\sigma = 1$, $\mu = 1$, the exact velocity $u_1(x,y) = -x^2(x-1)^2 y(y-1)(2y-1)$, $u_2(x,y) = -u_1(y,x)$ and the pressure $p(x,y) =$

Table 1: Error behavior for scheme (2.27).

h_d	h	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$\frac{\|\|\|\mathbf{u}-\mathbf{u}_{2h}\|\|\|}{\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|}$	$\|\mathbf{u}-\mathbf{u}_h\|$	$\frac{\|\mathbf{u}-\mathbf{u}_{2h}\|}{\|\mathbf{u}-\mathbf{u}_h\|}$	$\|p-p_h\|$	$\frac{\|p-p_{2h}\|}{\|p-p_h\|}$
1/8	$1.598e-1$	$2.082e-2$		$3.393e-4$		$1.068e-2$	
1/16	$8.372e-2$	$1.031e-2$	2.0	$9.649e-5$	3.5	$5.345e-3$	2.0
1/32	$3.679e-2$	$5.185e-3$	2.0	$2.598e-5$	3.7	$2.650e-3$	2.0
1/64	$1.899e-2$	$2.611e-3$	2.0	$6.795e-6$	3.8	$1.323e-3$	2.0
1/128	$9.413e-3$	$1.307e-3$	2.0	$1.730e-6$	3.9	$6.598e-4$	2.0

Table 2: Error behavior for scheme (2.42).

h_d	h	$\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|$	$\frac{\|\|\|\mathbf{u}-\mathbf{u}_{2h}\|\|\|}{\|\|\|\mathbf{u}-\mathbf{u}_h\|\|\|}$	$\|\mathbf{u}-\mathbf{u}_h\|$	$\frac{\|\mathbf{u}-\mathbf{u}_{2h}\|}{\|\mathbf{u}-\mathbf{u}_h\|}$	$\|p-p_h\|$	$\frac{\|p-p_{2h}\|}{\|p-p_h\|}$
1/8	$1.598e-1$	$2.071e-2$		$3.280e-4$		$1.079e-2$	
1/16	$8.372e-2$	$1.027e-2$	2.0	$9.204e-5$	3.5	$5.380e-3$	2.0
1/32	$3.679e-2$	$5.175e-3$	2.0	$2.476e-5$	3.7	$2.659e-3$	2.0
1/64	$1.899e-2$	$2.608e-3$	2.0	$6.361e-6$	3.8	$1.325e-3$	2.0
1/128	$9.413e-3$	$1.306e-3$	2.0	$1.613e-6$	3.9	$6.603e-4$	2.0

$(x-0.5)(y-0.5)$. Denote the numerical solution as $\mathbf{u}_h$ and p_h with step h_d which is used to generate the mesh data in the EasyMesh input file, and $h = \max\{h_e : e \in \Gamma\}$. For $\alpha = 2$, the numerical results are presented in Tables 1 and 2. It is observed from the tables that the numerical results support our theory.

Acknowledgments

This paper is supported by the Excellent Young and Middle-Aged Scientists Research Fund of Shandong Province (2008BS09026), National Natural Science Foundation of China (11171193), and National Natural Science Foundation of Shandong Province (ZR2011AM016).

References

[1] W. H. Reed and T. R. Hill, "Triangular mesh methods for the neutron transport equation," Tech. Rep. LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, NM, USA, 1973.

[2] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, "Unified analysis of discontinuous Galerkin methods for elliptic problems," *SIAM Journal on Numerical Analysis*, vol. 39, no. 5, pp. 1749–1779, 2002.

[3] X. Ye, "A new discontinuous finite volume method for elliptic problems," *SIAM Journal on Numerical Analysis*, vol. 42, no. 3, pp. 1062–1072, 2004.

[4] C. Bi and J. Geng, "Discontinuous finite volume element method for parabolic problems," *Numerical Methods for Partial Differential Equations*, vol. 26, no. 2, pp. 367–383, 2010.

[5] X. Ye, "A discontinuous finite volume method for the Stokes problems," *SIAM Journal on Numerical Analysis*, vol. 44, no. 1, pp. 183–198, 2006.

[6] E. Burman and P. Hansbo, "Stabilized Crouzeix-Raviart element for the Darcy-Stokes problem," *Numerical Methods for Partial Differential Equations*, vol. 21, no. 5, pp. 986–997, 2005.

[7] A. Masud, "A stabilized mixed finite element method for Darcy-Stokes flow," *International Journal for Numerical Methods in Fluids*, vol. 54, no. 6–8, pp. 665–681, 2007.

[8] R. Rannacher and S. Turek, "Simple nonconforming quadrilateral Stokes element," *Numerical Methods for Partial Differential Equations*, vol. 8, no. 2, pp. 97–111, 1992.

[9] M. Crouzeix and P.-A. Raviart, “Conforming and nonconforming finite element methods for solving the stationary Stokes equations,” *RAIRO. Modélisation Mathématique et Analyse Numérique*, vol. 7, pp. 33–75, 1973.

[10] EasyMesh, http://www-dinma.univ.trieste.it/nirftc/research/easymesh/.