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Research Article

Generalized AOR Method for Solving Absolute Complementarity Problems

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We introduce and consider a new class of complementarity problems, which is called the absolute value complementarity problem. We establish the equivalence between the absolute complementarity problems and the fixed point problem using the projection operator. This alternative equivalent formulation is used to discuss the existence of a solution of the absolute value complementarity problem. A generalized AOR method is suggested and analyzed for solving the absolute the complementarity problems. We discuss the convergence of generalized AOR method for the L-matrix. Several examples are given to illustrate the implementation and efficiency of the method. Results are very encouraging and may stimulate further research in this direction.

1. Introduction

Complementarity theory introduced and studied by Lemke [1] and Cottle and Dantzig [2] has enjoyed a vigorous growth for the last fifty years. It is well known that both the linear and nonlinear programs can be characterized by a class of complementarity problems. The complementarity problems have been generalized and extended to study a wide class of problems, which arise in pure and applied sciences; see [1–24] and the references therein. Equally important is the variational inequality problem, which was introduced and studied in the early sixties. The theory of variational inequality has been developed not only to study the fundamental facts on the qualitative behavior of solutions but also to provide highly efficient new numerical methods for solving various nonlinear problems. For the recent applications, formulation, numerical results, and other aspects of the variational inequalities, see [13–22].

Motivated and inspired by the research going on in these areas, we introduced and consider a new class of complmenetarity problems, which is called the absolute value complmenetarity problem. Related to the absolute value complementarity problem, we consider the problem of solving the absolute value variational inequality. We show that if the under lying set is a convex cone, then both these problems are equivalent. If the underlying set is the whole space, then the absolute value problem is equivalent to solving the absolute value equations, which have been studied extensively in recent years.

We use the projection technique to show that the absolute value complementarity problems are equivalent to the fixed point problem. This alternative equivalent form is used to study the existence of a unique solution of the absolute value complementarity problems under some suitable conditions. We again use the fixed point formulation to suggest and analyze a generalized AOR method for solving the absolute value complementarity problems. The convergence analysis of the proposed method is considered under some suitable conditions. Some examples are given to illustrate the efficiency and implementation of the proposed iterative methods. Results are very encouraging. The ideas and the technique of this paper may stimulate further research in these areas.

Let R^n be an inner product space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. For a given matrix $A \in R^{n\times n}$, a vector $b \in R^n$, we consider the problem of finding $x \in R^n$ such that

$$x \in K, \quad Ax - |x| - b \ \in K^*, \quad \langle Ax - |x| - b, x \rangle = 0, \tag{1.1}$$

where $K^* = \{x \in R^n : \ \langle x, y \rangle \geq 0, \ \forall y \in K\}$ is the polar cone of a closed and convex cone K, $A \in R^{n\times n}, \ b \in R^n$, and $|x|$ will denote the vector in R^n with absolute values of components of $x \in R^n$.

We remark that the absolute value complementarity problem (1.1) can be viewed as an extension of the complementarity problem considered by Lemke [1]. To solve the linear complementarity problems, several methods were proposed. These methods can be divided into two categories, the direct and indirect (iterative) methods. Lemke [1] and Cottle and Dantzig [2] developed the direct methods for solving linear complementarity problems based on the process of pivoting, whereas Mangasarian [10], Noor [14, 15], and Noor et al. [20–22] considered the iterative methods. For recent applications, formulations, numerical methods, and other aspects of the complementarity problems and variational inequalities, see [1–24].

Let K be a closed and convex set in the inner product space R^n. We consider the problem of finding $x \in K$ such that

$$\langle Ax - |x| - b, \ y - x \rangle \geq 0, \quad \forall y \in K. \tag{1.2}$$

The problem (1.2) is called the absolute value variational inequality, which is a special form of the mildly nonlinear variational inequalities, introduced and studied by Noor [13] in 1975.

If $K = R^n$, then the problem (1.2) is equivalent to find $x \in R^n$ such that

$$Ax - |x| - b = 0, \tag{1.3}$$

which are known as the absolute value equations. These equations have been considered and studied extensively in recent years; see [7–12, 17–20, 23, 24]. We would like to emphasize that

Mangasarian [1, 2, 4–11] has shown that the absolute value equations (1.3) are equivalent to the complementarity problems (1.1). Mangasarian [7–11] has used the complementarity approach to solve the absolute value equations (1.3). For other methods, see [7–12, 20–24] and the references therein.

In this paper, we suggest a generalized AOR method for solving absolute complementarity problem, which is easy to implement and gives almost exact solution of (1.3).

We also need the following definitions and concepts.

Definition 1.1. $B \in R^{n\times n}$ is called an L-matrix if $b_{ii} > 0$ for $i = 1,2,\dots,n$, and $b_{ij} \leq 0$ for $i \neq j,\ i,\ j = 1,2,\dots,n$.

Definition 1.2. A matrix $A \in R^{n\times n}$ is said to be positive definite matrix, if there exists a constant $\gamma > 0$, such that

$$\langle Ax,\ x\rangle \geq \gamma\|x\|^2, \quad \forall x \in R^n, \tag{1.4}$$

and bounded if there exists a constant $\beta > 0$ such that

$$\|Ax\| \leq \beta\|x\|, \quad \forall x \in R^n. \tag{1.5}$$

2. Iterative Methods

To propose and analyze algorithm for absolute complementarity problems, we need the following well-known results.

Lemma 2.1 (see [16]). *Let K be a nonempty closed convex set in R^n. For a given $z \in R^n$, $u \in K$ satisfies the inequality*

$$\langle u - z, u - v\rangle \geq 0, \quad v \in K, \tag{2.1}$$

if and only if

$$u = P_K z, \tag{2.2}$$

where P_K is the projection of R^n onto the closed convex set K.

Lemma 2.2. *If K is the positive cone in R^n, then $x \in K$ is a solution of absolute value variational inequality* (1.2) *if and only if $x \in K$ solves the absolute value complementarity problem* (1.1).

Proof. Let $x \in K$ be the solution of (1.2). Then

$$\langle Ax - |x| - b,\ y - x\rangle \geq 0, \quad \forall y \in K. \tag{2.3}$$

Since K is a cone, taking $y = 0 \in K$ and $y = 2x \in K$, we have

$$\langle Ax - |x| - b,\ x\rangle = 0. \tag{2.4}$$

From inequality (1.2), we have

$$0 \leq \langle Ax - |x| - b,\ y - x\rangle = \langle Ax - |x| - b, y\rangle - \langle Ax - |x| - b, x\rangle, \\ = \langle Ax - |x| - b, y\rangle, \quad \forall y \in K, \tag{2.5}$$

from which it follows that $Ax - |x| - b \in K^*$. Thus we conclude that $x \in K$ is the solution of absolute complementarity problems (1.1).

Conversely, let $x \in K$ be a solution of (1.1). Then

$$x \in K, \quad Ax - |x| - b \in K^*, \quad \langle Ax - |x| - b, x\rangle = 0. \tag{2.6}$$

From (2.5) and (2.6), it follows that

$$\langle Ax - |x| - b, y - x\rangle \geq 0, \quad \forall y \in K. \tag{2.7}$$

Hence $x \in K$ is the solution of absolute variational inequality (1.2). □

From Lemma 2.1, it follows that both the problems (1.1) and (1.2) are equivalent.

In the next result, we prove the equivalence between the absolute value variational inequality (1.2) and the fixed point.

Lemma 2.3. *Let K be closed convex set in R^n. Then, for a constant $\rho > 0$, $x \in K$ satisfies* (1.2) *if and only if $x \in K$ satisfies the relation*

$$x = P_K(x - \rho[Ax - |x| - b]), \tag{2.8}$$

where P_K is the projection of R^n onto the closed convex set K.

Proof. Let $x \in K$ be the solution of (1.2). Then, for a positive constant $\rho > 0$,

$$\langle x - (x - \rho(Ax - |x| - b)),\ y - x\rangle \geq 0, \quad \forall y \in K. \tag{2.9}$$

Using Lemma 2.1, we have

$$x = P_K(x - \rho[Ax - |x| - b]), \tag{2.10}$$

which is the required result. □

Now using Lemmas 2.2 and 2.3, we see that the absolute value complementarity problem (1.1) is equivalent to the fixed point problem of the following type:

$$x = P_K(x - \rho[Ax - |x| - b]). \tag{2.11}$$

We use this alternative fixed point formulation to study the existence of a unique solution of the absolute value complementarity problem. Equation (1.1) and this is the main motivation of our next result.

Theorem 2.4. *Let $A \in R^{n\times n}$ be a positive definite matrix with constant $\alpha > 0$ and continuous with constant $\beta > 0$. If $0 < \rho < 2(\gamma - 1)/(\beta^2 - 1)$, $\gamma > 1$, then there exists a unique solution $x \in K$ such that*

$$\langle Ax - |x| - b, y - x\rangle \geq 0, \quad \forall y \in K, \tag{2.12}$$

where K is a closed convex set in R^n.

Proof. Uniqueness: Let $x_1 \neq x_2 \in K$ be two solutions of (1.2). Then

$$\begin{aligned} \langle Ax_1 - |x_1| - b,\ y - x_1\rangle \geq 0, \quad \forall y \in K, \\ \langle Ax_2 - |x_2| - b,\ y - x_2\rangle \geq 0, \quad \forall y \in K. \end{aligned} \tag{2.13}$$

Taking $y = x_2 \in K$ in (2.13) and $y = x_1 \in K$ in (2.6), we have

$$\begin{aligned} \langle -Ax_1 + |x_1| + b,\ x_1 - x_2\rangle \geq 0, \\ \langle Ax_2 - |x_2| - b,\ x_1 - x_2\rangle \geq 0. \end{aligned} \tag{2.14}$$

Adding the previous inequalities, we obtain

$$\langle A(x_1 - x_2) - |x_1| + |x_2|,\ x_1 - x_2\rangle \leq 0, \tag{2.15}$$

which implies that

$$\langle A(x_1 - x_2),\ x_1 - x_2\rangle - \|x_1 - x_2\|^2 \leq 0. \tag{2.16}$$

Since A is positive definite, from (2.16), we have

$$(\gamma - 1)\|x_1 - x_2\|^2 \leq 0. \tag{2.17}$$

As $\gamma > 1$, therefore from (2.17) we have

$$\|x_1 - x_2\|^2 \leq 0, \tag{2.18}$$

which contradicts the fact that $\|x_1 - x_2\|^2 \geq 0$; hence $x_1 = x_2$.

Existence

Let $x \in K$ be the solution of (1.2). Then

$$\langle Ax - |x| - b,\ y - x\rangle \geq 0, \quad \forall y \in K. \tag{2.19}$$

From Lemma 2.3, we have

$$x = P_K\big(x - \rho[Ax - |x| - b]\big). \tag{2.20}$$

Define a mapping

$$F(x) = x = P_K\big(x - \rho[Ax - |x| - b]\big). \tag{2.21}$$

To show that the mapping $F(x)$ defined by (2.21) has a fixed point, it is enough to prove that $F(x)$ is a contraction mapping. For $x_1 \neq x_2 \in K$, consider

$$\begin{aligned} \|F(x_1) - F(x_2)\| &= \big\| P_K\big[x_1 - \rho(Ax_1 - |x_1| - b)\big] \; - P_K\big[x_2 - \rho(Ax_2 - |x_2| - b)\big]\big\| \\ &\leq \big\| \big[x_1 - \rho(Ax_1 - |x_1| - b)\big] - \big[x_2 - \rho(Ax_2 - |x_2| - b)\big]\big\| \\ &\leq \big\| x_1 - x_2 - \rho(Ax_1 - Ax_2)\big\| + \rho\||x_1| - |x_2|\| \\ &= \big\| x_1 - x_2 - \rho A(x_1 - x_2)\big\| + \rho\|x_1 - x_2\|, \end{aligned} \tag{2.22}$$

where we have used the fact that P_K is nonexpansive. Now using positive definiteness of A, we have

$$\begin{aligned} \big\| x_1 - x_2 - \rho A(x_1 - x_2)\big\|^2 &= \langle x_1 - x_2 - \rho A(x_1 - x_2),\; x_1 - x_2 - \rho A(x_1 - x_2)\rangle \\ &\leq \|x_1 - x_2\|^2 - 2\rho\gamma\langle x_1 - x_2, A(x_1 - x_2)\rangle + \rho^2\beta^2\|x_1 - x_2\|^2 \\ &= \left(1 - 2\alpha\rho + \beta^2\rho^2\right)\|x_1 - x_2\|^2. \end{aligned} \tag{2.23}$$

From (2.22) and (2.23), we have

$$\|F(x_1) - F(x_2)\| \leq \theta\|x_1 - x_2\|, \tag{2.24}$$

where $\theta = (\rho + \sqrt{1 - 2\gamma\rho + \beta^2\rho^2})$. Form $0 < \rho < 2(\gamma - 1)/(\beta^2 - 1)$ and $\rho < 1$, we have $\theta < 1$.

This shows that $F(x)$ is a contraction mapping and has a fixed point $x \in K$ satisfying the absolute value variational inequality (1.2). □

For the sake of simplicity, we consider the special case, when $K = [0, c]$ is a closed convex set in R^n and we define the projection $P_K x$ as

$$(P_K x)_i = \min\{\max(0, x_i),\; c_i\}, \qquad i = 1, 2, \ldots, n. \tag{2.25}$$

We recall the following well-known result.

Lemma 2.5 (see [3]). *For any x and y in R^n, the projection $P_K x$ has the following properties:*

(i) $P_K(x + y) \leq P_K x + P_K y$,

(ii) $P_K x - P_K y \leq P_K(x - y)$,

(iii) $x \leq y \Rightarrow P_K x \leq P_K y$,

(iv) $P_K x + P_K(-x) \leq |x|$, *with equality, if and only if* $-c \leq x \leq c$.

We now suggest the iterative methods for solving the absolute value complmentarity problems (1.1). For this purpose, we decompose the matrix A as,

$$A = D + L + U, \tag{2.26}$$

where D is the diagonal matrix, and L and U are strictly lower and strictly upper triangular matrices, respectively. Let $\Omega = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_n)$ with $\omega_i \in R_+$ and let α be a real number.

Algorithm 2.6.

Step 1. Choose an initial vector $x_0 \in R^n$ and a parameter $\Omega \in R_+$. Set $k = 0$.

Step 2. Calculate

$$x_{k+1} = P_K\left(x_k - D^{-1}\left[\alpha\Omega L u_{k+1} + (\Omega A - \alpha\Omega L)u_k - \Omega(|x_k| + b)\right]\right). \tag{2.27}$$

Step 3. If $x_{k+1} = x_k$, then stop. Else, set $k = k + 1$ and go to Step 2.

Now we define an operator $g : R^n \to R^n$ such that $g(x) = \xi$, where ξ is the fixed point of the system

$$\xi = P_K\left(x - D^{-1}\left[\alpha\Omega L\xi + (\Omega\ A - \alpha\Omega L)x - \Omega(|x| + b)\right]\right). \tag{2.28}$$

We also assume that the set

$$\varphi = \{x \in R^n : x \geq 0,\ Ax - |x| - b \geq 0\} \tag{2.29}$$

of the absolute value complementarity problem is nonempty.

To prove the convergence of Algorithm 2.6, we need the following result.

Theorem 2.7. *Consider the operator* $g : R^n \to R^n$ *as defined in* (2.28). *Assume that* $A \in R^{n\times n}$ *is an L-matrix. Also assume that* $0 < \omega_i \leq 1$, $0 \leq \alpha \leq 1$. *Then, for any* $x \in \varphi$, *the following holds:*

(i) $g(x) \leq x$,

(ii) $x \leq y \Rightarrow g(x) \leq g(y)$,

(iii) $\xi = g(x) \in \varphi$.

Proof. To prove (i), we need to verify that $\xi_i \leq x_i$, $i = 1, 2, \ldots, n$ hold with ξ_i satisfying

$$\xi_i = P_K\left(x_i - a_{ii}^{-1}\left[\alpha\omega_i \sum_{j=1}^{i-1} L_{ij}(\xi_j - x_j) + \omega_i(Ax - |x| - b)_i\right]\right). \tag{2.30}$$

To prove the required result, we use mathematical induction. For this let $i = 1$,

$$\xi_1 = P_K\Big(x_1 - a_{11}^{-1}\omega_1(Ax - |x| - b)_1\Big). \tag{2.31}$$

Since $Ax - |x| - b \geq 0,\ \omega_i > 0$, therefore $\xi_1 \leq x_1$. For $i = 2$, we have

$$\xi_2 = P_K\Big(x_2 - a_{22}^{-1}[\alpha\omega_2 L_{21}(\xi_1 - x_1) + \omega_2(Ax - |x| - b)_2]\Big). \tag{2.32}$$

Here $Ax - |x| - b \geq 0,\ \omega_i > 0,\ L_{21} \leq 0$, and $\xi_1 - x_1 \leq 0$. This implies that $\xi_2 \leq x_2$.

Suppose that

$$\xi_i \leq x_i \quad \text{for } i = 1, 2, \ldots, k-1. \tag{2.33}$$

We have to prove that the statement is true, for $i = k$, that is,

$$\xi_k \leq x_k. \tag{2.34}$$

Consider

$$\begin{aligned}\xi_k &= P_K\left(x_k - a_{kk}^{-1}\left[\alpha\omega_k\sum_{j=1}^{k-1}L_{kj}(\xi_j - x_j) + \omega_k(Ax - |x| - b)_k\right]\right),\\ &= P_K\Big(x_k - a_{kk}^{-1}[\alpha\omega_k(L_{k1}(\xi_1 - x_1) + L_{k2}(\xi_2 - x_2) + \cdots + L_{kk-1}(\xi_{k-1} - x_{k-1}))\\ &\qquad + \omega_k(Ax - |x| - b)_k]\Big).\end{aligned} \tag{2.35}$$

Since $Ax - |x| - b \geq 0,\ \omega_k > 0,\ \ L_{k1}, L_{k2}, \ldots, L_{kk-1} \leq 0$, and $\xi_i \leq x_i$ for $i = 1, 2, \ldots, k-1$, from (2.35) we can write $\xi_k \leq x_k$. Hence (i) is proved.

Now we prove (ii). For this let us suppose that $\xi = g(x)$ and $\phi = g(y)$. We will prove

$$x \leq y \Longrightarrow \xi \leq \phi. \tag{2.36}$$

As

$$\xi = P_K\Big(x - D^{-1}[\alpha\Omega L\xi + (\Omega A - \alpha\Omega L)x - \Omega(|x| + b)]\Big), \tag{2.37}$$

so ξ_i can be written as

$$\begin{aligned}\xi_i &= P_K\left(x_i - a_{ii}^{-1}\left[\alpha\omega_i\sum_{j=1}^{i-1}L_{ij}\xi_j + \omega_i a_{ii}x_i + (1-\alpha)\omega_i\sum_{j=1}^{i-1}L_{ij}x_j + \omega_i\sum_{\substack{j=1\\ j\neq i}}^{n}U_{ij}x_j - \omega_i|x_i| - \omega_i b_i\right]\right)\\ &= P_K\left((1-\omega_i)x_i - a_{ii}^{-1}\left[\alpha\omega_i\sum_{j=1}^{i-1}L_{ij}\xi_j + (1-\alpha)\omega_i\sum_{j=1}^{i-1}L_{ij}x_j + \omega_i\sum_{\substack{j=1\\ j\neq i}}^{n}U_{ij}x_j - \omega_i|x_i| - \omega_i b_i\right]\right).\end{aligned} \tag{2.38}$$

Similarly, for ϕ_i, we have

$$\phi_i = P_K\left((1-\omega_i)y_i - a_{ii}^{-1}\left[\alpha\omega_i\sum_{j=1}^{i-1}L_{ij}\phi_j + (1-\alpha)\omega_i\sum_{j=1}^{i-1}L_{ij}y_j + \omega_i\sum_{\substack{j=1\\ j\neq i}}^{n}U_{ij}y_j - \omega_i|y_i| - \omega_i b_i\right]\right), \tag{2.39}$$

and for $i = 1$,

$$\begin{aligned}\phi_1 &= P_K\left((1-\omega_1)y_1 - a_{11}^{-1}\omega_1\left[\sum_{\substack{j=1\\ j\neq i}}^{n}U_{1j}y_j - |y_1| - b_1\right]\right)\\ &\geq P_K\left((1-\omega_1)x_1 - a_{11}^{-1}\omega_1\left[\sum_{\substack{j=1\\ j\neq i}}^{n}U_{1j}x_j - |x_1| - b_1\right]\right)\\ &= \xi_1.\end{aligned} \tag{2.40}$$

Since $y_1 \geq x_1$, therefore $-|y_1| \leq -|x_1|$. Hence it is true for $i = 1$. Suppose it is true for $i = 1, 2, \dots k-1$; we will prove it for $i = k$; for this consider

$$\begin{aligned}\phi_k &= P_K\left((1-\omega_k)y_k - a_{kk}^{-1}\left[\alpha\omega_k\sum_{j=1}^{k-1}L_{kj}\phi_j + (1-\alpha)\omega_k\sum_{j=1}^{k-1}L_{kj}y_j + \omega_k\sum_{\substack{j=1\\ j\neq i}}^{n}U_{kj}y_j - \omega_k|y_k| - \omega_k b_k\right]\right)\\ &\geq P_K\left((1-\omega_k)x_k - a_{kk}^{-1}\left[\alpha\omega_k\sum_{j=1}^{k-1}L_{kj}\xi_j + (1-\alpha)\omega_k\sum_{j=1}^{k-1}L_{kj}x_j + \omega_k\sum_{\substack{j=1\\ j\neq i}}^{n}U_{kj}x_j - \omega_k|x_k| - \omega_k b_k\right]\right)\\ &= \xi_k.\end{aligned} \tag{2.41}$$

Since $x \leq y$, and $\xi_i \leq \phi_i$ for $i = 1, 2, \ldots k-1$, hence it is true for k and (ii) is verified.

Next we prove (iii), that is,

$$\xi = g(x) \in \varphi. \tag{2.42}$$

Let $\lambda = g(\xi) = P_K(\xi - D^{-1}\Omega[\alpha L(\lambda - \xi) + A\xi - |\xi| - b])$ from (i) $g(\xi) = \lambda \leq \xi$. Also by definition of g, $\xi = g(x) \geq 0$ and $\lambda = g(\xi) \geq 0$.

Now

$$\lambda_i = P_K\left(\xi_i - a_{ii}^{-1}\left[\alpha\omega_i \sum_{j=1}^{i-1} L_{ij}(\lambda_j - \xi_j) + \omega_i(A\xi - |\xi| - b)_i\right]\right). \tag{2.43}$$

For $i = 1$, $\xi_1 \geq 0$ by definition of g. Suppose that $(A\xi - |\xi| - b)_i < 0$, so

$$\begin{aligned} \lambda_1 &= P_K\left(\xi_1 - a_{11}^{-1}\omega_1(A\xi - |\xi| - b)_1\right) \\ &> P_K(\xi_1) = \xi_1, \end{aligned} \tag{2.44}$$

which contradicts the fact that $\lambda \leq \xi$. Therefore, $(A\xi - |\xi| - b)_i \geq 0$.

Now we prove it for any k in $i = 1, 2, \ldots, n$. Suppose the contrary $(A\xi - |\xi| - b)_i < 0$; then

$$\lambda_k = P_K\left(\xi_k - a_{kk}^{-1}\left[\alpha\omega_k \sum_{j=1}^{k-1} L_{kj}(\lambda_j - \xi_j) + \omega_k(A\xi - |\xi| - b)_k\right]\right). \tag{2.45}$$

As it is true for all $\alpha \in [0, 1]$, it should be true for $\alpha = 0$. That is,

$$\begin{aligned} \lambda_k &= P_K\left(\xi_k - a_{kk}^{-1}\omega_k(A\xi - |\xi| - b)_k\right) \\ &> P_K(\xi_k) = \xi_k, \end{aligned} \tag{2.46}$$

which contradicts the fact that $\lambda \leq \xi$. So $(A\xi - |\xi| - b)_k \geq 0$, for any k in $i = 1, 2, \ldots, n$. Hence $\xi = f(x) \in \varphi$. □

We now consider the convergence criteria of Algorithm 2.6 and this is the main motivation of our next result.

Theorem 2.8. *Assume that $A \in R^{n\times n}$ is an L-matrix. Also assume that $0 < \omega_i \leq 1$, $0 \leq \alpha \leq 1$. Then for any initial vector $x_0 \in \varphi$, the sequence $\{x_k\}$, $k = 0, 1, 2, \ldots$, defined by Algorithm 2.6 has the following properties:*

(i) $0 \leq x_{k+1} \leq x_k \leq x_0$; $k = 0, 1, 2, \ldots$,

(ii) $\lim_{k\to\infty} x_k = x^*$ *is a unique solution of the absolute value complementarity problem* (1.1).

Table 1: Computational results.

n	m	Number of iterations	TOC	Error
10	1	4	0.001	1.8204×10^{-8}
	10	40	0.011	7.4875×10^{-9}
50	1	4	0.003	1.2595×10^{-7}
	10	40	0.016	1.3834×10^{-7}
100	1	4	0.014	9.5625×10^{-7}
	10	41	0.031	6.6982×10^{-7}
500	1	5	0.203	1.3142×10^{-7}
	10	49	2.075	1.5168×10^{-7}
1000	1	6	1.076	7.2231×10^{-9}
	10	60	11.591	8.3961×10^{-9}

Proof. Since $x_0 \in \varphi$, by (i) of Theorem 2.7, we have $x_1 \leq x_0$ and $x_1 \in \varphi$. Recursively using Theorem 2.7 we obtain

$$0 \leq x_{k+1} \leq x_k \leq x_0; \quad k = 0, 1, 2, \ldots. \tag{2.47}$$

From (i) we observe that the sequence $\{x_k\}$ is monotone bounded; therefore, it converges to some $x^* \in R^n_+$ satisfying

$$\begin{aligned} x^* &= P_K\Big(x^* - D^{-1}[\alpha\Omega L x^* + (\Omega A - \alpha\Omega L)x^* - \Omega(|x^*| + b)]\Big) \\ &= P_K\Big(x^* - D^{-1}[\Omega A x^* - \Omega|x^*| - \Omega b]\Big). \end{aligned} \tag{2.48}$$

Hence $x^*$is the solution of the absolute value complementarity problem (1.1). □

3. Numerical Results

In this section, we consider several examples to show the efficiency of the proposed method. The convergence of the generalized AOR method is guaranteed for L-matrices only but it is also possible to solve different types of systems. The elements of the diagonal matrix Ω are chosen from the interval $[a, b]$ such that

$$\omega_i = a + \frac{(b-a)i}{n}, \quad i = 1,\ 2, \ldots, n, \tag{3.1}$$

where ω_i is the ith diagonal element of Ω. All the experiments are performed with Intel(R) Core(TM) 2 × 2.1 GHz, 1 GB RAM, and the codes are written in Matlab 7.

Example 3.1. We test Algorithm 2.6 on m consecutively generated solvable random problems $A \in R^{n\times n}$, and n ranging from 10 to 1000. We chose a random matrix A from a uniform distribution on $[0, 1]$, such that whose all diagonal elements are equal to 1000 and x is chosen randomly from a uniform distribution on $[0, 1]$. The constant vector is computed as $b = Ax - |x|$. The computational results are shown in Table 1.

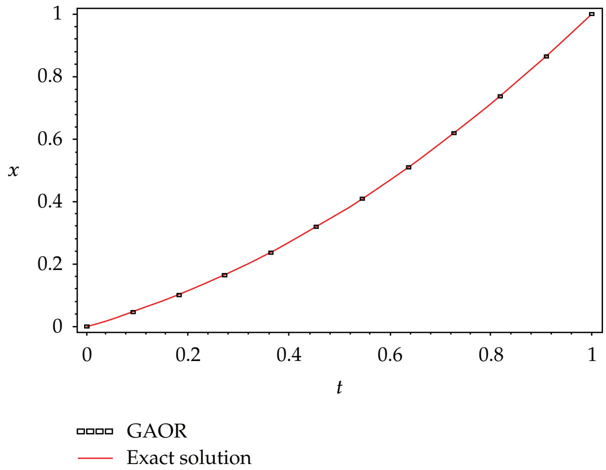

Figure 1

Example 3.2. Consider the ordinary differential equation:

$$\frac{d^2x}{dt^2} - |x| = \left(1 - x^2\right), \quad 0 \le x \le 1, \quad x(0) = 0, \quad x(1) = 1. \tag{3.2}$$

The exact solution is

$$x(t) = \begin{cases} 0.7378827425\sin(t) - 3\cos(t) + 3 - x^2, & x < 0, \\ -0.7310585786e^{-x} - 0.2689414214e^{x} + 1 + x^2, & x > 0. \end{cases} \tag{3.3}$$

We take $n = 10$; the matrix A is given by

$$a_{i,j} = \begin{cases} -242, & \text{for } j = i, \\ 121, & \text{for } \begin{cases} j = i+1, & i = 1,2,\dots,n-1, \\ j = i-1, & i = 2,3,\dots,n, \end{cases} \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

The constant vector b is given by

$$b = \left(\frac{120}{121}, \frac{117}{121}, \frac{112}{121}, \frac{105}{121}, \frac{96}{121}, \frac{85}{121}, \frac{72}{121}, \frac{57}{121}, \frac{40}{121}, \frac{-14620}{121}\right)^T. \tag{3.5}$$

Here A is not L-matrix. The comparison between the exact solution and the approximate solutions is given in Figure 1.

Table 2: Computational results.

Order	Iterative method [21]		AOR method	
	Number of iterations	TOC	Number of iterations	TOC
4	10	0.0168	10	0.001
8	11	0.018	11	0.001
16	11	0.143	11	0.002
32	12	3.319	11	0.008
64	12	7.145	11	0.082
128	12	11.342	11	0.330
256	12	25.014	11	2.298
512	12	98.317	11	19.230
1024	13	534.903	11	158.649

Example 3.3. Let the matrix A be given by

$$a_{i,j} = \begin{cases} 8, & \text{for } j = i, \\ -1, & \text{for } \begin{cases} j = i+1, & i = 1,2,\dots,n-1, \\ j = i-1, & i = 2,3,\dots,n, \end{cases} \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Let $b = (6,5,5,\dots,5,6)^T$, the problem size n ranging from 4 to 1024. The stopping criteria are $\|Ax - |x| - b\| < 10^{-6}$. We choose initial guess x_0 as $x_0 = (0,0,\dots,0)^T$. The computational results are shown in Table 2.

In Table 2 TOC denotes total time taken by CPU. The rate of convergence of AOR method is better than iterative method [21].

4. Conclusion

In this paper, we have introduced a new class of complementarity problems, known as the absolute value complementarity problems. We have used the projection technique to establish the equivalence between the absolute value variational inequalities, fixed point problems, and the absolute value complementarity problems. This equivalence is used to discuss the existence of a unique solution of the absolute value problems under some suitable conditions. We have also used this alternative equivalent formulation to suggest and analyze an iterative method for solving the absolute value complementarity problems. Some special cases are also discussed. The results and ideas of this paper may be used to solve the variational inequalities and related optimization problems. This is another direction for future research.

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