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Research Article

Common Fixed Point Theorems for a Class of Twice Power Type Contraction Maps in G-Metric Spaces

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We introduce a new twice power type contractive condition for three mappings in G-metric spaces, and several new common fixed point theorems are established in complete G-metric space. An example is provided to support our result. The results obtained in this paper differ from other comparable results already known.

1. Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been in the center of rigorous research activity. In 2006, a new structure of generalized metric space was introduced by Mustafa and Sims [1] as an appropriate notion of generalized metric space called G-metric space. Abbas and Rhoades [2] initiated the study of common fixed point in generalized metric space. Recently, many fixed point theorems for certain contractive conditions have been established in G-metric spaces, and for more details one can refer to [3–27]. Fixed point problems have also been considered in partially ordered G-metric spaces [28–31], cone metric spaces [32], and generalized cone metric spaces [33].

In 2006, Gu and He [34] introduced a class of twice power type contractive condition in metric space, proving some common fixed point theorems for four self-maps with twice power type Φ-contractive condition.

In this paper, motivated and inspired by the above results, we introduce a new twice power type contractive condition in G-metric space, and we prove some new common fixed point theorems in complete G-metric spaces. Our results obtained in this paper differ from other comparable results already known.

Throughout the paper, we mean by $\mathbb{N}$ the set of all natural numbers. Consistent with Mustafa and Sims [1], the following definitions and results will be needed in the sequel.

Definition 1.1 (see [1]). Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x, y, z = 0)$ if $x = y = z$;

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq$ y;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality);

then the function G is called a generalized metric, or, more specifically, a G-metric on X and the pair (X, G) are called a G-metric space.

Definition 1.2 (see [1]). Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n \to \infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Proposition 1.3 (see [1]). *Let (X, G) be a G-metric space, then the followings are equivalent.*

(1) *$\{x_n\}$ is G-convergent to x.*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*

(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Definition 1.4 (see [1]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 1.5 (see [1]). A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 1.6 (see [1]). *Let (X, G) be a G-metric space. Then the following are equivalent.*

(1) *The sequence $\{x_n\}$ is G-Cauchy.*

(2) *For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq k$.*

Proposition 1.7 (see [1]). *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.8 (see [1]). Let (X, G) and (X', G') be G-metric space, and $f : (X, G) \to (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.9 (see [1]). *Let (X,G) and (X',G') be G-metric space. Then $f : X \to X'$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Proposition 1.10 (see, [1]). *Let (X,G) be a G-metric space. Then, for any x, y, z, a in X it follows that:*

(i) *if $G(x,y,z) = 0$, then $x = y = z$;*

(ii) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$;

(iii) $G(x,y,y) \le 2G(y,x,x)$;

(iv) $G(x,y,z) \le G(x,a,z) + G(a,y,z)$;

(v) $G(x,y,z) \le (2/3)(G(x,y,a) + G(x,a,z) + G(a,y,z))$;

(vi) $G(x,y,z) \le (G(x,a,a) + G(y,a,a) + G(z,a,a))$.

2. Main Results

Theorem 2.1. *Let (X,G) be a complete G-metric space. Suppose the three self-mappings $T, S, R : X \to X$ satisfy the following condition:*

$$\begin{aligned} G^2(Tx,Sy,Rz) \le{}& \alpha G(x,Tx,Tx)G(y,Sy,Sy) + \beta G(y,Sy,Sy)G(z,Rz,Rz) \\ &+ \gamma G(x,Tx,Tx)G(z,Rz,Rz), \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, where α, β, γ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then T, S, and R have a unique common fixed point (say u) and T, S, R are all G-continuous at u.

Proof . We will proceed in two steps.

Step 1. We prove any fixed point of T is a fixed point of S and R and conversely. Assume that $p \in X$ is such that $Tp = p$. However, by (2.1), we have

$$\begin{aligned} G^2(Tp,Sp,Rp) &\le \alpha G(p,Tp,Tp)G(p,Sp,Sp) + \beta G(p,Sp,Sp)G(p,Rp,Rp) \\ &\quad + \gamma G(p,Tp,Tp)G(p,Rp,Rp) \\ &= \alpha G(p,p,p)G(p,Sp,Sp) + \beta G(p,Sp,Sp)G(p,Rp,Rp) \\ &\quad + \gamma G(p,p,p)G(p,Rp,Rp) \\ &= \beta G(p,Sp,Sp)G(p,Rp,Rp). \end{aligned} \tag{2.2}$$

Now we discuss the above inequality in three cases.

Case (i). If $p \ne Sp$ and $p \ne Rp$, then, by (G3), we have

$$G(p,Sp,Sp) \le G(p,Sp,Rp), \qquad G(p,Rp,Rp) \le G(p,Sp,Rp). \tag{2.3}$$

So, the above inequality becomes

$$G^2(p,Sp,Rp) = G^2(Tp,Sp,Rp) \le \beta G^2(p,Sp,Rp). \tag{2.4}$$

Since $G^2(p, Sp, Rp) > 0$, hence we have $\beta \geq 1$; however, it contradicts with $0 \leq \beta \leq \alpha+\beta+\gamma < 1$, so we get $p = Sp = Rp$.

Case (ii). If $p = Rp$, then we have

$$G^2(p, Sp, Rp) = G^2(Tp, Sp, Rp) \leq \beta G(p, Sp, Sp)G(p, Rp, Rp) = 0. \tag{2.5}$$

Hence we have $G^2(p, Sp, Rp) = 0$ and so $p = Sp = Rp$.

Case (iii). If $p = Sp$, we can also get $G^2(p, Sp, Rp) = 0$. Hence we have $p = Sp = Rp$. Therefore p is a common fixed point of T, S and R.

The same conclusion holds if $p = Sp$ or $p = Rp$.

Step 2. We prove that T, S, and R have a unique common fixed point.

Let $x_0 \in X$ be an arbitrary point, and define the sequence $\{x_n\}$ by $x_{3n+1} = Tx_{3n}$, $x_{3n+2} = Sx_{3n+1}$, $x_{3n+3} = Rx_{3n+2}$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$, for some n, with $n = 3m$, then $p = x_{3m}$ is a fixed point of T and, by the first step, p is a common fixed point of S, T, and R. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Without loss of generality, we can assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.

Next, we prove sequence $\{x_n\}$ is a G-Cauchy sequence. In fact, by (2.1) and ($G3$), we have

$$\begin{aligned}
G^2(x_{3n+1}, x_{3n+2}, x_{3n+3}) &= G^2(Tx_{3n}, Sx_{3n+1}, Rx_{3n+2}) \\
&\leq \alpha G(x_{3n}, Tx_{3n}, Tx_{3n})G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1}) \\
&\quad + \beta G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\
&\quad + \gamma G(x_{3n}, Tx_{3n}, Tx_{3n})G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\
&= \alpha G(x_{3n}, x_{3n+1}, x_{3n+1})G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
&\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
&\quad + \gamma G(x_{3n}, x_{3n+1}, x_{3n+1})G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
&\leq \alpha G(x_{3n}, x_{3n+1}, x_{3n+2})G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3})G(x_{3n+2}, x_{3n+3}, x_{3n+1}) \\
&\quad + \gamma G(x_{3n}, x_{3n+1}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+1}).
\end{aligned} \tag{2.6}$$

Which gives that

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (\alpha+\gamma)G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \tag{2.7}$$

It follows that

$$(1-\beta)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (\alpha+\gamma)G(x_{3n}, x_{3n+1}, x_{3n+2}). \tag{2.8}$$

From $0 \leq \beta < 1$ we know that $1 - \beta > 0$. Then, we have

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \frac{\alpha + \gamma}{1 - \beta} G(x_{3n}, x_{3n+1}, x_{3n+2}). \tag{2.9}$$

On the other hand, by using (2.1) and $(G3)$, we have

$$\begin{aligned}
G^2(x_{3n+2}, x_{3n+3}, x_{3n+4}) &= G^2(Tx_{3n+3}, Sx_{3n+1}, Rx_{3n+2}) \\
&\leq \alpha G(x_{3n+3}, Tx_{3n+3}, Tx_{3n+3}) G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1}) \\
&\quad + \beta G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1}) G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\
&\quad + \gamma G(x_{3n+3}, Tx_{3n+3}, Tx_{3n+3}) G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\
&= \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+4}) G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
&\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2}) G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
&\quad + \gamma G(x_{3n+3}, x_{3n+4}, x_{3n+4}) G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
&\leq \alpha G(x_{3n+2}, x_{3n+3}, x_{3n+4}) G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3}) G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \\
&\quad + \gamma G(x_{3n+2}, x_{3n+3}, x_{3n+4}) G(x_{3n+2}, x_{3n+3}, x_{3n+4}).
\end{aligned} \tag{2.10}$$

Which implies that

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq (\alpha + \beta) G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + \gamma G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{2.11}$$

It follows that

$$(1 - \gamma) G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq (\alpha + \beta) G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \tag{2.12}$$

Form the condition $0 \leq \gamma \leq \alpha + \beta + \gamma < 1$, we know that $1 - \gamma > 0$. Therefore, we have

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq \frac{\alpha + \beta}{1 - \gamma} G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \tag{2.13}$$

Again, using (2.1) and ($G3$), we can get

$$\begin{aligned}
G^2(x_{3n+3}, x_{3n+4}, x_{3n+5}) &= G^2(Tx_{3n+3}, Sx_{3n+4}, Rx_{3n+2}) \\
&\le \alpha G(x_{3n+3}, Tx_{3n+3}, Tx_{3n+3}) G(x_{3n+4}, Sx_{3n+4}, Sx_{3n+4}) \\
&\quad + \beta G(x_{3n+4}, Sx_{3n+4}, Sx_{3n+4}) G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\
&\quad + \gamma G(x_{3n+3}, Tx_{3n+3}, Tx_{3n+3}) G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\
&= \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+4}) G(x_{3n+4}, x_{3n+5}, x_{3n+5}) \\
&\quad + \beta G(x_{3n+4}, x_{3n+5}, x_{3n+5}) G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
&\quad + \gamma G(x_{3n+3}, x_{3n+4}, x_{3n+4}) G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
&\le \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+5}) G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \\
&\quad + \beta G(x_{3n+3}, x_{3n+4}, x_{3n+5}) G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \\
&\quad + \gamma G(x_{3n+5}, x_{3n+3}, x_{3n+4}) G(x_{3n+2}, x_{3n+3}, x_{3n+4}).
\end{aligned} \tag{2.14}$$

Which implies that

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \le \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+5}) + (\beta + \gamma) G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{2.15}$$

It follows that

$$(1-\alpha) G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \le (\beta + \gamma) G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{2.16}$$

By the condition $0 \le \alpha \le \alpha + \beta + \gamma < 1$, we know that $1 - \alpha > 0$. Hence, we have

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \le \frac{\beta + \gamma}{1 - \alpha} G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{2.17}$$

Let $q = \max\{(\alpha + \gamma)/(1 - \beta), (\alpha + \beta)/(1 - \gamma), (\beta + \gamma)/(1 - \alpha)\}$, then from $0 \le \alpha + \beta + \gamma < 1$ we know that $0 \le q < 1$. Combining (2.9), (2.13), and (2.17), we have

$$G(x_n, x_{n+1}, x_{n+2}) \le q G(x_{n-1}, x_n, x_{n+1}) \le \cdots \le q^n G(x_0, x_1, x_2). \tag{2.18}$$

Thus, by ($G3$) and ($G5$), for every $m, n \in \mathbb{N}$, $m > n$, noting that $0 \le q < 1$, we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m), \\
&\le G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_{m+1}) \\
&\le \left(q^n + q^{n+1} + \cdots + q^{m-1}\right) G(x_0, x_1, x_2) \\
&\le \frac{q^n}{1-q} G(x_0, x_1, x_2).
\end{aligned} \tag{2.19}$$

Which implies that $G(x_n, x_m, x_m) \to 0$, as $n, m \to \infty$. Thus $\{x_n\}$ is a G-Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$, such that $\{x_n\}$ is G-convergent to u.

Next we prove u is a common fixed point of T, S, and R. By using (2.1), we have

$$\begin{aligned} G^2(Tu, x_{3n+2}, x_{3n+3}) &= G^2(Tu, Sx_{3n+1}, Rx_{3n+2}) \\ &\leq \alpha G(u, Tu, Tu) G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1}) \\ &\quad + \beta G(x_{3n+1}, Sx_{3n+1}, Sx_{3n+1}) G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\ &\quad + \gamma G(u, Tu, Tu) G(x_{3n+2}, Rx_{3n+2}, Rx_{3n+2}) \\ &= \alpha G(u, Tu, Tu) G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\ &\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2}) G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\ &\quad + \gamma G(u, Tu, Tu) G(x_{3n+2}, x_{3n+3}, x_{3n+3}). \end{aligned} \tag{2.20}$$

Letting $n \to \infty$, and using the fact that G is continuous on its variables, we can get

$$G^2(Tu, u, u) = 0. \tag{2.21}$$

Which gives that $Tu = u$, that is u is a fixed point of T. By using (2.1) again, we have

$$\begin{aligned} G^2(x_{3n+1}, Su, x_{3n+3}) &= G^2(Tx_{3n}, Su, Rx_{3n+2}) \\ &\leq \alpha G(x_{3n}, x_{3n+1}, x_{3n+1}) G(u, Su, Su) \\ &\quad + \beta G(u, Su, Su) G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\ &\quad + \gamma G(x_{3n}, x_{3n+1}, x_{3n+1}) G(x_{3n+2}, x_{3n+3}, x_{3n+3}). \end{aligned} \tag{2.22}$$

Letting $n \to \infty$ at both sides, for G is continuous on its variables, it follows that

$$G^2(u, Su, u) = 0. \tag{2.23}$$

Therefore, $Su = u$; that is, u is a fixed point of S. Similarly, by (2.1), we can also get

$$\begin{aligned} G^2(x_{3n+1}, x_{3n+2}, Ru) &= G^2(Tx_{3n}, Sx_{3n+1}, Ru) \\ &\leq \alpha G(x_{3n}, x_{3n+1}, x_{3n+1}) G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\ &\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+2}) G(u, Ru, Ru) \\ &\quad + \gamma G(x_{3n}, x_{3n+1}, x_{3n+1}) G(u, Ru, Ru). \end{aligned} \tag{2.24}$$

On taking $n \to \infty$ at both sides, since G is continuous on its variables, we get that

$$G^2(u, u, Ru) = 0. \tag{2.25}$$

Which gives that $u = Ru$, therefore, u is fixed point of R. Consequently, we have $u = Tu = Su = Ru$, and u is a common fixed point of T, S and R. Suppose v is another common fixed point of T, S, and R, and we have $v = Tv = Sv = Rv$, then by (2.1), we have

$$\begin{aligned} G^2(u,u,v) &= G^2(Tu,Su,Rv) \\ &\le \alpha G(u,Tu,Tu)G(u,Su,Su) + \beta G(u,Su,Su)G(v,Rv,Rv) \\ &\quad + \gamma G(u,Tu,Tu)G(v,Rv,Rv) \\ &= \alpha G(u,u,u)G(u,u,u) + \beta G(u,u,u)G(v,v,v) \\ &\quad + \gamma G(u,u,u)G(v,v,v) \\ &= 0. \end{aligned} \tag{2.26}$$

Which implies that $G^2(u,u,v) = 0$, hence, $u = v$. Then we know the common fixed point of T, S, and R is unique.

To show that T is G-continuous at u, let $\{y_n\}$ be any sequence in X such that $\{y_n\}$ is G-convergent to u. For $n \in \mathbb{N}$, we have

$$\begin{aligned} G^2(Ty_n,u,u) &= G^2(Ty_n,Su,Ru) \\ &\le \alpha G(y_n,Ty_n,Ty_n)G(u,Su,Su) + \beta G(u,Su,Su)G(u,Ru,Ru) \\ &\quad + \gamma G(y_n,Ty_n,Ty_n)G(u,Ru,Ru) \\ &= \alpha G(y_n,Ty_n,Ty_n)G(u,u,u) + \beta G(u,u,u)G(u,u,u) \\ &\quad + \gamma G(y_n,Ty_n,Ty_n)G(u,u,u) \\ &= 0. \end{aligned} \tag{2.27}$$

Which implies that $\lim_{n\to\infty} G^2(Ty_n,u,u) = 0$. Hence $\{Ty_n\}$ is G-convergent to $u = Tu$. So T is G-continuous at u. Similarly, we can also prove that S, R are G-continuous at u. Therefore, we complete the proof. □

Corollary 2.2. *Let (X, G) be a complete G-metric space. Suppose the three self-mappings $T, S, R : X \to X$ satisfy the condition:*

$$\begin{aligned} G^2(T^p x, S^s y, R^r z) \le{}& \alpha G(x,T^p x,T^p x)G(y,S^s y,S^s y) + \beta G(y,S^s y,S^s y)G(z,R^r z,R^r z) \\ &+ \gamma G(x,T^p x,T^p x)G(z,R^r z,R^r z), \end{aligned} \tag{2.28}$$

for all $x, y, z \in X$, where $p, s, r \in \mathbb{N}$, α, β, γ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then T, S, and R have a unique common fixed point (say u) and T^p, S^s, R^r are all G-continuous at u.

Proof. From Theorem 2.1 we know that T^p, S^s, R^r have a unique common fixed point (say u); that is, $T^p u = u$, $S^s u = u$, $R^r u = u$, and T^p, S^s, R^r are G-continuous at u. Since $Tu = TT^p u = T^{p+1}u = T^p Tu$, so Tu is another fixed point of T^p, $Su = SS^s u = S^{s+1}u = g^s gu$, so Su is another fixed point of S^s, and $Ru = RR^r u = R^{r+1}u = R^r Ru$, so Ru is another fixed point of R^r. By (G3) and the condition (2.28) in Corollary 2.2, we have

$$\begin{aligned} G^2(Tu, S^sTu, R^rTu) &= G^2(T^pTu, S^sTu, R^rTu) \\ &\le \alpha G(Tu, T^pTu, T^pTu)G(Tu, S^sTu, S^sTu) \\ &\quad + \beta G(Tu, S^sTu, S^sTu)G(Tu, R^rTu, R^rTu) \\ &\quad + \gamma G(Tu, T^pTu, T^pTu)G(Tu, R^rTu, R^rTu) \\ &= \beta G(Tu, S^sTu, S^sTu)G(Tu, R^rTu, R^rTu) \\ &\le \beta G(Tu, S^sTu, R^rTu)G(Tu, S^sTu, R^rTu). \end{aligned} \tag{2.29}$$

Since $0 \le \beta < 1$, we can get $G^2(Tu, S^sTu, R^rTu) = 0$. That means $Tu = T^pTu = S^sTu = R^rTu$, hence Tu is another common fixed point of T^p, S^s-and R^r. Since the common fixed point of T^p, S^s-and R^r is unique, we deduce that $u = Tu$. By the same argument, we can prove $u = Su, u = Ru$. Thus, we have $u = Tu = Su = Ru$. Suppose v is another common fixed point of T, S, and R, then $v = T^pv = S^sv = R^rv$, and by using the condition (2.28) in Corollary 2.2 again, we have

$$\begin{aligned} G^2(v, u, u) &= G^2(T^pv, S^su, R^ru) \\ &\le \alpha G(v, T^pv, T^pv)G(u, S^su, S^su) + \beta G(u, S^su, S^su)G(u, R^ru, R^ru) \\ &\quad + \gamma G(v, T^pv, T^pv)G(u, R^ru, R^ru) \\ &= \alpha G(v, v, v)G(u, u, u) + \beta G(u, u, u)G(u, u, u) + \gamma G(v, v, v)G(u, u, u) \\ &= 0. \end{aligned} \tag{2.30}$$

Which implies that $G^2(v, u, u) = 0$, hence $v = u$. So the common fixed of T, S, and R is unique. It is obvious that every fixed point of T is a fixed point of S and R and conversely. □

Corollary 2.3. *Let (X, G) be a complete G-metric space. Suppose the self-mapping $T : X \to X$ satisfies the following condition:*

$$\begin{aligned} G^2(Tx, Ty, Tz) \le{}& \alpha G(x, Tx, Tx)G(y, Ty, Ty) + \beta G(y, Ty, Ty)G(z, Tz, Tz) \\ &+ \gamma G(x, Tx, Tx)G(z, Tz, Tz), \end{aligned} \tag{2.31}$$

for all $x, y, z \in X$, where α, β, γ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Let $T = S = R$ in Theorem 2.1, we can get this conclusion holds. □

Corollary 2.4. *Let (X,G) be a complete G-metric space. Suppose the self-mapping $T : X \to X$ satisfies the following condition:*

$$\begin{aligned} G^2(T^p x, T^p y, T^p z) \le{}& \alpha G(x, T^p x, T^p x) G(y, T^p y, T^p y) + \beta G(y, T^p y, T^p y) G(z, T^p z, T^p z) \\ &+ \gamma G(x, T^p x, T^p x) G(z, T^p z, T^p z). \end{aligned} \tag{2.32}$$

for all $x, y, z \in X$, where α, β, γ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point (say u) and T^p is G-continuous at u.

Proof. Let $T = S = R$, $p = s = r$ in Corollary 2.2, we can get this conclusion holds. □

Theorem 2.5. *Let (X,G) be a complete G-metric space, and let $T, S, R : X \to X$ be three self-mappings in X, which satisfy the following condition.*

$$\begin{aligned} G^2(Tx, Sy, Rz) \le{}& \alpha G(x, Tx, Sy) G(y, Sy, Rz) + \beta G(y, Sy, Rz) G(z, Rz, Tx) \\ &+ \gamma G(x, Tx, Sy) G(z, Rz, Tx). \end{aligned} \tag{2.33}$$

for all $x, y, z \in X$, α, β, γ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then T, S and R have a unique common fixed point (say u) and T, S, R are all G-continuous at u.

Proof. We will proceed in two steps.
Step 1. We prove any fixed point of T is a fixed point of S and R and conversely. Assume that $p \in X$ is such that $Tp = p$. Now we prove that $p = Sp$ and $p = Rp$. If it is not the case, then for $p \neq Sp$ and $p \neq Rp$, by (2.33) and (G3) we have

$$\begin{aligned} G^2(Tp, Sp, Rp) &\le \alpha G(p, Tp, Sp) G(p, Sp, Rp) + \beta G(p, Sp, Rp) G(p, Rp, Tp) \\ &\quad + \gamma G(p, Tp, Sp) G(p, Rp, Tp) \\ &= \alpha G(p, p, Sp) G(p, Sp, Rp) + \beta G(p, Sp, Rp) G(p, Rp, p) \\ &\quad + \gamma G(p, p, Sp) G(p, Rp, p) \\ &\le \alpha G(p, Rp, Sp) G(p, Sp, Rp) + \beta G(p, Sp, Rp) G(p, Rp, Sp) \\ &\quad + \gamma G(p, Rp, Sp) G(p, Rp, Sp) \\ &= (\alpha + \beta + \gamma) G^2(p, Rp, Sp). \end{aligned} \tag{2.34}$$

It follows that

$$G^2(p, Sp, Rp) = G^2(Tp, Sp, Rp) \le (\alpha + \beta + \gamma) G^2(p, Sp, Rp). \tag{2.35}$$

Since $G^2(p, Sp, Rp) > 0$, hence we have $\alpha + \beta + \gamma \ge 1$, however it contradicts with the condition $0 \le \alpha + \beta + \gamma < 1$, so we can have $p = Sp = Rp$, hence p is a common fixed point of T, S, and R.

Analogously, following the similar arguments to those given above, we can obtain a contradiction for $p \neq Sp$ and $p = Rp$ or $p = Sp$ and $p \neq Rp$. Hence in all the cases, we conclude that $p = Sp = Rp$. The same conclusion holds if $p = Sp$ or $p = Rp$.

Step 2. We prove that T, S and R have a unique common fixed point. Let $x_0 \in X$ be an arbitrary point, and define the sequence $\{x_n\}$ by $x_{3n+1} = Tx_{3n}, x_{3n+2} = Sx_{3n+1}, x_{3n+3} = Rx_{3n+2}$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$, for some n, with $n = 3m$, then $p = x_{3m}$ is a fixed point of T and, by the first step, p is a common fixed point of S, T, and R. The same holds if $n = 3m+1$ or $n = 3m+2$. Without loss of generality, we can assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. We first prove the sequence $\{x_n\}$ is a G-Cauchy sequence. In fact, by using (2.33) and ($G3$), we have

$$\begin{aligned} G^2(x_{3n+1}, x_{3n+2}, x_{3n+3}) &= G^2(Tx_{3n}, Sx_{3n+1}, Rx_{3n+2}) \\ &\leq \alpha G(x_{3n}, x_{3n+1}, x_{3n+2})G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3})G(x_{3n+2}, x_{3n+3}, x_{3n+1}) \\ &\quad + \gamma G(x_{3n}, x_{3n+1}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+1}). \end{aligned} \tag{2.36}$$

Which gives that

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (\alpha + \gamma)G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \tag{2.37}$$

It follows that

$$(1-\beta)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (\alpha + \gamma)G(x_{3n}, x_{3n+1}, x_{3n+2}). \tag{2.38}$$

From $0 \leq \beta < 1$, we know that $1-\beta > 0$. Then, we have

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \frac{\alpha + \gamma}{1-\beta}G(x_{3n}, x_{3n+1}, x_{3n+2}). \tag{2.39}$$

On the other hand, by using (2.33) and ($G3$), we have

$$\begin{aligned} G^2(x_{3n+2}, x_{3n+3}, x_{3n+4}) &= G^2(Tx_{3n+3}, Sx_{3n+1}, Rx_{3n+2}) \\ &\leq \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+2})G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3})G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \\ &\quad + \gamma G(x_{3n+3}, x_{3n+4}, x_{3n+2})G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \end{aligned} \tag{2.40}$$

Which implies that

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq (\alpha + \beta)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + \gamma G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{2.41}$$

It follows that

$$(1-\gamma)G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq (\alpha + \beta)G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \tag{2.42}$$

Since $0 \le \gamma < 1$, we know that $1-\gamma > 0$. So, we have

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \le \frac{\alpha+\beta}{1-\gamma} G(x_{3n+1}, x_{3n+2}, x_{3n+3}). \tag{2.43}$$

Again, using (2.33) and $(G3)$, we can get

$$\begin{aligned} G^2(x_{3n+3}, x_{3n+4}, x_{3n+5}) &= G^2(Tx_{3n+3}, Sx_{3n+4}, Rx_{3n+2}) \\ &\le \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+5}) G(x_{3n+4}, x_{3n+5}, x_{3n+3}) \\ &\quad + \beta G(x_{3n+4}, x_{3n+5}, x_{3n+3}) G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \\ &\quad + \gamma G(x_{3n+2}, x_{3n+3}, x_{3n+4}) G(x_{3n+3}, x_{3n+4}, x_{3n+5}). \end{aligned} \tag{2.44}$$

Which implies that

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \le \alpha G(x_{3n+3}, x_{3n+4}, x_{3n+5}) + (\beta+\gamma) G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{2.45}$$

It follows that

$$(1-\alpha) G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \le (\beta+\gamma) G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{2.46}$$

Since $0 \le \alpha \le \alpha+\beta+\gamma < 1$, we know that $1-\alpha > 0$. So we have

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \le \frac{\beta+\gamma}{1-\alpha} G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{2.47}$$

Let $q = \max\{(\alpha+\gamma)/(1-\beta), (\alpha+\beta)/(1-\gamma), (\beta+\gamma)/(1-\alpha)\}$, then $0 \le q < 1$, and by combining (2.39), (2.43), and (2.47), we have

$$G(x_n, x_{n+1}, x_{n+2}) \le q G(x_{n-1}, x_n, x_{n+1}) \le \cdots \le q^n G(x_0, x_1, x_2). \tag{2.48}$$

Thus, by $(G3)$ and $(G5)$, for every $m, n \in \mathbb{N}$, if $m > n$, noting that $0 \le q < 1$, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\le G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_{m+1}) \\ &\le \left(q^n + q^{n+1} + \cdots + q^{m-1}\right) G(x_0, x_1, x_2) \\ &\le \frac{q^n}{1-q} G(x_0, x_1, x_2). \end{aligned} \tag{2.49}$$

Which implies that $G(x_n, x_m, x_m) \to 0$, as $n, m \to \infty$. Thus $\{x_n\}$ is a G-Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$, such that $\{x_n\}$ is G-convergent to u.

Now we prove u is a common fixed point of T, S, and R. By using (2.33), we have

$$\begin{aligned} G^2(Tu, x_{3n+2}, x_{3n+3}) &= G^2(Tu, Sx_{3n+1}, Rx_{3n+2}) \\ &\le \alpha G(u, Tu, Sx_{3n+1})G(x_{3n+1}, Sx_{3n+1}, Rx_{3n+2}) \\ &\quad + \beta G(x_{3n+1}, Sx_{3n+1}, Rx_{3n+2})G(x_{3n+2}, Rx_{3n+2}, Tu) \\ &\quad + \gamma G(u, Tu, Sx_{3n+1})G(x_{3n+2}, Rx_{3n+2}, Tu) \\ &= \alpha G(u, Tu, x_{3n+2})G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &\quad + \beta G(x_{3n+1}, x_{3n+2}, x_{3n+3})G(x_{3n+2}, x_{3n+3}, Tu) \\ &\quad + \gamma G(u, Tu, x_{3n+2})G(x_{3n+2}, x_{3n+3}, Tu). \end{aligned} \tag{2.50}$$

Letting $n \to \infty$, and using the fact that G is continuous on its variables and $\gamma < 1$, we can get

$$G^2(Tu, u, u) \le \gamma G^2(u, u, Tu). \tag{2.51}$$

Which gives that $Tu = u$, hence u is a fixed point of T. By using (2.33) again, we have

$$\begin{aligned} G^2(x_{3n+1}, Su, x_{3n+3}) &= G^2(Tx_{3n}, Su, Rx_{3n+2}) \\ &\le \alpha G(x_{3n}, x_{3n+1}, Su)G(u, Su, x_{3n+3}) + \beta G(u, Su, x_{3n+3})G(x_{3n+2}, x_{3n+3}, x_{3n+1}) \\ &\quad + \gamma G(x_{3n}, x_{3n+1}, Su)G(x_{3n+2}, x_{3n+3}, x_{3n+1}). \end{aligned} \tag{2.52}$$

Letting $n \to \infty$ at both sides, for G is continuous in its variables, it follows that

$$G^2(u, Su, u) \le \alpha G^2(u, Su, u). \tag{2.53}$$

For $0 \le \alpha < 1$, Therefore, we can get $G^2(u, Su, u) = 0$, hence $Su = u$, hence u is a fixed point of S. Similarly, by (2.33), we can also get

$$\begin{aligned} G^2(x_{3n+1}, x_{3n+2}, Ru) &= G^2(Tx_{3n}, Sx_{3n+1}, Ru) \\ &\le \alpha G(x_{3n}, x_{3n+1}, x_{3n+2})G(x_{3n+1}, x_{3n+2}, Ru) \\ &\quad + \beta G(x_{3n+1}, x_{3n+2}, Ru)G(u, Ru, x_{3n+1}) \\ &\quad + \gamma G(x_{3n}, x_{3n+1}, x_{3n+2})G(u, Ru, x_{3n+1}). \end{aligned} \tag{2.54}$$

On taking $n \to \infty$ at both sides, since G is continuous in its variables, we get that

$$G^2(u, u, Ru) \le \beta G^2(u, u, Ru). \tag{2.55}$$

Since $0 \le \beta < 1$, so we get $G^2(u,u,Ru) = 0$, hence $u = Ru$, therefore, u is a fixed point of R. Consequently, we have $u = Tu = Su = Ru$, and u is a common fixed point of T, S, and R. Suppose $v \ne u$ is another common fixed point of T, S, and R, and we have $v = Tv = Sv = Rv$, then by (2.33), we have

$$\begin{aligned} G^2(u,u,v) &= G^2(Tu,Su,Rv) \\ &\le \alpha G(u,Tu,Su)G(u,Su,Rv) + \beta G(u,Su,Rv)G(v,Rv,Tu) \\ &\quad + \gamma G(u,Tu,Su)G(v,Rv,Tu) \\ &= \alpha G(u,u,u)G(u,u,v) + \beta G(u,u,v)G(v,v,u) + \gamma G(u,u,u)G(v,v,u). \end{aligned} \tag{2.56}$$

Which gives that

$$G^2(u,u,v) \le \beta G(u,u,v)G(v,v,u). \tag{2.57}$$

Hence, we can get $G(u,u,v) \le \beta G(v,v,u)$. By using (2.33) again, we get

$$\begin{aligned} G^2(u,v,v) &= G^2(Tu,Sv,Rv) \\ &\le \alpha G(u,Tu,Sv)G(v,Sv,Rv) + \beta G(v,Sv,Rv)G(v,Rv,Tu) \\ &\quad + \gamma G(u,Tu,Sv)G(v,Rv,Tu) \\ &= \alpha G(u,u,v)G(v,v,v) + \beta G(v,v,v)G(v,v,u) + \gamma G(u,u,v)G(v,v,u). \end{aligned} \tag{2.58}$$

Which implies that

$$G^2(u,v,v) \le \gamma G(u,u,v)G(v,v,u). \tag{2.59}$$

Hence, we can get

$$G(u,v,v) \le \gamma G(u,u,v). \tag{2.60}$$

By combining $G(u,u,v) \le \beta G(v,v,u)$, we can have

$$G(u,v,v) \le \gamma G(u,u,v) \le \beta\gamma G(v,v,u). \tag{2.61}$$

Since $v \ne u, G(u,v,v) > 0$, so we have that $\beta\gamma \ge 1$. Since $0 \le \beta,\ \gamma < 1$, we know $0 \le \beta\gamma < 1$, so it's a contradiction. Hence, we get $u = v$. Then we know the common fixed point of T, S, and R is unique.

To show that T is G-continuous at u, let $\{y_n\}$ be any sequence in X such that $\{y_n\}$ is G-convergent to u. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
G^2(Ty_n, u, u) &= G^2(Ty_n, Su, Ru) \\
&\le \alpha G(y_n, Ty_n, Su)G(u, Su, Ru) + \beta G(u, Su, Ru)G(u, Ru, Ty_n) \\
&\quad + \gamma G(y_n, Ty_n, Su)G(u, Ru, Ty_n) \\
&= \alpha G(y_n, Ty_n, u)G(u, u, u) + \beta G(u, u, u)G(u, u, Ty_n) \\
&\quad + \gamma G(y_n, Ty_n, u)G(u, u, Ty_n) \\
&= \gamma G(y_n, Ty_n, u)G(u, u, Ty_n).
\end{aligned} \tag{2.62}$$

Which implies that

$$G(Ty_n, u, u) \le \gamma G(y_n, Ty_n, u). \tag{2.63}$$

On taking $n \to \infty$ at both sides, considering $\gamma < 1$, we get $\lim_{n\to\infty} G(Ty_n, u, u) = 0$. Hence $\{Ty_n\}$ is G-convergent to $u = Tu$. So T is G-continuous at u. Similarly, we can also prove that S, R are G-continuous at u. Therefore, we complete the proof. □

Now we introduce an example to support Theorem 2.5.

Example 2.6. Let X = $[0,1]$, and let (X, G) be a G-metric space defined by $G(x, y, z) = |x-y| + |y-z| + |z-x|$, for all x, y, z in X. Let T, S, and R be three self-mappings defined by

$$Tx = \begin{cases} 1, & x \in \left[0, \dfrac{1}{2}\right] \\ \dfrac{6}{7}, & x \in \left(\dfrac{1}{2}, 1\right] \end{cases}, \qquad Sx = \begin{cases} \dfrac{7}{8}, & x \in \left[0, \dfrac{1}{2}\right] \\ \dfrac{6}{7}, & x \in \left(\dfrac{1}{2}, 1\right] \end{cases}, \qquad Rx = \frac{6}{7}, \quad x \in [0,1]. \tag{2.64}$$

Next we proof the mappings T, S, and R are satisfying Condition (2.33) of Theorem 2.5 with $\alpha = 1/7$, $\beta = 1/7$ and $\gamma = 4/7$.

Case 1. If $x, y \in [0, 1/2]$, $z \in [0,1]$, then

$$\begin{gathered}
G^2(Tx, Sy, Rz) = G^2\left(1, \frac{7}{8}, \frac{6}{7}\right) = \frac{4}{49}, \\
G(x, Tx, Sy) = G\left(x, 1, \frac{7}{8}\right) = |x-1| + \left|x - \frac{7}{8}\right| + \frac{1}{8} \ge \frac{1}{2} + \frac{3}{8} + \frac{1}{8} = 1, \\
G(y, Sy, Rz) = G\left(y, \frac{7}{8}, \frac{6}{7}\right) = \left|y - \frac{7}{8}\right| + \left|y - \frac{6}{7}\right| + \frac{1}{56} \ge \frac{3}{8} + \frac{5}{14} + \frac{1}{56} = \frac{3}{4}, \\
G(z, Rz, Tx) = G\left(z, \frac{6}{7}, 1\right) = \left|z - \frac{6}{7}\right| + \frac{1}{7} + |z-1| \ge 0 + \frac{1}{7} + 0 = \frac{1}{7}.
\end{gathered} \tag{2.65}$$

Thus, we have

$$\begin{aligned} G^2(Tx,Sy,Rz) &= \frac{4}{49} \le \alpha\cdot 1\cdot\frac{3}{4}+\beta\cdot\frac{3}{4}\cdot\frac{1}{7}+\gamma\cdot 1\cdot\frac{1}{7} \\ &\le \alpha G(x,Tx,Sy)G(y,Sy,Rz)+\beta G(y,Sy,Rz)G(z,Rz,Tx) \\ &\quad +\gamma G(x,Tx,Sy)G(z,Rz,Tx). \end{aligned} \tag{2.66}$$

Case 2. If $x\in[0,1/2],\ y\in(1/2,1],\ z\in[0,1]$, then we can get

$$\begin{gathered} G^2(Tx,Sy,Rz) = G^2\left(1,\frac{6}{7},\frac{6}{7}\right) = \frac{4}{49}, \\ G(x,Tx,Sy) = G\left(x,1,\frac{6}{7}\right) = |x-1|+\left|x-\frac{6}{7}\right|+\frac{1}{7} \ge \frac{1}{2}+\frac{5}{14}+\frac{1}{7} = 1, \\ G(y,Sy,Rz) = G\left(y,\frac{6}{7},\frac{6}{7}\right) = \left|y-\frac{6}{7}\right|+\left|y-\frac{6}{7}\right| \ge 0+0 = 0, \\ G(z,Rz,Tx) = G\left(z,\frac{6}{7},1\right) = \left|z-\frac{6}{7}\right|+\frac{1}{7}+|z-1| \ge 0+\frac{1}{7}+0 = \frac{1}{7}. \end{gathered} \tag{2.67}$$

Thus, we have

$$\begin{aligned} G^2(Tx,Sy,Rz) &= \frac{4}{49} \le \alpha\cdot 1\cdot 0+\beta\cdot 0\cdot\frac{1}{7}+\gamma\cdot 1\cdot\frac{1}{7} \\ &\le \alpha G(x,Tx,Sy)G(y,Sy,Rz)+\beta G(y,Sy,Rz)G(z,Rz,Tx) \\ &\quad +\gamma G(x,Tx,Sy)G(z,Rz,Tx). \end{aligned} \tag{2.68}$$

Case 3. If $x\in(1/2,1],\ y\in[0,1/2],\ z\in[0,1]$, then we have

$$\begin{gathered} G^2(Tx,Sy,Rz) = G^2\left(\frac{6}{7},\frac{7}{8},\frac{6}{7}\right) = \frac{1}{784}, \\ G(x,Tx,Sy) = G\left(x,\frac{6}{7},\frac{7}{8}\right) = \left|x-\frac{6}{7}\right|+\left|x-\frac{7}{8}\right|+\frac{1}{56} \ge 0+0+\frac{1}{56} = \frac{1}{56}, \\ G(y,Sy,Rz) = G\left(y,\frac{7}{8},\frac{6}{7}\right) = \left|y-\frac{7}{8}\right|+\left|y-\frac{6}{7}\right|+\frac{1}{56} \ge \frac{3}{8}+\frac{5}{14}+\frac{1}{56} = \frac{3}{4}, \\ G(z,Rz,Tx) = G\left(z,\frac{6}{7},\frac{6}{7}\right) = \left|z-\frac{6}{7}\right|+\left|z-\frac{6}{7}\right| \ge 0+0 = 0. \end{gathered} \tag{2.69}$$

Thus, we have

$$G^2(Tx,Sy,Rz) = \frac{1}{784} \leq \alpha \cdot \frac{1}{56} \cdot \frac{3}{4} + \beta \cdot \frac{3}{4} \cdot 0 + \gamma \cdot \frac{1}{56} \cdot 0$$
$$\leq \alpha G(x,Tx,Sy)G(y,Sy,Rz) + \beta G(y,Sy,Rz)G(z,Rz,Tx)$$
$$+ \gamma G(x,Tx,Sy)G(z,Rz,Tx). \tag{2.70}$$

Case 4. If $x,\ y \in (1/2,1],\ z \in [0,1]$, then we have

$$G^2(Tx,Sy,Rz) = G^2\left(\frac{6}{7},\frac{6}{7},\frac{6}{7}\right) = 0. \tag{2.71}$$

Thus, we have

$$G^2(Tx,Sy,Rz) = 0$$
$$\leq \alpha G(x,Tx,Sy)G(y,Sy,Rz) + \beta G(y,Sy,Rz)G(z,Rz,Tx)$$
$$+ \gamma G(x,Tx,Sy)G(z,Rz,Tx). \tag{2.72}$$

Then in all of the above cases, the mappings T, S, and R satisfy the contractive condition (2.33) of Theorem 2.5 with $\alpha = 1/7,\ \beta = 1/7,\ \gamma = 4/7$. So that all the conditions of Theorem 2.5 are satisfied. Moreover, $6/7$ is the unique common fixed point for all of the three mappings T, S, and R.

At last, we prove T, S, and R are all G-continuous at the common fixed point $6/7$. Since $6/7 \in (1/2,1]$, and let the sequence $\{y_n\} \subset (0,1]$ and $y_n \to (6/7)(n \to \infty)$, then there exists $N \in \mathbb{N}$ such that $\{y_n\} \subset (1/2,1]$, for all $n > N$. Without loss of generality, we can assume that $\{y_n\} \subset (1/2,1]$, and so $Ty_n = 6/7,\ Sy_n = 6/7$ and $Ry_n = 6/7$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ry_n = \frac{6}{7}. \tag{2.73}$$

Which implies that T, S, and R are all G-continuous at the common fixed point $6/7$.

Corollary 2.7. *Let (X,G) be a complete G-metric space. Suppose the three self-mappings $T, S, R : X \to X$ satisfy the condition:*

$$G^2(T^p x, S^s y, R^r z) \leq \alpha G(x, T^p x, S^s y)G(y, S^s y, R^r z) + \beta G(y, S^s y, R^r z)G(z, R^r z, T^p x)$$
$$+ \gamma G(x, T^p x, S^s y)G(z, R^r z, T^p x), \tag{2.74}$$

for all $x, y, z \in X$, where $p, s, r \in \mathbb{N}$, α, β, γ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then T, S, and R have a unique common fixed point (say u) and T^p, S^s, R^r are all G-continuous at u.

Proof. Since the proof of Corollary 2.7 is very similar to that of Corollary 2.2, so we delete it. □

Corollary 2.8. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a self-mapping in X, which satisfies the following condition:*

$$G^2(Tx, Ty, Tz) \leq \alpha G(x, Tx, Ty)G(y, Ty, Tz) + \beta G(y, Ty, Tz)G(z, Tz, Tx) + \gamma G(x, Tx, Ty)G(z, Tz, Tx). \tag{2.75}$$

for all $x, y, z \in X$, where α, β, γ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Corollary 2.9. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a self-mapping in X, which satisfies the following condition:*

$$G^2(T^p x, T^p y, T^p z) \leq \alpha G(x, T^p x, T^p y)G(y, T^p y, T^p z) + \beta G(y, T^p y, T^p z)G(z, T^p z, T^p x) + \gamma G(x, T^p x, T^p y)G(z, T^p z, T^p x). \tag{2.76}$$

for all $x, y, z \in X$, where $p \in N$, α, β, γ are nonnegative real numbers and $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point (say u) and T^p is G-continuous at u.

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