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Research Article

Surfaces of a Constant Negative Curvature

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I study the geometric notion of a differential system describing surfaces of a constant negative curvature and describe a family of pseudospherical surfaces for the nonlinear partial differential equations with constant Gaussian curvature −1.

1. Introduction

In recent decades, a class of transformations having their origin in the work by Bäcklund in the late nineteenth century has provided a basis for remarkable advances in the study of nonlinear partial differential equations (NLPDEs) [1]. The importance of Bäcklund transformations (BTs) and their generalizations is basically twofold. Thus, on one hand, invariance under a BT may be used to generate an infinite sequence of solutions for certain NLPDEs by purely algebraic superposition principles. On the other hand, BTs may also be used to link certain NLPDEs (particularly nonlinear evolution equations (NLEEs) modelling nonlinear waves) to canonical forms whose properties are well known [2, 3]. Nonlinear wave phenomena have attracted the attention of physicists for a long time. Investigation of a certain kind of NLPDEs has made great progress in the last decades. These equations have a wide range of physical applications and share several remarkable properties [4–6]: (i) the initial value problem can be solved exactly in terms of linear procedures, the so-called "inverse scattering method (ISM);" (ii) they have an infinite number of "conservation laws;" (iii) they have "BTs;" (iv) they describe pseudo-spherical surfaces (pss), and hence one may interpret the other properties (i)–(iii) from a geometrical point of view; (v) they are completely integrable [1, 3]. This geometrical interpretation is a natural generalization of a classical example given by Chern and Tenenblat [2] who introduced the notion of a differential equation (DE) for a function that describes a pss, and they obtained a classification for such equations of type $u_t = F(u, u_x, \ldots, u_{x^k})$ $(u_{x^k} =^k u/x^k)$. These results provide a systematic procedure to obtain a linear eigenvalue problem associated to any NLPDE of this type [7].

Sasaki [6] gave a geometrical interpretation for inverse scattering problem (ISP), considered by Ablowitz et al. [4], in terms of pss. Based on this interpretation, one may consider the following definition.

Let M^2 be a two-dimensional differentiable manifold with coordinates (x, t). A DE for a real function $u(x, t)$ describes a pss if it is a necessary and sufficient condition for the existence of differentiable functions:

$$f_{ij}, \quad 1 \le i \le 3,\ 1 \le j \le 2, \tag{1.1}$$

depending on u and its derivatives such that the one-forms

$$\omega_1 = f_{11}dx + f_{12}dt, \qquad \omega_2 = f_{21}dx + f_{22}dt, \qquad \omega_3 = f_{31}dx + f_{32}dt, \tag{1.2}$$

satisfy the structure equations of a pss, that is,

$$dw_1 = w_3 \wedge w_2, \qquad dw_2 = w_1 \wedge w_3, \qquad dw_3 = w_1 \wedge w_2. \tag{1.3}$$

This structure was considered for the first time by Chern and Tenenblat [2], motivated by Sasaki's observation [6] that the equations which are the necessary and sufficient condition for the integrability of a linear problem of Ablowitz, Kaup, Newell, Segur-(AKNS-) type [4, 7–12] do describe pss. Its importance, in the present context, arises from the fact that the connection between pss and integrability of DEs goes well beyond the AKNS framework, as will be explained in Section 2. A DE for a real-valued function $u(x, t)$ is kinematically integrable if it is the integrability condition of a one-parameter family of linear problems [13–20]:

$$v_x = P(\eta)v, \qquad v_t = Q(\eta)v, \tag{1.4}$$

in which $P(\eta)$ and $Q(\eta)$ are $\mathrm{SL}(2, R)$-valued functions of x, t, and (u and its derivatives) up to a finite order. Thus, an equation is kinematically integrable if it is equivalent to the zero curvature condition:

$$\frac{\partial P(\eta)}{\partial t} - \frac{\partial Q(\eta)}{\partial x} + [P(\eta), Q(\eta)] = 0, \tag{1.5}$$

where $\operatorname{tr} P(\eta) = \operatorname{tr} Q(\eta) = 0$, for each η (spectral parameter or eigenvalue). In addition, a DE will be said to strictly kinematically integrable if it's kinematically integrable and diagonal entries of the matrix $P(\eta)$ introduced above are η and $-\eta$.

The main aim of this paper is to use the geometric properties and differentiable functions in the construction of BTs for some NLEEs which describe pss.

The paper is organized as follows. In Section 2 we summarize the AKNS formulation of the ISM using the language of exterior differential forms; this language is very useful for geometry. The correspondence between NLEEs and their families of pss is established in Section 3. In Section 4 we find the BTs for some NLEEs (Liouville, Burgers, and sinh-Gordon equations, a third-order evolution equation (TOEE), a modified Korteweg-de Vries (mKdV) equation, and both families of equations I and II) which describe pss. Finally, we give some conclusions in Section 5.

2. The AKNS System for Some NLEEs

The ISM was first devised for the Korteweg-de Vries (KdV) equation [5]. Later, it was extended by Zakharov and Shabat [21] to a 2×2 scattering problem for the nonlinear Schrödinger equation (NLSE) and subsequently generalized by Ablowitz et al. [4] to include a variety of NLEEs. The AKNS method consists of the following steps: (i) set up an appropriate, 2×2 linear scattering (eigenvalue) problem in the "space" variable in which the solution of the NLEE plays the role of the potential; (ii) choose the "time" dependence of the eigenfunctions in such a way that the eigenvalues remain invariant as the potential evolves according to the NLEEs; (iii) solve the direct scattering problem at the initial "time" and determine the "time" dependence of the scattering data; (iv) do the ISP at later "times," namely, reconstruct the potential from the scattering data. In this section, we concentrate on the first step of the AKNS method. As a consequence, each solution of the DE provides a metric on M^2, whose Gaussian curvature is constant, equal to -1. Moreover, the above definition of a DE is equivalent to saying that the DE for u is the integrability condition for the problem:

$$dv = \Omega v, \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{2.1}$$

where d denotes exterior differentiation, v is a vector, and the 2×2 matrix Ω ($\Omega_{ij},\ i, j = 1, 2$) is traceless:

$$\operatorname{tr} \Omega = 0 \tag{2.2}$$

and consists of a one-parameter (η), family of one-forms in the independent variables (x, t), the dependent variable u and its derivatives. Equation (1.2) has three one-forms ω_1, ω_2, and ω_3 consisting of independent and dependent variables and their derivatives, such that the NLPDE is given by

$$\Theta \equiv d\Omega - \Omega\Omega = 0, \qquad \Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}, \tag{2.3}$$

which is, by construction, the original NLPDE to be solved. We illustrate here the following examples given by AKNS system.

(a) Liouville's equation:

$$u_{xt} = -2e^{u}, \tag{2.4}$$

$$\Omega = \frac{1}{2}\begin{pmatrix} \eta dx - \dfrac{1}{\eta}e^{u}dt & u_x dx + \dfrac{1}{\eta}e^{u}dt \\ u_x dx - \dfrac{1}{2\eta}e^{u}dt & -\eta dx + \dfrac{1}{\eta}e^{u}dt \end{pmatrix}. \tag{2.5}$$

(b) Burgers' equation:

$$2u_t - 2uu_x - u_{xx} = 0, \tag{2.6}$$

$$\Omega = \frac{1}{2}\begin{pmatrix} \eta dx + \dfrac{\eta u}{2}dt & (u+\eta)dx + \left(\dfrac{\eta u}{2} + \dfrac{u^2}{2} + \dfrac{u_x}{2}\right)dt \\ (u-\eta)dx + \left(\dfrac{-\eta u}{2} + \dfrac{u^2}{2} + \dfrac{u_x}{2}\right)dt & -\eta dx - \dfrac{\eta u}{2}dt \end{pmatrix}. \tag{2.7}$$

(c) sinh-Gordon equation:

$$u_{xt} = \sinh u, \tag{2.8}$$

$$\Omega = \frac{1}{2}\begin{pmatrix} \eta dx + \dfrac{1}{\eta}\cosh u dt & u_x dx - \dfrac{1}{\eta}\sinh u dt \\ u_x dx + \dfrac{1}{\eta}\sinh u dt & -\eta dx - \dfrac{1}{\eta}\cosh u dt \end{pmatrix}. \tag{2.9}$$

(d) A TOEE [22]:

$$u_t = \left(u_x^{-1/2}\right)_{xx} + u_x^{3/2}, \tag{2.10}$$

$$\Omega = \frac{1}{2}\begin{pmatrix} \eta dx - \eta^2 u_x^{-1/2}dt & -\eta e^{-u}dx + B_1 dt \\ \eta e^{u}dx + C_1 dt & -\eta dx + \eta^2 u_x^{-1/2}dt \end{pmatrix}, \tag{2.11}$$

where $B_1 = \eta((u_x^{-1/2})_x - u_x^{1/2} + \eta u_x^{-1/2})e^{-u}$, $C_1 = \eta((u_x^{-1/2})_x + u_x^{1/2} - \eta u_x^{-1/2})e^{u}$.

(e) A mKdV equation:

$$u_t = u_{xxx} + \left(a + u^2\right)u_x, \tag{2.12}$$

where a is a constant,

$$\Omega = \frac{1}{2}\begin{pmatrix} \eta dx + \left(\eta^3 + \dfrac{\eta u^2}{3} + a\eta\right)dt & -\sqrt{\dfrac{2}{3}}udx + B_2 dt \\ \sqrt{\dfrac{2}{3}}udx + C_2 dt & -\eta dx - \left(\eta^3 + \dfrac{\eta u^2}{3} + a\eta\right)dt \end{pmatrix}, \tag{2.13}$$

where $B_2 = \sqrt{2/3}(\eta^2 u + \eta u_x + u^3/3 + u_{xx} + au)$, $C_2 = \sqrt{2/3}(\eta^2 u + u^3/3 - \eta u_x + u_{xx} + au)$.

(f) A family of equations I [23]:

$$\left[u_t - (\alpha g(u) + \beta)u_x\right]_x = -g'(u), \tag{2.14}$$

where $g(u)$ is a differentiable function of u which satisfies $g'' + \mu g = \theta$, with α, β, μ, and θ are real constants, such that $\xi^2 = \alpha\eta^2 + \mu$,

$$\Omega = \frac{1}{2}\begin{pmatrix} \eta dx + \left(\dfrac{\xi^2 g - \theta}{\eta} + \beta\eta\right)dt & -\xi u_x dx + \left(\xi(\alpha g + \beta)u_x + \dfrac{\xi}{\eta}g'\right)dt \\ \xi u_x dx + \left(\xi(\alpha g + \beta)u_x - \dfrac{\xi}{\eta}g'\right)dt & -\eta dx - \left(\dfrac{\xi^2 g - \theta}{\eta} + \beta\eta\right)dt \end{pmatrix}. \tag{2.15}$$

(g) A family of equations II [23] similar to the family I, but with some signs changed:

$$\left[u_t - (\alpha g(u) + \beta)u_x\right]_x = g'(u), \tag{2.16}$$

where $g(u)$ is a differentiable function of u which satisfies $g'' + \mu g = \theta$, with α, β, μ, and θ are real constants, such that $\xi^2 = \alpha\eta^2 - \mu$,

$$\Omega = \frac{1}{2}\begin{pmatrix} \eta dx + \left(\dfrac{\xi^2 g - \theta}{\eta} + \beta\eta\right)dt & \xi u_x dx + \left(\xi(\alpha g + \beta)u_x + \dfrac{\xi}{\eta}g'\right)dt \\ \xi u_x dx + \left(\xi(\alpha g + \beta)u_x - \dfrac{\xi}{\eta}g'\right)dt & -\eta dx - \left(\dfrac{\xi^2 g - \theta}{\eta} + \beta\eta\right)dt \end{pmatrix}, \tag{2.17}$$

keeping in mind that the parameter η plays the role of the eigenvalue for the scattering problem in (2.1). Note that the one-form, Ω, is not unique for a given NLPDE, for the scattering equations (2.1), (2.2), and (2.3) are form invariant under the "gauge" transformation:

$$\nu \longrightarrow \nu' = A\nu, \quad \Omega \longrightarrow \Omega' = dAA^{-1} + A\Omega A^{-1}, \quad \Theta \longrightarrow \Theta' = A\Theta A^{-1}, \tag{2.18}$$

where A is an arbitrary 2×2 matrix with determinant unity,

$$\det A = 1. \tag{2.19}$$

Integrability of (2.1) is,

$$0 = d^2 v = d\Omega v - \Omega dv = (d\Omega - \Omega\Omega)v, \tag{2.20}$$

requires the vanishing of the two form

$$d\Omega - \Omega\Omega = 0. \tag{2.21}$$

It should be noted that the solution of these equations is of a very special kind. In general, (2.21) gives three different equations, which cannot be satisfied simultaneously by one-dependent variable u. It has been pointed out [17, 24] that Ω can be interpreted as a connection one form for the principle $\mathrm{SL}(2, R)$ bundle on R^2 and Θ as its curvature two form. The geometrical explanation of the $\mathrm{SL}(2, R)$ is given in [6, 16].

3. The NLEEs Which Describe pss

Whenever the functions are real, Sasaki [6] gave a geometrical interpretation for the problem. Consider the one-forms defined by

$$\begin{aligned} \omega_1 &= (r + q)dx + (C + B)dt, \\ \omega_2 &= \eta dx + 2A dt, \\ \omega_3 &= (r - q)dx + (C - B)dt, \end{aligned} \tag{3.1}$$

where $\eta = -2i\xi$.

Let M^2 be a two-dimensional differentiable manifold parametrized by coordinates x, t. We consider a metric on M^2 defined by ω_1, ω_2. The first two equations in (1.3) are the structure equations which determine the connection form ω_3, and the last equation in (1.3), the Gauss equation, determines that the Gaussian curvature of M^2 is -1, that is, M^2 is a pss. Moreover, an EE must be satisfied for the existence of forms (3.1) satisfying (1.3). This justifies the definition of a DE which describes a pss that we considered in the introduction.

We will restrict ourselves to the case where $f_{21} = \eta$. More precisely, we say that a DE for $u(x, t)$ describes a pss if it is a necessary and sufficient condition for the existence of functions f_{ij}, $1 \le i \le 3$, $1 \le j \le 2$, depending on $u(x, t)$ and its derivatives, $f_{21} = \eta$, such that the one-forms in (1.2), satisfy the structure equations (1.3) of a pss. It follows from this definition that for each nontrivial solution u of the DE, one gets a metric defined on M^2, whose Gaussian curvature is -1.

It has been known, for along time, that the sinh-Gordon (SG) equation describes a pss. In this paper, we extend the same analysis to include the Liouville, Burgers, sinh-Gordon equations, a TOEE, a mKdV equation, and both families of equations I and II.

Example 3.1. Let M^2 be a differentiable surface, parametrized by coordinates x, t.

(a) Liouville's equation.

Consider

$$\omega_1 = u_x dx, \quad \omega_2 = \eta dx - \frac{e^u}{\eta} dt, \quad \omega_3 = -\frac{e^u}{\eta} dt. \tag{3.2}$$

Then M^2 is a pss if and only if u satisfies Liouville's equation (2.4).

(b) Burgers' equation.

Consider

$$\omega_1 = u dx + \left(\frac{u^2}{2} + \frac{u_x}{2} \right) dt, \quad \omega_2 = \eta dx + \frac{\eta u}{2} dt, \quad \omega_3 = -\eta dx - \frac{\eta u}{2} dt. \tag{3.3}$$

Then M^2 is a pss if and only if u satisfies the Burgers' equation (2.6).

(c) sinh-Gordon equation.

Consider

$$\omega_1 = u_x dx, \quad \omega_2 = \eta dx + \frac{\cosh u}{\eta} dt, \quad \omega_3 = \frac{\sinh u}{\eta} dt. \tag{3.4}$$

Then M^2 is a pss if and only if u satisfies the sinh-Gordon equation (2.8).

(d) A TOEE.

Consider

$$\begin{aligned} \omega_1 &= \eta \sinh u dx + \left[\eta \left(u_x^{-1/2} \right)_x \cosh u + \eta \left(u_x^{1/2} - \eta u_x^{-1/2} \right) \sinh u \right] dt, \\ \omega_2 &= \eta dx - \eta^2 u_x^{-1/2} dt, \\ \omega_3 &= \eta \cosh u dx + \left[\eta \left(u_x^{-1/2} \right)_x \sinh u + \eta \left(u_x^{1/2} - \eta u_x^{-1/2} \right) \cosh u \right] dt. \end{aligned} \tag{3.5}$$

Then M^2 is a pss iff u satisfies a TOEE (2.10).

(e) A mKdV equation.

Consider

$$\omega_1 = -\eta\sqrt{\frac{2}{3}}u_x dt,$$
$$\omega_2 = \eta dx + \left(\eta^3 + \frac{\eta u^2}{3} + a\eta\right)dt, \tag{3.6}$$
$$\omega_3 = \sqrt{\frac{2}{3}}u dx + \sqrt{\frac{2}{3}}\left(\eta^2 u + \frac{u^3}{3} + u_{xx} + au\right)dt.$$

Then M^2 is a pss iff u satisfies a mKdV equation (2.12).

(f) A family of equations I.

Consider

$$\omega_1 = -\frac{\xi}{\eta}g' dt,$$
$$\omega_2 = \eta dx + \left(\frac{\xi^2 g - \theta}{\eta} + \beta\eta\right)dt, \tag{3.7}$$
$$\omega_3 = \xi u_x dx + \xi(\alpha g + \beta)u_x dt.$$

Then M^2 is a pss if and only if u satisfies the family of equations I (2.14).

(g) A family of equations II.

Consider

$$\omega_1 = \xi u_x dx + \xi(\alpha g + \beta)u_x dt,$$
$$\omega_2 = \eta dx + \left(\frac{\xi^2 g - \theta}{\eta} + \beta\eta\right)dt, \tag{3.8}$$
$$\omega_3 = \xi\eta g' dt.$$

Then M^2 is a pss if and only if u satisfies the family of equations II (2.16).

4. A Geometric Method Which Provides BTs

In this section, we show how the geometric properties of a pss may be applied to obtain analytic results for some NLEEs which describe pss.

The classical Bäcklund theorem originated in the study of pss, relating solutions of the SG equation. Other transformations have been found relating solutions of specific equations in [15, 17, 24, 25]. Such transformations are called BTs after the classical one. A BT which relates solutions of the same equation is called a self-Bäcklund transformation (sBT). An interesting fact which has been observed is that DEs which have sBT also admit a superposition formula. The importance of such formulas is due to the following: if u_0 is a solution of the NLEE and u_1, u_2 are solutions of the same equation obtained by the sBT, then the superposition formula provides a new solution u' algebraically. By this procedure one obtains the soliton solutions of a NLEE. In what follows we show that geometrical properties of pss provide a systematic method to obtain the BTs for some NLEEs which describe pss.

Proposition 4.1. *Given a coframe $\{\overline{\omega}_1, \overline{\omega}_2\}$ and corresponding connection one-form $\overline{\omega}_3$ on a smooth Riemannian surfaces M^2, there exists a new coframe $\{\overline{\omega}'_1, \overline{\omega}'_2\}$ and new connection one-form $\overline{\omega}'_3$ satisfying the following:*

$$d\overline{\omega}'_1 = 0, \qquad d\overline{\omega}'_2 = \overline{\omega}'_2 \overline{\omega}'_1, \qquad \overline{\omega}'_3 + \overline{\omega}'_2 = 0, \tag{4.1}$$

if and only if the surface M^2 is pss. For the sake of clarity, we give a revised proof of [26].

Proof. Assume that the orthonormal dual to the coframes $\{\overline{\omega}_1, \overline{\omega}_2\}$ and $\{\overline{\omega}'_1, \overline{\omega}'_2\}$ possesses the same orientation. The one-forms $\overline{\omega}_i$ and $\overline{\omega}'_i$ $(i = 1, 2, 3)$ are connected by means of [27–31]:

$$\overline{\omega}'_1 = \overline{\omega}_1 \cos\phi - \overline{\omega}_2 \sin\phi, \qquad \overline{\omega}'_2 = \overline{\omega}_1 \sin\phi + \overline{\omega}_2 \cos\phi, \qquad \overline{\omega}'_3 = \overline{\omega}_3 - d\phi. \tag{4.2}$$

It follows that $\overline{\omega}'_1$, $\overline{\omega}'_2$, $\overline{\omega}'_3$ satisfying (4.1) exist if and only if the Pfaffian system,

$$\overline{\omega}_3 - d\phi + \overline{\omega}_1 \sin\phi + \overline{\omega}_2 \cos\phi = 0, \tag{4.3}$$

on the space of coordinates (x, t, ϕ) is completely integrable for $\phi(x, t)$, and this happens if and only if M^2 is pss.

Geometrically, (4.1) and (4.3) determine geodesic coordinates on M^2. Now, if $u_t = F(u, u_x, \ldots, u_{x^k})$ describes pss with associated one-forms $\omega_i = f_{i1}dx + f_{i2}dt$, (4.1) and (4.3) imply that the Pfaffian system,

$$\omega_3 - d\phi + \omega_1 \sin\phi + \omega_2 \cos\phi = 0, \tag{4.4}$$

is completely integrable for $\phi(x, t)$ whenever $u(x, t)$ is a local solution of $u_t = F(u, u_x, \ldots, u_{x^k})$ [2, 32]. □

Proposition 4.2. *Let $u_t = F(u, u_x, \ldots, u_{x^k})$ be a NLEE which describe a pss with associated one-forms* (1.2)*. Then, for each solution $u(x, t)$ of $u_t = F(u, u_x, \ldots, u_{x^k})$, the system of equations for $\phi(x, t)$,*

$$\phi_x - f_{31} + f_{11} \sin\phi + \eta \cos\phi = 0, \qquad \phi_t - f_{32} + f_{12} \sin\phi + f_{22} \cos\phi = 0, \tag{4.5}$$

is completely integrable. Moreover, for each solution of $u(x, t)$ of $u_t = F(u, u_x, \ldots, u_{x^k})$ and corresponding solution ϕ,

$$(f_{11} \cos\phi - \eta \sin\phi)dx + (f_{12} \cos\phi - f_{22} \sin\phi)dt \tag{4.6}$$

is a closed one-form [2].

Eliminating $\phi(x,t)$ from (4.5), by using the substitution

$$\cos\phi = \frac{2\Gamma}{1+\Gamma^2}, \tag{4.7}$$

where

$$\Gamma = \frac{v_1}{v_2}, \tag{4.8}$$

then (4.5) is reduced to the Riccati equations:

$$\frac{\partial\Gamma}{\partial x} = \eta\Gamma + \frac{1}{2}f_{11}\left(1-\Gamma^2\right) - \frac{1}{2}f_{31}\left(1+\Gamma^2\right), \tag{4.9}$$

$$\frac{\partial\Gamma}{\partial t} = f_{22}\Gamma + \frac{1}{2}f_{12}\left(1-\Gamma^2\right) - \frac{1}{2}f_{32}\left(1+\Gamma^2\right). \tag{4.10}$$

The procedure in the following is that one constructs a transformation Γ' satisfying the same equation as (4.10) with a potential $u'(x)$ where

$$u'(x) = u(x) + f(\Gamma,\eta). \tag{4.11}$$

Thus, eliminating Γ in (4.9), (4.10) and (4.11), we have a BT to a desired NLEE. We consider the following examples [33].

(a) BT for Liouville's equation.

For (2.4) we consider the functions defined by

$$\begin{aligned} f_{11} &= u_x, & f_{12} &= 0,\\ f_{21} &= \eta, & f_{22} &= -\frac{e^u}{\eta},\\ f_{31} &= 0, & f_{32} &= -\frac{e^u}{\eta}, \end{aligned} \tag{4.12}$$

for any solution $u(x,t)$ of (2.4), the above functions satisfy (2.21). Then (4.9) becomes

$$\frac{\partial\Gamma}{\partial x} = \eta\Gamma + \frac{u_x}{2}\left(1-\Gamma^2\right). \tag{4.13}$$

If we choose Γ' and u' as

$$\begin{aligned} \Gamma' &= \frac{1}{\Gamma},\\ u' &= -u - 2\ln\frac{1-\Gamma}{1+\Gamma}, \end{aligned} \tag{4.14}$$

then Γ' and u' satisfy (4.13). If we eliminate Γ in (4.13) and (4.10) with (4.14), we get the BT:

$$(u' - u)_x = 2\eta \sinh \frac{1}{2}(u' + u), \qquad (u' + u)_t = \frac{2}{\eta} e^{1/2(u-u')}. \tag{4.15}$$

Equation (4.15) is the BT for Liouville's equation (2.4) with f_{11}, f_{22}, and f_{32} given in (4.12).

(b) BT for Burgers' equation.

For any solution $u(x,t)$ of the Burgers' equation (2.6), the functions

$$\begin{gathered} f_{11} = u, \qquad f_{12} = \left(\frac{u^2}{2} + \frac{u_x}{2}\right), \\ f_{21} = \eta, \qquad f_{22} = \frac{\eta u}{2}, \\ f_{31} = -\eta, \qquad f_{32} = -\frac{\eta u}{2}. \end{gathered} \tag{4.16}$$

The above functions f_{ij} satisfy (2.21). Then (4.9) becomes [27]

$$\frac{\partial \Gamma}{\partial x} = \frac{\eta}{2}\left(1 + 2\Gamma + \Gamma^2\right) + \frac{u}{2}\left(1 - \Gamma^2\right). \tag{4.17}$$

If we choose Γ' and u' as

$$\begin{gathered} \Gamma' = \frac{1}{\Gamma}, \\ u' = -u + 4\frac{\partial}{\partial x}\tanh^{-1}\Gamma, \end{gathered} \tag{4.18}$$

then Γ' and u' satisfy (4.17). If we eliminate Γ in (4.17) and (4.10) with (4.18), we get the BT:

$$\begin{gathered} (w' - w)_x = \frac{\eta}{2}\left[1 + \sinh 2(w' + w) + 2\sinh^2(w' + w)\right], \\ (w' + w)_t = 4w_x^2 + 4w_{xx} + 2\eta w_x e^{2(w+w')}, \end{gathered} \tag{4.19}$$

where we put $u' = 4w'_x$ and $u = 4w_x$. Equation (4.19) is the BT for the Burgers' equation (2.6) with f_{11}, f_{22}, and f_{32} given in (4.16).

(c) BT for sinh-Gordon equation.

For (2.8) we consider the following functions of $u(x,t)$ defined by [28]

$$\begin{aligned} f_{11} &= u_x, & f_{12} &= 0, \\ f_{21} &= \eta, & f_{22} &= \frac{\cosh u}{\eta}, \\ f_{31} &= 0, & f_{32} &= \frac{\sinh u}{\eta}, \end{aligned} \tag{4.20}$$

for any solution $u(x,t)$ of (2.8), the above functions f_{ij} satisfy (2.21). Then (4.9) becomes [29]

$$\frac{\partial \Gamma}{\partial x} = \eta\Gamma + \frac{u_x}{2}\left(1 - \Gamma^2\right). \tag{4.21}$$

If we choose Γ' and u' as

$$\begin{aligned} \Gamma' &= \frac{1}{\Gamma}, \\ u' &= -u + 4\tanh^{-1}\Gamma, \end{aligned} \tag{4.22}$$

then Γ' and u' satisfy (4.21). If we eliminate Γ in (4.21) and (4.10) with (4.22), we get the BT:

$$\begin{aligned} \left(u' - u\right)_x &= 2\eta \sinh\frac{(u' + u)}{2}, \\ \left(u' + u\right)_t &= \frac{-2}{\eta}\left[\sinh u \cosh\frac{(u' + u)}{2} - \cosh u \sinh\frac{(u' + u)}{2}\right]. \end{aligned} \tag{4.23}$$

Equation (4.23) is the BT for the sinh-Gordon equation (2.8) with f_{11}, f_{22}, and f_{32} given in (4.20).

(d) BT for a TOEE.

For (2.10) we consider the functions defined by

$$\begin{aligned} f_{11} &= \eta \sinh u, & f_{12} &= \left[\eta\left(u_x^{-1/2}\right)_x \cosh u + \eta\left(u_x^{1/2} - \eta u_x^{-1/2}\right)\sinh u\right], \\ f_{21} &= \eta, & f_{22} &= -\eta^2 u_x^{-1/2}, \\ f_{31} &= \eta \cosh u, & f_{32} &= \left[\eta\left(u_x^{-1/2}\right)_x \sinh u + \eta\left(u_x^{1/2} - \eta u_x^{-1/2}\right)\cosh u\right]. \end{aligned} \tag{4.24}$$

The above functions f_{ij} satisfy (2.21). Then (4.9) becomes [30]

$$\frac{\partial\Gamma}{\partial x} = \eta\Gamma - \frac{-\eta}{2}e^{-u} - \frac{\eta}{2}e^{u}\Gamma^2. \tag{4.25}$$

If we choose Γ' and u' as

$$\begin{aligned} \Gamma' &= \Gamma, \\ u' &= -u - 2\ln\Gamma, \end{aligned} \tag{4.26}$$

then Γ' and u' satisfy (4.25). If we eliminate Γ in (4.25) and (4.10) with (4.26), we get the BT:

$$\begin{aligned} (u' + u)_x &= -2\eta + 2\eta\cosh\frac{(u' - u)}{2}, \\ (u' + u)_t &= 2\eta^2 u_x^{-1/2} - 2\eta\left(u_x^{-1/2}\right)_x \sinh\frac{(u' - u)}{2} \\ &\quad + 2\eta\left[u_x^{1/2} - \eta u_x^{-1/2}\right]\cosh\frac{(u' - u)}{2}. \end{aligned} \tag{4.27}$$

Equation (4.27) is the BT for a TOEE (2.10) with f_{11}, f_{22}, and f_{32} given in (4.24).

(e) BT for a mKdV equation.

For (2.12) we consider the following functions of $u(x,t)$ defined by

$$\begin{aligned} f_{11} &= 0, \qquad f_{12} = -\eta\sqrt{\frac{2}{3}}u_x, \\ f_{21} &= \eta, \qquad f_{22} = \left(\eta^3 + \frac{\eta u^2}{3} + a\eta\right), \\ f_{31} &= \sqrt{\frac{2}{3}}u, \qquad f_{32} = \sqrt{\frac{2}{3}}\left(\eta^2 u + \frac{u^3}{3} + u_{xx} + au\right), \end{aligned} \tag{4.28}$$

for any solution $u(x,t)$ of (2.12), the above functions f_{ij} satisfy (2.21). Then (4.9) becomes [31]

$$\frac{\partial\Gamma}{\partial x} = \eta\Gamma - \sqrt{\frac{1}{6}}u\left(1 + \Gamma^2\right). \tag{4.29}$$

If we choose Γ' and u' as

$$\Gamma' = \frac{1}{\Gamma},$$
$$u' = u + 2\sqrt{6}\frac{\partial}{\partial x}\tan^{-1}\Gamma, \tag{4.30}$$

then Γ' and u' satisfy (4.29). If we eliminate Γ in (4.29) and (4.10) with (4.30), we get the BT:

$$\left(w' + w\right)_x = \frac{\eta}{2}\sin 2\left(w' - w\right),$$
$$\left(w' - w\right)_t = -\left(2w_{xxx} + 16w_x^3 + 2\eta^2 w_x + \frac{a\eta}{\sqrt{6}}\right) + \frac{1}{2}\left(\eta^3 + 8\eta w_x^2 + a\eta\right)\sin 2\left(w' - w\right) - 2\eta w_{xx}\cos 2\left(w' - w\right), \tag{4.31}$$

where we put $u' = 2\sqrt{6}w'_x$ and $u = 2\sqrt{6}w_x$. Equation (4.31) is the BT for an mKdV equation (2.12) with f_{11}, f_{22}, and f_{32} given in (4.28).

(f) BT for the family of equations I.

For any solution u of the family of equations I (2.14), the functions

$$f_{11} = 0, \qquad f_{12} = -\frac{\xi}{\eta}g',$$
$$f_{21} = \eta, \qquad f_{22} = \left(\frac{\xi^2 g - \theta}{\eta} + \beta\eta\right), \tag{4.32}$$
$$f_{31} = \xi u_x, \qquad f_{32} = \xi(\alpha g + \beta)u_x.$$

The above functions f_{ij} satisfy (2.21). Then (4.9) becomes [34]

$$\frac{\partial\Gamma}{\partial x} = \eta\Gamma - \frac{\xi}{2}u_x\left(1 + \Gamma^2\right). \tag{4.33}$$

If we choose Γ' and u' as

$$\Gamma' = \frac{1}{\Gamma},$$
$$u' = u + \frac{4}{\xi}\tan^{-1}\Gamma, \tag{4.34}$$

then Γ' and u' satisfy (4.33). If we eliminate Γ in (4.33) and (4.10) with (4.34), we get the BT:

$$\begin{gathered}
(u'+u)_x = \frac{2\eta}{\xi}\sin\xi\frac{(u'-u)}{2}, \\
(u'-u)_t = \frac{2}{\xi}\left(\frac{\xi^2 g-\theta}{\eta}+\beta\eta\right)\sin\xi\frac{(u'-u)}{2} - 2(\alpha g+\beta)u_x \\
+\frac{2}{\eta}g'\cos\xi\frac{(u'-u)}{2}.
\end{gathered} \tag{4.35}$$

Equation (4.35) is the BT for the family of equations I (2.14) with f_{11}, f_{22}, and f_{32} given in (4.32).

(g) BT for the family of equations II.

For (2.16) we consider the functions of $u(x,t)$ defined by

$$\begin{gathered}
f_{11} = \xi u_x, \qquad f_{12} = \xi(\alpha g+\beta)u_x, \\
f_{21} = \eta, \qquad f_{22} = \left(\frac{\xi^2 g-\theta}{\eta}+\beta\eta\right), \\
f_{31} = 0, \qquad f_{32} = \frac{\xi}{\eta}g',
\end{gathered} \tag{4.36}$$

for any solution $u(x,t)$ of (2.16), the above functions f_{ij} satisfy (2.21). Then (4.9) becomes

$$\frac{\partial\Gamma}{\partial x} = \eta\Gamma + \frac{\xi}{2}u_x\left(1-\Gamma^2\right). \tag{4.37}$$

If we choose Γ' and u' as

$$\begin{gathered}
\Gamma' = \frac{1}{\Gamma}, \\
u' = -u + \frac{4}{\xi}\tanh^{-1}\Gamma,
\end{gathered} \tag{4.38}$$

then Γ' and u' satisfy (4.37). If we eliminate Γ in (4.37) and (4.10) with (4.38), we get the BT:

$$\begin{aligned} (u'-u)_x &= \frac{2\eta}{\xi}\sinh\xi\frac{(u'+u)}{2}, \\ (u'+u)_t &= \frac{2}{\xi}\left(\frac{\xi^2 g-\theta}{\eta}+\beta\eta\right)\sinh\xi\frac{(u'+u)}{2}+2(\alpha g+\beta)u_x \\ &\quad -\frac{2}{\eta}g'\cosh\xi\frac{(u'+u)}{2}. \end{aligned} \tag{4.39}$$

Equation (4.39) is the BT for the family of equations II (2.16) with f_{11}, f_{22}, and f_{32} given in (4.36).

We have previously discussed the relationships among the geometrical properties and the BT. There, we restrict our discussion to the NLEEs which can be reduced to the Liouville's form of the geometrical properties such as the Burgers, the sinh-Gordon equations, a TOEE, a mKdV, and the two family of equations I and II.

5. Conclusions

We may hope to find some relationships among various soliton equations which describe pss. The latter yields directly the curvature condition (Gaussian curvature equal to −1, corresponding to pseudo-spherical surfaces). This geometrical method is considered for several NLPDEs which describe pss: Liouville, Burgers, sinh-Gordon equations, a TOEE, a mKdV, and the two families of equations I and II. We show how the geometric properties of a pss may be applied to obtain analytic results for some NLEEs which describe pss. This geometrical method allows some further generalization of the work on Bäcklund transformations given by Wadati et al. [3]. The Bäcklund transformations for all seven NLPDEs mentioned above are derived in this way.

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