Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 749084, 10 pages
doi:10.1155/2012/749084

Research Article

On Generalized Hyers-Ulam Stability of Admissible Functions

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Received 8 December 2011; Accepted 24 January 2012

Academic Editor: Árpád Baricz

We consider the Hyers-Ulam stability for the following fractional differential equations in sense of Srivastava-Owa fractional operators (derivative and integral) defined in the unit disk: $D_z^{\beta} f(z) = G(f(z), D_z^{\alpha} f(z), zf'(z); z), 0 < \alpha < 1 < \beta \leq 2$, in a complex Banach space. Furthermore, a generalization of the admissible functions in complex Banach spaces is imposed, and applications are illustrated.

1. Introduction

A classical problem in the theory of functional equations is the following: if a function f approximately satisfies functional equation $\mathcal{E}$, when does there exist an exact solution of $\mathcal{E}$ which f approximates? In 1940, Ulam [1, 2] imposed the question of the stability of Cauchy equation, and in 1941, Hyers solved it [3]. In 1978, Rassias [4] provided a generalization of Hyers theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (see [5–7]). Li and Hua [8] discussed and proved the Hyers-Ulam stability of spacial type of finite polynomial equation, and Bidkham et al. [9] introduced the Hyers-Ulam stability of generalized finite polynomial equation. Rassias [10] imposed a Cauchy type additive functional equation and investigated the generalized Hyers-Ulam "product-sum" stability of this equation.

Recently, Jung presented a book [11], which complements the books of Hyers, Isac, and Rassias (Stability of Functional Equations in Several Variables, Birkhäuser, 1998) and of Czerwik (Functional Equations and Inequalities in Several Variables, World Scientific, 2002) by covering and offering almost all classical results on the Hyers-Ulam-Rassias stability such as the Hyers-Ulam-Rassias stability of the additive Cauchy equation, generalized additive functional equations, Hosszú's functional equation, Hosszú's equation of Pexider type, homogeneous functional equation, Jensen's functional equation, the quadratic functional equations, the exponential functional equations, Wigner equation, Fibonacci functional equation, the gamma functional equation, and the multiplicative functional equations. Furthermore, the concept of superstability for some problems is defined and studied.

The Ulam stability and data dependence for fractional differential equations in sense of Caputo derivative has been posed by Wang et al. [12] while in sense of Riemann-Liouville derivative has been discussed by Ibrahim [13]. Finally, the author generalized the Ulam-Hyers stability for fractional differential equation including infinite power series [14, 15].

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators [16], Erdèlyi-Kober operators [17], Weyl-Riesz operators [18], Grünwald-Letnikov operators [19] and Caputo fractional derivative [20–24], have appeared during the past three decades. The existence of positive solution and multipositive solutions for nonlinear fractional differential equation are established and studied [25]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain is suggested and posed in [26–28].

2. Preliminaries

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ denote the space of all analytic functions on U. Here we suppose that $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\mathcal{H}[a, m]$ be the subspace of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots, \quad z \in U. \tag{2.1}$$

Let $\mathcal{A}$ be the class of functions f, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is called univalent ($\mathcal{S}$) if it is one-one in U. A function $f \in \mathcal{A}$ is called convex if it satisfies the following inequality:

$$\Re\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0, \quad (z \in U). \tag{2.2}$$

We denoted this class $\mathcal{C}$.

In [29], Srivastava and Owa, posed definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows.

Definition 2.1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} \, d\zeta, \tag{2.3}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Definition 2.2. The fractional integral of order $\alpha>0$ is defined, for a function $f(z)$, by

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1}\, d\zeta; \quad \alpha>0, \tag{2.4}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane ($\mathbb{C}$) containing the origin, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Remark 2.3. We have the following:

$$\begin{aligned} D_z^{\alpha} z^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \quad \mu>-1, \\ I_z^{\alpha} z^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad \mu>-1. \end{aligned} \tag{2.5}$$

In [27], it was shown the relation

$$I_z^{\alpha} D_z^{\alpha} f(z) = D_z^{\alpha} I_z^{\alpha} f(z) = f(z), \qquad f(0)=0. \tag{2.6}$$

More details on fractional derivatives and their properties and applications can be found in [30, 31].

We next introduce the generalized Hyers-Ulam stability depending on the properties of the fractional operators.

Definition 2.4. Let $p\in(0,1)$. We say that

$$\sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z) \tag{2.7}$$

has the generalized Hyers-Ulam stability if there exists a constant $K>0$ with the following property: for every $\epsilon>0$, $w\in\overline{U}=U\bigcup\partial U$, if

$$\left|\sum_{n=0}^{\infty} a_n w^{n+\alpha}\right| \le \epsilon\left(\sum_{n=0}^{\infty}\frac{|a_n|^p}{p(n+1)^2}\right), \tag{2.8}$$

then there exists some $z\in\overline{U}$ that satisfies (2.7) such that

$$\left|z^i - w^i\right| \le \epsilon K, \quad \left(z,w\in\overline{U},\ i\in\mathbb{N}\right). \tag{2.9}$$

In the present paper, we study the generalized Hyers-Ulam stability for holomorphic solutions of the fractional differential equation in complex Banach spaces X and Y

$$D_z^{\beta} f(z) = G\big(f(z), D_z^{\alpha} f(z), zf'(z); z\big), \quad 0 < \alpha < 1 < \beta \le 2, \tag{2.10}$$

where $G : X^3 \times U \to Y$ and $f : U \to X$ are holomorphic functions such that $f(0) = \Theta$ (Θ is the zero vector in X).

3. Generalized Hyers-Ulam Stability

In this section we present extensions of the generalized Hyers-Ulam stability to holomorphic vector-valued functions. Let X, Y represent complex Banach space. The class of admissible functions $\mathcal{G}(X, Y)$ consists of those functions $g : X^3 \times U \to Y$ that satisfy the admissibility conditions:

$$\big\| g(r, ks, lt; z) \big\| \ge 1, \quad \text{when } \|r\| = \|s\| = \|t\| = 1, \ (z \in U, \ k, l \ge 1). \tag{3.1}$$

We need the following results.

Lemma 3.1 (see [32]). *If $f : D \to X$ is holomorphic, then $\|f\|$ is a subharmonic of $z \in D \subset \mathbb{C}$. It follows that $\|f\|$ can have no maximum in D unless $\|f\|$ is of constant value throughout D.*

Lemma 3.2 (see [33]). *Let $f : U \to X$ be the holomorphic vector-valued function defined in the unit disk U with $f(0) = \Theta$ (the zero element of X). If there exists a $z_0 \in U$ such that*

$$\big\| f(z_0) \big\| = \max_{|z| = |z_0|} \big\| f \big\|, \tag{3.2}$$

then

$$\big\| z_0 f'(z_0) \big\| = \kappa \big\| f(z_0) \big\|, \quad \kappa \ge 1. \tag{3.3}$$

Lemma 3.3 (see [34, page 88]). *If the function $f(z)$ is in the class $\mathcal{S}$, then*

$$\big| D_z^{\alpha+n} f(z) \big| \le \frac{(n + \alpha + |z|)\Gamma(n + \alpha + 1)}{(1 - |z|)^{n+\alpha+2}}, \quad (z \in U; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \ 0 \le \alpha < 1). \tag{3.4}$$

Lemma 3.4 (see [29, page 225]). *If the function $f(z)$ is in the class $\mathcal{C}$, then*

$$\big| D_z^{\alpha+n} f(z) \big| \le \frac{\Gamma(n + \alpha + 1)}{(1 - |z|)^{n+\alpha+1}}, \quad (z \in U; n \in \mathbb{N}_0; \ 0 \le \alpha < 1). \tag{3.5}$$

Theorem 3.5. *Let $G \in \mathcal{G}(X, Y)$ and $f : U \to X$ be a holomorphic vector-valued function defined in the unit disk U, with $f(0) = \Theta$. If $f \in \mathcal{S}$, then*

$$\big\| G\big(f(z), D_z^{\alpha} f(z); z\big) \big\| < 1 \Longrightarrow \big\| f(z) \big\| < 1. \tag{3.6}$$

Proof. Since $f \in \mathcal{S}$, then from Lemma 3.3, we observe that

$$\left|D_z^\alpha f(z)\right| \leq \frac{(\alpha + |z|)\Gamma(\alpha + 1)}{(1 - |z|)^{\alpha+2}}. \tag{3.7}$$

Assume that $\|f(z)\| \not< 1$ for $z \in U$. Thus, there exists a point $z_0 \in U$ for which $\|f(z_0)\| = 1$. According to Lemma 3.1, we have

$$\begin{gathered} \|f(z)\| < 1, \quad z \in U_{r_0} = \{z : |z| < |z_0| = r_0\}, \\ \max_{|z| \leq |z_0|} \|f(z)\| = \|f(z_0)\| = 1. \end{gathered} \tag{3.8}$$

In view of Lemma 3.2, at the point z_0, there is a constant $\kappa \geq 1$ such that

$$\|z_0 f'(z_0)\| = \kappa \|f(z_0)\| = \kappa. \tag{3.9}$$

Consequently, we obtain that

$$\|f(z_0)\| = \frac{(1 - |z|)^{\alpha+2}}{(\alpha + |z|)\Gamma(\alpha + 1)} \qquad \|D_{z_0}^\alpha f(z_0)\| = \frac{1}{\kappa} \|z_0 f'(z_0)\| = 1. \tag{3.10}$$

We put $k := ((\alpha + |z|)\Gamma(\alpha + 1)/(1 - |z|)^{\alpha+2}) \geq 1$, for some $0 < \alpha < 1$ and $z \in U$ and $l := \kappa \geq 1$; hence from (3.1), we deduce

$$\|G(f(z_0), D_{z_0}^\alpha f(z_0), z_0 f'(z_0); z_0)\| = \left\| G\left(f(z_0), k\left[\frac{D_{z_0}^\alpha f(z_0)}{k}\right], l\left[\frac{z_0 f'(z_0)}{l}\right]; z_0 \right) \right\| \geq 1, \tag{3.11}$$

which contradicts the hypothesis in (3.6) that we must have $\|f(z)\| < 1$. □

Corollary 3.6. *Assume the problem* (2.10). *If* $G \in \mathcal{G}(X, Y)$ *is a holomorphic univalent vector-valued function defined in the unit disk* U, *then*

$$\|G(f(z), D_z^\alpha f(z), z f'(z); z)\| < 1 \Longrightarrow \left\| I_z^\beta G(f(z), D_z^\alpha f(z), z f'(z); z) \right\| < 1. \tag{3.12}$$

Proof. By univalency of G, the fractional differential equation (2.10) has at least one holomorphic univalent solution f. Thus, according to Remark 2.3, the solution $f(z)$ of the problem (2.10) takes the form

$$f(z) = I_z^\beta G(f(z), D_z^\alpha f(z), z f'(z); z). \tag{3.13}$$

Therefore, in virtue of Theorem 3.5, we obtain the assertion (3.12). □

Theorem 3.7. *Let $G \in \mathcal{G}(X, Y)$ be holomorphic univalent vector-valued functions defined in the unit disk U then* (2.10) *has the generalized Hyers-Ulam stability for $z \to \partial U$.*

Proof. Assume that

$$G(z) := \sum_{n=0}^{\infty} \varphi_n z^n, \quad z \in U \tag{3.14}$$

therefore, by Remark 2.3, we have

$$I_z^{\alpha} G(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z). \tag{3.15}$$

Also, $z \to \partial U$ and thus $|z| \to 1$. According to Theorem 3.5, we have

$$\|f(z)\| < 1 = |z|. \tag{3.16}$$

Let $\epsilon > 0$ and $w \in \overline{U}$ be such that

$$\left| \sum_{n=1}^{\infty} a_n w^{n+\alpha} \right| \leq \epsilon \left(\sum_{n=1}^{\infty} \frac{|a_n|^p}{p(n+1)^2} \right). \tag{3.17}$$

We will show that there exists a constant K independent of ϵ such that

$$\left| w^i - u^i \right| \leq \epsilon K, \quad w \in \overline{U}, \; u \in U \tag{3.18}$$

and satisfies (2.7). We put the function

$$f(w) = \frac{-1}{\lambda a_i} \sum_{n=1, n \neq i}^{\infty} a_n w^{n+\alpha}, \quad a_i \neq 0, \; 0 < \lambda < 1, \tag{3.19}$$

thus, for $w \in \partial U$, we obtain

$$\begin{aligned} \left| w^i - u^i \right| &= \left| w^i - \lambda f(w) + \lambda f(w) - u^i \right| \\ &\leq \left| w^i - \lambda f(w) \right| + \lambda \left| f(w) - u^i \right| \\ &< \left| w^i - \lambda f(w) \right| + \lambda \left| w^i - u^i \right| \\ &= \left| w^i + \frac{1}{a_i} \sum_{n=1, n \neq i}^{\infty} a_n w^{n+\alpha} \right| + \lambda \left| w^i - u^i \right| \\ &= \frac{1}{|a_i|} \left| \sum_{n=1}^{\infty} a_n w^{n+\alpha} \right| + \lambda \left| w^i - u^i \right|. \end{aligned} \tag{3.20}$$

Without loss of generality, we consider $|a_i| = \max_{n\geq 1}(|a_n|)$ yielding

$$\begin{aligned}
\left|w^i - u^i\right| &\leq \frac{1}{|a_i|(1-\lambda)}\left|\sum_{n=1}^{\infty} a_n w^{n+\alpha}\right| \\
&\leq \frac{\epsilon}{|a_i|(1-\lambda)}\left(\sum_{n=0}^{\infty}\frac{|a_n|^p}{p(n+1)^2}\right) \\
&\leq \frac{\epsilon|a_i|^{p-1}}{p(1-\lambda)}\left(\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\right) \\
&= \frac{\pi^2\epsilon|a_i|^{p-1}}{6p(1-\lambda)} \\
&:= K\epsilon.
\end{aligned} \tag{3.21}$$

This completes the proof. □

In the same manner of Theorem 3.5, and by using Lemma 3.4, we have the following result.

Theorem 3.8. *Let $G \in \mathcal{G}(X,Y)$ and $f : U \to X$ be a holomorphic vector-valued function defined in the unit disk U, with $f(0) = \Theta$. If $f \in \mathcal{C}$, then*

$$\left\|G\left(f(z), D_z^\alpha f(z), zf'(z); z\right)\right\| < 1 \Longrightarrow \left\|f(z)\right\| < 1. \tag{3.22}$$

4. Applications

In this section, we introduce some applications of functions to achieve the generalized Hyers-Ulam stability.

Example 4.1. Consider the function $G : X^3 \times U \to \mathbb{R}$ by

$$G(r,s,t;z) = a(\|r\| + \|s\| + \|t\|)^n + b|z|^2, \quad n \in \mathbb{R}_+ \tag{4.1}$$

with $a \geq 0.5$, $b \geq 0$ and $G(\Theta,\Theta,\Theta;\ 0) = 0$. Our aim is to apply Theorem 3.5, this follows since

$$\|G(r,ks,lt;z)\| = a(\|r\| + k\|s\| + l\|t\|)^n + b|z|^2 = a(1+k+l)^n + b|z|^2 \geq 1, \tag{4.2}$$

when $\|r\| = \|s\| = \|t\| = 1$, $z \in U$. Hence by Theorem 3.5, we have the following. If $a \geq 0.5$, $b \geq 0$ and $f : U \to X$ is a holomorphic univalent vector-valued function defined in U, with $f(0) = \Theta$, then

$$a\left(\left\|f(z)\right\| + \left\|D_z^\alpha f(z)\right\| + \left\|zf'(z)\right\|\right)^n + b|z|^2 < 1 \Longrightarrow \left\|f(z)\right\| < 1. \tag{4.3}$$

Consequently, $\|I_z^\alpha G(f(z), D_z^\alpha f(z), zf'(z); z)\| < 1$, thus in view of Theorem 3.7, f has the generalized Hyers-Ulam stability.

Example 4.2. Assume that the function $G : X^3 \longrightarrow X$ by

$$G(r,s,t;z) = G(r,s,t) = re^{\|s\|\|t\|-1}, \tag{4.4}$$

with $G(\Theta,\Theta,\Theta) = \Theta$. By applying Corollary 3.6, we need to show that $G \in \mathcal{G}(X,X)$. Since

$$\|G(r,ks,lt)\| = \left\|re^{\|ks\|\|lt\|-1}\right\| = e^{kl-1} \geq 1, \tag{4.5}$$

when $\|r\| = \|s\| = \|t\| = 1$, $k \geq 1$ and $l \geq 1$. Hence by Corollary 3.6, we have the following. For $f : U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\left\|f(z)e^{\|D_z^\alpha f(z)\|\|zf'(z)\|-1}\right\| < 1 \Longrightarrow \|f(z)\| < 1. \tag{4.6}$$

Consequently, $\|I_z^\alpha G(f(z), D_z^\alpha f(z), zf'(z); z)\| < 1$, thus in view of Theorem 3.7, f has the generalized Hyers-Ulam stability.

Example 4.3. Let $a, b, c : U \to \mathbb{C}$ satisfy the following:

$$|a(z) + \mu b(z) + \nu c(z)| \geq 1, \tag{4.7}$$

for every $\mu \geq 1$, $\nu > 1$ and $z \in U$. Consider the function $G : X^3 \longrightarrow Y$ by

$$G(r,s,t;z) = a(z)r + \mu b(z)s + \nu c(z)t, \tag{4.8}$$

with $G(\Theta,\Theta,\Theta) = \Theta$. Now for $\|r\| = \|s\| = \|t\| = 1$, we have

$$\|G(r,\mu s,\nu t;z)\| = |a(z) + \mu b(z) + \nu c(z)| \geq 1 \tag{4.9}$$

and thus $G \in \mathcal{G}(X,Y)$. If $f : U \to X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\|a(z)f(z) + b(z)D_z^\alpha f(z) + zc(z)f'(z)\| < 1 \Longrightarrow \|f(z)\| < 1. \tag{4.10}$$

Hence according to Theorem 3.7, f has the generalized Hyers-Ulam stability.

Acknowledgment

This research has been funded by University Malaya, under Grant no. (RG208-11AFR).

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