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Research Article

Homoclinic Orbits for a Class of Noncoercive Discrete Hamiltonian Systems

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A class of first-order noncoercive discrete Hamiltonian systems are considered. Based on a generalized mountain pass theorem, some existence results of homoclinic orbits are obtained when the discrete Hamiltonian system is not periodical and need not satisfy the global Ambrosetti-Rabinowitz condition.

1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{R}$ denote the set of all natural numbers, integers, and real numbers, respectively. Throughout this paper, without special statement, $|\cdot|$ denotes the usual norm in $\mathbf{R}^N$ with $N \in \mathbf{N}$, $u \cdot v$ denotes the inner product of $u \in \mathbf{R}^N$ and $v \in \mathbf{R}^N$.

Consider the noncoercive discrete Hamiltonian systems

$$J\Delta x(t) - M(t)Sx(t) + v^* \times v^* H'(t, v \times v(Sx(t))) = 0, \quad t \in \mathbf{Z}, \tag{1.1}$$

where $v : \mathbf{R}^N \to \mathbf{R}^m (1 \le m \le N)$ is a nontrivial linear operator, v^* is its adjoint, $v \times v$ is the tensorial product of $v, v : (v \times v)(p, q) = (v(p), v(q))$

$$M(t) = \begin{pmatrix} 0 & v^* L(t) v \\ v^* L(t) v & 0 \end{pmatrix}, \tag{1.2}$$

with $L(t)$ is an $(N \times N)$ symmetric matrix valued function and $H : \mathbf{R} \times \mathbf{R}^N$, $(t, y) \mapsto H(t, y)$ is a continuous function, differentiable with respect to the second variable with continuous derivative $H'(t, y) = (\partial H/\partial y)(t, y)$. S is the shift operator defined as $Sx(t) = \binom{x_1(t+1)}{x_2(t)}$ and $x(t) = \binom{x_1(t)}{x_2(t)}$, where $x_1, x_2 \in \mathbf{R}^N$. $\Delta x_i(t) = x_i(t+1) - x_i(t)$, $i = 1, 2$, is the forward difference operator. J is the standard symplectic matrix $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$, where I_N is the identity matrix on $\mathbf{R}^N$.

As usual, assuming that a solution $x(t) = 0$ an equilibrium for (1.1), we say that a solution $x(t)$ is homoclinic to 0 if $x(t)$ satisfies $x(t) \neq 0$, and the asymptotic condition $x(t) \to 0$ as $|t| \to +\infty$. Such solutions have been found in various models of continuous dynamical systems and frequently have tremendous effects on the dynamics of such nonlinear systems. So the homoclinic orbits have been extensively studied since the time of Poincaré , see [1–7] and references therein.

In recent years, there has been much research activity concerning the theory of difference equations. To a large extent, this due to the realization that difference equations are important in applications. New applications that involve difference equations continue to arise with frequency in the modelling of computer science, economics, neural network, ecology, cybernetics, and so forth, we can refer to [8–13] for detail. Many scholars have investigated discrete Hamiltonian systems independently main for two reasons. The first one is that the behaviour of discrete Hamiltonian systems is sometimes sharply different from the behaviour of the corresponding continuous systems. The second one is that there is a fundamental relationship between solutions of continuous systems and the corresponding discrete systems by employing discrete variable methods (see [8] for detail).

The general form of (1.1) is

$$J\Delta x(t) - H'(t, x(t)) = 0, \quad t \in \mathbf{Z}, \tag{1.3}$$

which was studied by many scholars in various fields. By making use of minimax theory and geometrical index theory, [14] gave results on subharmonic solutions with prescribed minimal periods. When (1.3) are superquadratic systems, Guo and Yu [15] obtained some existence and multiplicity results by Z_2 index theory and linking theorem. In [16], when H is subquadratic at infinity, the authors gave some existence results of periodic solutions. As to homoclinic orbits for discrete systems, [17–19] studied the second order discrete systems by critical point theory recently. While for the first order discrete systems, such as (1.1) or (1.3), to the authors' best knowledge, it seems there exists no similar results.

Moreover, we may regard (1.1) as being a discrete analogue of Hamiltonian systems

$$J\dot{x}(t) - M(t)x + H'(t, x(t)) = 0, \quad t \in \mathbf{R}. \tag{1.4}$$

Equation (1.1) is the best approximation of (1.4) when one lets the step size not be equal to 1 but the variable's step size go to zero, so solutions of (1.1) can give some desirable numerical features for (1.4). (1.4) is one form of classical Hamiltonian systems appearing in the study of various fields and many well-known results were given.

In view of above reasons, the goal of this paper is to study the existence of homoclinic orbits for the first order discrete Hamiltonian system (1.1) when H satisfies superquadratic conditions and need not satisfy the global Ambrosetti-Rabinowitz (AR) condition:

(AR): there exist two constants $\mu > 2$ and $r > 0$ such that for all $t \in \mathbf{Z}$ and $x \in \mathbf{R}^{2N}$, $|x| \geq r$

$$0 < \mu H(t,x) \leq H'(t,x(t)) \cdot x. \tag{1.5}$$

Let $l(t)$ denotes the smallest eigenvalue of $v^* L(t) v$, that is,

$$l(t) = \inf_{\xi \in \mathbf{R}^N, |\xi|=1} v^* L(t) v(\xi) \cdot \xi, \quad \forall t \in \mathbf{Z}. \tag{1.6}$$

For later use, we need the following assumptions:

(L_1) there exists $1 < \gamma < 2$ such that $l(t)|t|^{\gamma-2} \to +\infty$ as $|t| \to +\infty$;

(H_1) $H(t,y)/|y|^2 \to +\infty$ as $|y| \to +\infty$, $t \in \mathbf{Z}$;

(H_2) $|H'(t,y)|/|y| \to 0$ as $|y| \to +0$ and $t \in \mathbf{Z}$;

(H_3) there exist $a > 0$ and $\alpha > 1$ such that

$$\left|H'(t,y)\right| \leq a\left(|y|^{\alpha} + 1\right), \quad \forall t \in \mathbf{Z},\ \forall y \in \mathbf{R}^{2N}; \tag{1.7}$$

(H_4) there exist $\beta > \alpha$, $b > 0$ and $r > 0$ such that

$$H'(t,y) \cdot y - 2H(t,y) \geq b|y|^{\beta}, \quad \forall t \in \mathbf{Z},\ \forall |y| \geq r; \tag{1.8}$$

(H_5) for all $t \in \mathbf{Z}$ and all $y \in \mathbf{R}^{2N}$

$$H'(t,y)y \geq 2H(t,y) \geq 0. \tag{1.9}$$

Remark 1.1. By assumption (H_1) and (H_2), we know that $H(t,y)$ satisfies the superquadratic condition at both infinity and 0 respect to the second variable y.

The rest of the paper is organized as follows. In Section 2, we shall establish the variational structure for (1.1) and turn the problem of looking for homoclinic orbits for (1.1) to the problem for seeking critical points of the corresponding functional. In order to apply the generalized mountain pass theorem, we give some preliminary results in Section 3. In Section 4, we shall state our main result and complete the proof of our result.

2. Variational Structure

Set $S = \{x = \{x(t)\} \mid x(t) = \binom{x_1(t)}{x_2(t)} \in \mathbf{R}^{2N}, x_j(t) \in \mathbf{R}^N, j = 1,2, t \in \mathbf{Z}\}$ is a space which is composed of the following vectors,

$$x = \{x(t)\}_{t \in \mathbf{Z}} = \left\{\ldots, x^T(-t), \ldots, x^T(-1), x^T(0), x^T(1), \ldots, x^T(t), \ldots\right\}. \tag{2.1}$$

Define the subspace X of S as

$$X = \left\{ x \in S \mid \sum_{t\in\mathbf{Z}} [(J\Delta Sx(t-1), x(t)) + (M(t)Sx(t), x(t))] < +\infty \right\}. \tag{2.2}$$

Denote $u = v \times v$, $u^* = v^* \times v^*$, define another subspace E of X as follows:

$$E = \left\{ x \in X \mid x(t) \in (\operatorname{Ker} u)^{\perp} \right\}. \tag{2.3}$$

The space E is a Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{t\in\mathbf{Z}} \left[\left(J\Delta Sx(t-1), y(t)\right) + \left(M(t)Sx(t), y(t)\right) \right], \quad \forall x, y \in E, \tag{2.4}$$

and the norm introduced from the inner product as follows:

$$\|x\|^2 = \langle x, x\rangle = \sum_{t\in\mathbf{Z}} [(J\Delta Sx(t-1), x(t)) + (M(t)Sx(t), x(t))], \quad \forall x \in E. \tag{2.5}$$

Define a functional $F(x)$ on E as follows:

$$F(x) = -\frac{1}{2}\sum_{t\in\mathbf{Z}} [(J\Delta Sx(t-1), x(t)) + (M(t)Sx(t), x(t))] + \sum_{t\in\mathbf{Z}} G(t, u(Sx(t-1))), \tag{2.6}$$

according to the definition of $\|x\|$, $F(x)$ can be written in another form as follows:

$$F(x) = -\frac{1}{2}\|x\|^2 + \sum_{t\in\mathbf{Z}} G(t, u(Sx(t-1))). \tag{2.7}$$

The functional $F(x)$ is a well-defined C^1 on E, and next we prove that the problem of looking for homoclinic orbits for (1.1) can be turned to the problem for seeking critical points of the corresponding functional $F(x)$ (see (2.6) or (2.7)).

Let

$$\begin{aligned} F_1(x) &= -\frac{1}{2}\sum_{t\in\mathbf{Z}} (J\Delta Sx(t-1), x(t)) \\ &= -\frac{1}{2}\sum_{t\in\mathbf{Z}} \left(\begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1(t) \\ \Delta x_2(t-1) \end{pmatrix}, \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right) \\ &= -\frac{1}{2}\sum_{t\in\mathbf{Z}} [(x_2(t-1), x_1(t)) - (x_2(t), x_1(t)) + (x_1(t+1), x_2(t)) - (x_1(t), x_2(t))], \end{aligned} \tag{2.8}$$

while

$$\begin{aligned}
&\sum_{t\in\mathbf{Z}}(x_2(t-1),x_1(t))-(x_2(t),x_1(t))\\
&\quad=\sum_{t\in\mathbf{Z}}(x_2(t-1),x_1(t))-\sum_{t\in\mathbf{Z}}(x_2(t),x_1(t))\\
&\quad=\sum_{t\in\mathbf{Z}}(x_2(t),x_1(t+1))-\left(\sum_{t\in\mathbf{Z}}(x_2(t),x_1(t))\right)\\
&\quad=\sum_{t\in\mathbf{Z}}(x_2(t),\Delta x_1(t)),
\end{aligned}\tag{2.9}$$

then

$$F_1(x)=-\sum_{t\in\mathbf{Z}}(\Delta x_1(t),x_2(t)),\tag{2.10}$$

it follows that

$$F(x)=-\sum_{t\in\mathbf{Z}}(\Delta x_1(t),x_2(t))-\frac{1}{2}\sum_{t\in\mathbf{Z}}(M(t)Sx(t),x(t))+\sum_{t\in\mathbf{Z}}G(t,u(Sx(t-1))).\tag{2.11}$$

Write $F'_{x_i(t)}=\partial F(x)/\partial x_i(t)$, $i=1,2$, for any given $t\in\mathbf{Z}$, there holds

$$\begin{aligned}
F'_{x_1(t)}&=x_2(t-1)-x_2(t)-v^*Lvx_2(t-1)+u^*G_{x_1(t)}(t-1,u(Sx(t-1))),\\
F'_{x_2(t)}&=x_1(t+1)-x_1(t)-v^*Lvx_1(t+1)+u^*G_{x_2(t)}(t,u(Sx(t))).
\end{aligned}\tag{2.12}$$

Then we can draw a conclusion that $F'(x)=0$ is true if and only if

$$\begin{aligned}
F'_{x_1(t)}&=0,\\
F'_{x_2(t)}&=0,
\end{aligned}\tag{2.13}$$

so

$$\begin{aligned}
x_2(t-1)-x_2(t)-v^*Lvx_2(t-1)+u^*G_{x_1(t)}(t-1,u(Sx(t-1)))&=0,\\
x_1(t+1)-x_1(t)-v^*Lvx_1(t+1)+u^*G_{x_2(t)}(t,u(Sx(t)))&=0,
\end{aligned}\tag{2.14}$$

which can be reformed as

$$\begin{pmatrix}0&-I_N\\I_N&0\end{pmatrix}\begin{pmatrix}\Delta x_1(t)\\\Delta x_2(t)\end{pmatrix}-\begin{pmatrix}0&v^*L(t)v\\v^*L(t)v&0\end{pmatrix}\begin{pmatrix}x_1(t+1)\\x_2(t)\end{pmatrix}+u^*\begin{pmatrix}G_{x_1(t)}(t-1,u(Sx(t)))\\G_{x_2(t)}(t,u(Sx(t)))\end{pmatrix}=0,\tag{2.15}$$

that is

$$J\Delta x(t) - M(t)Sx(t) + u^* H'(t, u(Sx(t))) = 0, \tag{2.16}$$

which is just (1.1). Therefore, we obtain the following lemma.

Lemma 2.1. *$x = \{x(t)\} \neq 0$ is a homoclinic orbit of (1.1) if and only if x is a critical point of functional $F(x)$ in E.*

3. Preliminary Results

In order to apply the critical point theory to look for critical points for (2.6), we give some lemmas which will be of fundamental importance in proving our main result.

Let E be a real Hilbert space with the norm $\|\cdot\|$. Suppose that E has an orthogonal decomposition $E = E^1 \oplus E^2$ with both E^1 and E^2 being infinite dimensional. Suppose (v_n) (resp., (ω_n)) is an orthogonal basis for E^1 (resp. E^2), and set $E_n^1 = \text{span}\{v_1, v_2, \ldots, v_n\}$, $E_n^1 = \text{span}\{\omega_1, \omega_2, \ldots, \omega_n\}$, $E_n = E_n^1 \oplus E_n^2$, and $f_n = f|_{E_n}$, the restriction of f on E_n. We say that f satisfies the $(\text{PS})^*$ condition if any sequence $(x^{(n)})$ in E, $(x^{(n)}) \in E_n$ such that $f_n(x^{(n)}) \leq C$ a constant, and $f_n'(x^{(n)}) \to 0$ possesses a convergent subsequence.

We state a basic theorem introduced in [20] by Rabinowitz which is used to obtain the critical points of the functional $F(x)$.

Lemma 3.1 (Generalized mountain pass lemma). *Let $f \in C^1(E, \mathbf{R})$ satisfy*

(f_1) the $(\text{PS})^*$condition;

(f_2) there are $\rho, \delta > 0$ such that

$$f(x) \geq \delta, \tag{3.1}$$

for all $x \in S_\rho = \{x \in E^2 \mid \|x\| = \rho\}$;

(f_3) there are $r > \rho$, $M > 0$, $e \in E_1^2$ with $\|e\| = 1$ such that

$$f|_{\partial Q} \leq 0, \qquad f|_Q \leq M, \tag{3.2}$$

where $Q = \{(B_r \bigcap E^1)\} \oplus \{se \mid 0 \leq s \leq r\}\}$.

Then f has a critical point x with $f(x) \geq \delta$.

Next we consider the eigenvalue problem.

$$J\Delta Sx(t-1) + M(t)Sx(t) = \lambda x(t). \tag{3.3}$$

Equation (3.3) can be reformed as follows:

$$\begin{aligned} x_1(t+1) &= (I + v^* L v)^{-1} x_1(t) + \lambda (I + v^* L v)^{-1} x_2(t), \\ x_2(t+1) &= \lambda (v^* L v - I)^{-1} (I + v^* L v)^{-1} x_1(t) + \left[\lambda^2 (v^* L v - I)^{-1} (I + v^* L v)^{-1} - I\right] x_2(t), \end{aligned} \tag{3.4}$$

Denote

$$A(\lambda) = \begin{pmatrix} (I + v^*Lv)^{-1} & \lambda(I + v^*Lv)^{-1} \\ \lambda(v^*Lv - I)^{-1}(I + v^*Lv)^{-1} & \lambda^2(v^*Lv - I)^{-1}(I + v^*Lv)^{-1} - I \end{pmatrix}, \tag{3.5}$$

then (3.3) can be expressed by the following:

$$x(t+1) = A(\lambda)x(t). \tag{3.6}$$

Therefore a standard argument shows that $\sigma(A)$, the spectrum of A, consists of eigenvalues numbered by, (counted in their multiplicities) the following:

$$\cdots \lambda_{-2} \le \lambda_{-1} \le 0 \le \lambda_1 \le \lambda_2 \cdots, \tag{3.7}$$

with $\lambda_k \to \pm\infty$ as $k \to \pm\infty$, and denote the corresponding system of eigenfunctions of A by (e_k).

Let $E^0 = \operatorname{Ker} A$, $E^+ = \operatorname{span}\{e_1, \ldots, e_n\}$ and $E^- = (E^0 \oplus E^+)^{\perp_E}$, where $S^{\perp_E}$ stands for the orthogonal complementary subspace of S in E. Then

$$E = E^- \oplus E^0 \oplus E^+, \tag{3.8}$$

so the functional (2.7) can be rewritten as follows

$$F(x) = -\frac{1}{2}\left(\|x^+\|^2 - \|x^-\|^2\right) + \sum_{t \in \mathbf{Z}} G(t, u(Sx(t-1))), \tag{3.9}$$

for all $x = x^- + x^0 + x^+ \in E^- + E^0 + E^+$.

Set $l^2 = \{x = \{x(t)\} \in S \mid \sum_{t\in\mathbf{Z}} |x(t)|^2 < +\infty\}$ and $l^\infty = \{x = \{x(t)\} \in S \mid |x(t)| < +\infty$, for all $t \in \mathbf{Z}\}$ and their norms are defined by the following:

$$\begin{aligned} \|x\|_{l^2}^2 &= \sum_{t\in\mathbf{Z}} |x(t)|^2 = (x, x)_{l^2}, \\ \|x\|_{l^\infty} &= \sup_{t\in\mathbf{Z}} |x(t)|, \end{aligned} \tag{3.10}$$

respectively. For any given $1 \le r < +\infty$, define $l^r = \{x = \{x(t)\} \in S \mid \sum_{t\in\mathbf{Z}} |x(t)|^r < +\infty\}$ with the norm

$$\|x\|_{l^r}^r = \left(\sum_{t\in\mathbf{Z}} |x(t)|^r\right) = (x, x)_{l^r}. \tag{3.11}$$

Define a selfadjoint operator A on E by the following:

$$(Ax, x) = \sum_{t\in\mathbf{Z}} (J\Delta Sx(t-1) + M(t)Sx(t), x(t)) = \|x\|^2,$$
$$(Ax, Jx)_{l^2} = \sum_{t\in\mathbf{Z}} (M(t)Sx(t), Jx(t)), \tag{3.12}$$

$|A|$ is the absolute value. Give another norm the domain of A by the following:

$$\|x\|_1 = \|(I + |A|)x\|_{l^2}, \tag{3.13}$$

it is easy to get, for all $x \in E$,

$$(Ax, Jx)_{l^2} \le \|x\|_1^2$$
$$(x, x)_{l^2} \le \|x\|_1^2. \tag{3.14}$$

Now we state a fundamental proposition, which will be used in the later.

Proposition 3.2. *Let L satisfy (L_1). Then for all $1 \le p \in (2/(3-\gamma), +\infty)$ there exists a constant $\lambda_p > 0$ such that*

$$\|x\|_{l^p} \le \lambda_p \|x\|, \quad \forall x \in E. \tag{3.15}$$

Proof. We complete the proof of Proposition 3.2 by 3 steps.
Step 1. When (L_1) holds and $p = 2$, we prove that

$$\|x\|_{l^2} \le \lambda_2 \|x\|, \quad \forall x \in E, \tag{3.16}$$

Note that, by (L_1), $l(t) \to +\infty$ as $|t| \to +\infty$, that is, $l(t)$ is bounded from below and so there is a $\widetilde{a} > 0$ such that

$$v^* L(t) v + \widetilde{a} I_N \ge 0, \quad t \in \mathbf{Z},$$
$$\beta(R) = \inf_{|t|\ge R} l(t) \longrightarrow \infty, \quad \text{as } |t| \longrightarrow \infty. \tag{3.17}$$

For $R > 0$, choose a subsequence $x^{(k)}(t) \in E$, one has

$$\begin{aligned}
\sum_{|t|\geq R} \left| x^{(k)} \right|^2 &\leq \sum_{|t|\geq R} \frac{Jx^{(k)}(t) \cdot (v^* L(t) v + \tilde{a} J) x^{(k)}(t)}{l(t)} \\
&\leq \frac{1}{\beta(R)} \sum_{t\in\mathbf{Z}} Jx^{(k)}(t) \cdot (v^* L(t) v + \tilde{a} J) x^{(k)}(t) \\
&= \frac{1}{\beta(R)} \sum_{t\in\mathbf{Z}} \left(Jx^{(k)}(t) \cdot v^* L(t) v \cdot x^{(k)}(t) + \tilde{a} J x^{(k)}(t) \cdot J x^{(k)}(t) \right) \\
&= \frac{1}{\beta(R)} \left(Ax^{(k)}, Jx^{(k)} \right)_{l^2} + \frac{1}{\beta(R)} \tilde{a} \left(Jx^{(k)}, Jx^{(k)} \right)_{l^2} \\
&\leq \frac{1}{\beta(R)} \left\| x^{(k)} \right\|_1^2 .
\end{aligned} \tag{3.18}$$

For any given $\epsilon > 0$, by (3.18), one can take R_0 so large that

$$\sum_{|t|\geq R_0} \left| x^{(k)} \right|^2 \leq \frac{\epsilon^2}{4}. \tag{3.19}$$

Without loss of generality, we can assume that $x^{(k)} \rightharpoonup 0$ in E. Define $E_I = \{x \in S_I \mid \sum_{t\in\mathbf{Z}} [(J\Delta S x(t-1), x(t)) + (M(t) S x(t), x(t))] < +\infty\}$, $S_I = \{x = \{x(t)\} \mid x(t) \in E, t \in I\}$ and $I = \{t \mid |t| \leq R_0\}$. So $x^{(k)}$ is bounded in E_I, which implies that $x^{(k)}$ is bounded in l_I^2. This together with the uniqueness of the weak limit in l_I^2, we have $x^{(k)} \to 0$ in E_I, so there exists a k_0 such that

$$\sum_{t\in I} \left| x^{(k)} \right|^2 \leq \frac{\epsilon^2}{4}, \quad \forall k \geq k_0. \tag{3.20}$$

Combing (3.19) and (3.20), we have $x^{(k)} \to 0$ in l^2. It follows that (3.16) is true.

Step 2. For all $p > 2$, there exists a constant $\lambda_p > 0$ such that (3.15) holds.

For any $p > 2$ and $x \in E$, by the Hölder inequality, we have

$$\begin{aligned}
\sum_{t\in\mathbf{Z}} |x|^p &= \sum_{t\in\mathbf{Z}} |x| \cdot |x|^{p-1} \\
&\leq \left(\sum_{t\in\mathbf{Z}} |x|^2 \right)^{1/2} \cdot \left(\sum_{t\in\mathbf{Z}} |x|^{2(p-1)} \right)^{(p-1)/2(p-1)} \\
&\leq C \|x\|^{p-1} \|x\|_{l^2},
\end{aligned} \tag{3.21}$$

which together with (3.16) yields (3.15).

Step 3. Since (L_1) implies $l(t) \to +\infty$ as $|t| \to \infty$, by Step 1 and 2, it remains to consider the case for $1 \leq p \in (2/(3-\gamma), 2)$.

Let

$$\widetilde{\beta}(R) = \inf_{|t|>R} l(t)|t|^{\gamma-2}. \tag{3.22}$$

By (L_1), $\widetilde{\beta}(R) \to \infty$ as $R \to \infty$.

Write $\alpha = (2-\gamma)/(2-p)$, then $\alpha p > 1$. Set for $R > 0$ and $x \in E$, $E_R^1(x) = \{t \mid |t| \geq R$ and $|t|^\alpha \cdot |x(t)| > 1\}$ and $E_R^2(x) = \{t \mid |t| \geq R$ and $|t|^\alpha \cdot |x(t)| \leq 1\}$. Then

$$\begin{aligned}
\sum_{t\in E_R^1(x)} |x|^p &= \sum_{t\in E_R^1(x)} \left(|t|^\alpha |x(t)|\right)^p \cdot |t|^{-\alpha p} \\
&\leq \sum_{t\in E_R^1(x)} \left(|t|^\alpha |x(t)|\right)^2 \cdot |t|^{-\alpha p} \\
&= \sum_{t\in E_R^1(x)} |t|^{2\alpha} |x(t)|^2 |t|^{-\alpha p} \\
&= \sum_{t\in E_R^1(x)} |t|^{2\alpha-\alpha p} |x(t)|^2 \\
&= \sum_{t\in E_R^1(x)} |t|^{2-\gamma} |x(t)|^2,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\sum_{t\in\mathbf{Z}} (M(t)Sx(t), Jx(t)) &= (Ax, Jx)_{l^2} \\
&\leq \left\| |A|^{1/2} x \right\|_{l^2} \cdot \left\| |A|^{1/2} Jx \right\|_{l^2} \\
&\leq C\|x\|^2 \\
\sum_{t\in E_R^1(x)} |x|^2 &\leq \sum_{t\in E_R^1(x)} \frac{Jx \cdot (M + \widetilde{a}J)x}{l(t)} \\
&\leq \frac{1}{\widetilde{\beta}(R)} |t|^{\gamma-2} ((M + \widetilde{a}J)x, Jx)_{l^2},
\end{aligned} \tag{3.24}$$

From (3.24), we get

$$\begin{aligned}
\sum_{t\in E_R^1(x)} |x|^p &\leq \sum_{t\in E_R^1(x)} |x|^2 |t|^{2-\gamma} \\
&\leq \frac{1}{\widetilde{\beta}(R)} ((M + \widetilde{a}J)x, Jx)_{l^2} \\
&\leq \frac{1}{\widetilde{\beta}(R)} C\|x\|^2
\end{aligned} \tag{3.25}$$

and so

$$\begin{aligned}
\sum_{|t|\geq R} |x|^p &= \sum_{t\in E_R^1(x)} |x|^p + \sum_{t\in E_R^2(x)} |x|^p \\
&= \sum_{t\in E_R^1(x)} \left(|t|^{\alpha}|x(t)|\right)^p |t|^{-\alpha p} + \sum_{t\in E_R^2(x)} \left(|t|^{\alpha}|x(t)|\right)^p |t|^{-\alpha p} \\
&\leq \frac{1}{\widetilde{\beta}(R)} C\|x\|^2 + \sum_{t\in E_R^2(x)} \left(|t|^{\alpha}|x(t)|\right)^p |t|^{-\alpha p} \\
&\leq \frac{1}{\widetilde{\beta}(R)} C\|x\|^2 + \sum_{t\in E_R^2(x)} |t|^{-\alpha p}.
\end{aligned} \tag{3.26}$$

Since $\alpha p > 1$ and $\sum_{t\in E_R^2(x)} |t|^{-\alpha p} = \sum_{|t|\geq R} |t|^{-\alpha p}$ then there exists a constant s such that

$$\sum_{|t|\geq R} |t|^{-\alpha p} = s, \tag{3.27}$$

which together with (3.26) yields

$$\sum_{|t|\geq R} |x|^p \leq \frac{1}{\widetilde{\beta}(R)} C\|x\|^2 + s. \tag{3.28}$$

Give $\epsilon > 0$, by (3.28), choose $R_0 > 0$ so large that

$$\sum_{|t|\geq R_0} |x|^p < \frac{\epsilon^p}{2}. \tag{3.29}$$

Denote $I = \{t \in \mathbf{Z} \mid |t| \leq R_0\}$, $E_I = \{x \mid x(t) \in E, t \in I\}$. Any given subsequence $(x^{(k)}) \in E$, we can suppose $x^{(k)} \rightharpoonup 0$ on E, now $x^{(k)}$ is bounded in l_I^p. This together with the uniqueness of the weak limit in l_I^p on I. For any $x \in E_I$, we have

$$\|x\|_{l_I^p}^p < \frac{\epsilon^p}{2}. \tag{3.30}$$

Combining (3.29) and (3.30), it follows

$$\sum_{t\in\mathbf{Z}} |x|^p < \epsilon, \tag{3.31}$$

that is, $\|x\|_{l^p} < \epsilon$, there has a constant $\lambda_p > 0$ such that

$$\|x\|_{l^p} \leq \lambda_p \|x\|, \tag{3.32}$$

while $1 \leq p \in (2/(3-\gamma), 2)$. □

4. Main Results and Proofs

In the previous section, we turned the homoclinic orbits problem of (1.1) to the corresponding critical point problem of the functional (2.6) or (2.7). Next, we state our main results and complete their proofs by Lemma 3.1.

Our main result is as follows.

Theorem 4.1. *Suppose that H satisfies (L_1) and (H_1)–(H_5). Then the discrete Hamiltonian system* (1.1) *has a nontrivial homoclinic orbit.*

Remark 4.2. Observe that if $x(t)$ is a homoclinic solution of (1.1), then $y(t) = x(-t)$ is a homoclinic solution of the following:

$$J\Delta y(t) + M(-t)Sy(t) - v^* \times v^* H'(-t, v \times v(Sx(t))) = 0. \tag{4.1}$$

Moreover, $-H(-t, y)$ satisfies (H_1)–(H_5) whenever $H(t, y)$ satisfies (H_1'), (H_2), (H_3), (H_4') and (H_5'), where

(H_1') $H(t, y)/|y|^2 \to -\infty$ as $|y| \to +\infty$, $t \in \mathbf{Z}$;

(H_4') there exist $\beta > \alpha$, $b > 0$ and $r > 0$ such that

$$H'(t, y) \cdot y - 2H(t, y) \le -b|y|^{\beta}, \quad \forall t \in \mathbf{Z},\ \forall |y| \ge r; \tag{4.2}$$

(H_5') for all $t \in \mathbf{Z}$ and all $y \in \mathbf{R}^{2N}$

$$H'(t, y)y \le 2H(t, y) \le 0. \tag{4.3}$$

So in the following, we will give another theorem and can omit its proof.

Theorem 4.3. *The conclusion of Theorem 4.1 holds when replacing (H_1), (H_4), and (H_5) with (H_1'), (H_4'), and (H_5').*

With the aid of previous sections, we will verify that $F(x)$ satisfies the assumptions of Lemma 3.1. We will proceed by successive lemmas.

Lemma 4.4. *If H satisfies assumptions of Theorem 4.1, then there are constants $\rho > 0$, $\delta > 0$ such that*

$$F|_B \ge \delta, \tag{4.4}$$

where $B = \{x \in E^2 \mid \|x\| = \rho\}$.

Proof. For any $x \in E$, it is easy to see that there exist two constants $0 < m_0 < M_0$ such that

$$m_0|x| \le |u(x)| \le M_0|x|. \tag{4.5}$$

By (H_1), (H_3), and (4.5), for all $\epsilon > 0$, there exist a constant $C_\epsilon > 0$ such that

$$\left|H'(t,u(x))\right| \leq \frac{1}{M_0}\left[2\epsilon|x| + (\alpha+1)C_\epsilon|x|^\alpha\right], \quad \forall x \in E. \tag{4.6}$$

Now by mean value Theorem, (4.5) and (4.6), for all $x \in E$ and $t \in \mathbf{Z}$, we have

$$\begin{aligned} |H(t,u(x))| &= \int_0^1 H'(t,su(x)) \cdot u(x)ds \\ &\leq \epsilon|x|^2 + C_\epsilon|x|^{\alpha+1}, \end{aligned} \tag{4.7}$$

on the other hand,

$$\|Sx\|_{l^2}^2 = \sum_{t\in\mathbf{Z}} |Sx(t)|^2 = \sum_{t\in\mathbf{Z}} |x_1(t+1)|^2 + |x_2(t)|^2 = \|x\|_{l^2}^2, \tag{4.8}$$

similarly,

$$\|Sx\|_{l^{\alpha+1}}^{\alpha+1} = \|x\|_{l^{\alpha+1}}^{\alpha+1}, \tag{4.9}$$

it follows that

$$\begin{aligned} \sum_{t\in\mathbf{Z}}|H(t,u(Sx(t)))| &\leq \sum_{t\in\mathbf{Z}}\left[\epsilon|Sx|^2 + C_\epsilon|Sx|^{\alpha+1}\right] \\ &= \epsilon\sum_{t\in\mathbf{Z}}|Sx|^2 + C_\epsilon\sum_{t\in\mathbf{Z}}|Sx|^{\alpha+1} \\ &= \epsilon\|x\|_{l^2}^2 + C_\epsilon\|x\|_{l^{\alpha+1}}^{\alpha+1}, \end{aligned} \tag{4.10}$$

By Proposition 3.2 and (4.10), for any $x \in E$, it holds

$$\begin{aligned} F(x) &= \frac{1}{2}\|x\|^2 - \sum_{t\in\mathbf{Z}} H(t,u(Sx(t))) \\ &\geq \frac{1}{2}\|x\|^2 - \epsilon\|x\|_{l^2}^2 - C_\epsilon\|x\|_{l^{\alpha+1}}^{\alpha+1} \\ &\geq \frac{1}{2}\|x\|^2 - \epsilon\lambda_2^2\|x\|^2 - C_\epsilon\lambda_{\alpha+1}^{\alpha+1}\|x\|^{\alpha+1}, \end{aligned} \tag{4.11}$$

Choosing ϵ such that $\epsilon\lambda_2^2 = 1/4$, we obtain, for any $x \in E^2$,

$$F(x) \geq \frac{1}{4}\|x\|^2 - C_\epsilon\lambda_{\alpha+1}^{\alpha+1}\|x\|^{\alpha+1}. \tag{4.12}$$

Since $\alpha > 1$, then there are constants $\rho > 0$, $\delta > 0$ such that

$$F|_B \geq \delta, \tag{4.13}$$

which completes the proof of Lemma 4.4. □

Lemma 4.5. *Under assumptions of Theorem 4.1, let $e \in E_1^2$ with $\|e\| = 1$, there exist $r_1, r_2 > 0$ such that*

$$F(x) \leq 0, \quad x \in \partial Q, \tag{4.14}$$

where $Q = \{se \mid 0 \leq s \leq r_1\} \oplus \{x \in E^1 \mid \|x\| \leq r_2\}$.

Proof. Let $e \in E_1^2$ with $\|e\| = 1$ and $F = E^1 \oplus \operatorname{span}\{e\}$. For $x = x^- + x^0 + x^+ \in F - \{0\}$ and $\epsilon > 0$, denote

$$\Omega_x = \{t \in \mathbf{Z} \mid |x(t)| \geq \epsilon\|x\|\}, \tag{4.15}$$

then there exists $\epsilon_1 > 0$ such that

$$\sharp\Omega_x = \sharp\{t \in \mathbf{Z} \mid |x(t)| \geq \epsilon_1\|x\|\} \geq \left[\frac{1}{\epsilon_1}\right] + 1, \tag{4.16}$$

where $\sharp\Omega_x$ is the number of t in Ω_x and $[\cdot]$ is the greatest integer function.

By (H_1), for $d = 1/2\epsilon_1^2 m_0^2$, there exists $R_1 > 0$ such that

$$H(t,y) \geq d|y|^2, \quad \forall|y| \geq R_1,\ t \in \mathbf{Z}, \tag{4.17}$$

where m_0 was defined by (4.5). Then it follows

$$\begin{aligned} H(t, u(Sx(t))) &\geq d|u(Sx(t))|^2 \geq \frac{1}{2\epsilon_1^3 m_0^2}|u(Sx(t))|^2 \\ &\geq \frac{1}{2\epsilon_1^3}|Sx(t)|^2 \geq \frac{1}{2\epsilon_1}\|Sx(t)\|^2 \\ &= \frac{1}{2\epsilon_1}\|x\|^2, \end{aligned} \tag{4.18}$$

for all $x \in F - \{0\}$ with $\|x\| \geq R_1/m_0\epsilon_1$ and $t \in \Omega_x$. Hence, from (4.16) and (4.18), one has

$$\begin{aligned} F(x) &= \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2 - \sum_{t\in\mathbf{Z}} H(t, u(Sx(t))) \\ &\leq \frac{1}{2}\|x^+\|^2 - \sum_{t\in\Omega_x} H(t, u(Sx(t))) \\ &\leq \frac{1}{2}\|x^+\|^2 - \frac{1}{2\epsilon_1}\|Sx(t)\|^2\left(\left[\frac{1}{\epsilon_1}\right] + 1\right) \\ &\leq \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x\|^2 \\ &\leq 0, \end{aligned} \tag{4.19}$$

is true for all $x \in F - \{0\}$ with $\|x\| \geq R_1/m_0\epsilon_1$. Let $r_1 > 0$ and denote

$$Q = \{se \mid 0 \leq s \leq r_1\} \oplus \left\{x \in E^1 \|x\| \leq r_2\right\}. \tag{4.20}$$

Then by (4.19), for all $r_1 > \max\{\rho, R_1/m_0\epsilon_1\}$, we have

$$F(x) \leq 0, \quad \forall x \in \partial Q, \tag{4.21}$$

where ρ is defined by Lemma 4.5, this is just (4.14). We completed the proof of Lemma 4.5. □

In order to verify that $F(x)$ satisfies (f_1) of Lemma 3.1, we need the following lemma.

Lemma 4.6. *Write*

$$h(x) = \sum_{t\in\mathbf{Z}} H(t, u(Sx(t))), \quad \forall x \in E. \tag{4.22}$$

Then $h(x) \in C^1(E, \mathbf{R})$.

Proof. Let $\varphi(z) = h(t, x + zy)$, $0 \leq z \leq 1$, for all $x, y \in E$, since $H \in C^1(E, \mathbf{R})$

$$\begin{aligned} \varphi'(0) &= \lim_{z\to 0}\frac{\varphi(z) - \varphi(0)}{z} = \lim_{z\to 0}\frac{h(t, x + zy) - h(t, x)}{z} \\ &= \lim_{z\to 0}\frac{1}{z}\sum_{t\in\mathbf{Z}}\left[H(t, u(S(x + zy)(t))) - H(t, u(Sx(t)))\right] \\ &= \lim_{z\to 0}\sum_{t\in\mathbf{Z}}\left[\nabla H(t, u(S(x_t + \theta_t z y_t))) \cdot u(Sy_t)\right] \\ &= \sum_{t\in\mathbf{Z}}\left[\nabla H(t, u(S(x_t + \theta_t z y_t))) \cdot u(Sy_t)\right], \end{aligned} \tag{4.23}$$

where $0 < \theta_t < 1$, then $h(x)$ is Gâteaux differential on E and

$$h'(x)y = \sum_{t\in\mathbf{Z}} \nabla H(t, u(Sx(t))) \cdot u(Sy(t)), \quad \forall x, y \in E. \tag{4.24}$$

Let $x^{(k)} \rightharpoonup x$ weakly in E, by Proposition 3.2, one can assume that $x^{(k)} \to x$ strongly in l^p for $p \in [1, +\infty)$. By (4.24), we have

$$\left\| h'\left(x^{(k)}\right) - h'(x) \right\| = \sup_{\|y\|=1} \left| \sum_{t\in\mathbf{Z}} \left(\nabla H\left(t, u\left(Sx^{(k)}(t)\right)\right) - \nabla H(t, u(Sx(t))) \cdot u(Sy(t)) \right) \right|. \tag{4.25}$$

By (4.5) and (4.6), there exists a constant $C_1 > 0$ such that for any $R > 0$, there holds

$$\begin{aligned}
&\left| \sum_{|t|\geq R} \left(\nabla H\left(t, u\left(Sx^{(k)}(t)\right)\right) - \nabla H(t, u(Sx(t))) \cdot u(Sy(t)) \right) \right| \\
&\quad \leq \sum_{|t|\geq R} \left[\left| \nabla H\left(t, u\left(Sx^{(k)}(t)\right)\right) \right| + |\nabla H(t, u(Sx(t)))| \right] \cdot |u(Sy(t))| \\
&\quad \leq \sum_{|t|\geq R} C_1 \left[\left| x^{(k)} \right| + |x| + \left| x^{(k)} \right|^{\alpha} + |x|^{\alpha} \right] |y| \\
&\quad \leq C_1 \left(\sum_{|t|\geq R} \left(\left| x^{(k)} \right| + |x| \right)^2 \right)^{1/2} \left(\sum_{|t|\geq R} |y|^2 \right)^{1/2} \\
&\qquad + C_1 \left(\sum_{|t|\geq R} \left(\left| x^{(k)} \right| + |x| \right)^{\alpha+1} \right)^{\alpha/(\alpha+1)} \left(\sum_{|t|\geq R} |y|^{\alpha+1} \right)^{1/(\alpha+1)} \\
&\quad \leq C_1 \|y\|_{l^2} \left(\sum_{|t|\geq R} \left(\left| x^{(k)} \right| + |x| \right)^2 \right)^{1/2} + C_1 \|y\|_{l^{\alpha+1}} \left(\sum_{|t|\geq R} \left(\left| x^{(k)} \right| + |x| \right)^{\alpha+1} \right)^{\alpha/(\alpha+1)},
\end{aligned} \tag{4.26}$$

so by Proposition 3.2, there exists a constant $C_2 > 0$ such that for any $\|y\| = 1$,

$$\begin{aligned}
&\left| \sum_{|t|\geq R} \left(\nabla H\left(t, u\left(Sx^{(k)}(t)\right)\right) - \nabla H(t, u(Sx(t))) \cdot u(Sy(t)) \right) \right| \\
&\quad \leq C_2 \left[\left(\sum_{|t|\geq R} \left| x^{(k)} \right|^2 + |x|^2 \right)^{1/2} + \left(\sum_{|t|\geq R} \left| x^{(k)} \right|^{\alpha+1} + |x|^{\alpha+1} \right)^{\alpha/(\alpha+1)} \right].
\end{aligned} \tag{4.27}$$

We deduce from (4.27) that for any $\epsilon > 0$, there has $R > 0$ so large that

$$\left| \sum_{|t|\geq R} \left(\nabla H\left(t, u\left(Sx^{(k)}(t)\right)\right) - \nabla H(t, u(Sx(t))) \cdot u(Sy(t)) \right) \right| < \frac{\epsilon}{2}, \tag{4.28}$$

for all $k \in \mathbf{N}$ and $y \in E$ with $\|y\| = 1$. On the hand, it is well known that since $x^{(k)} \to x$ strongly in l^2, then when $x \in E_{I_R}$

$$\left\| \nabla H\left(t, u\left(Sx^{(k)}\right)\right) - \nabla H(t, uS(x)) \right\|_{l^2} \longrightarrow 0, \tag{4.29}$$

as $k \to \infty$, where $E_{I_R} = \{x \mid x(t) \in E, t \in I_R\}$ and $I_R = \{t \mid t \in \mathbf{Z}, |t| < R\}$. Thus there is $k_0 \in \mathbf{N}$ such that

$$\left| \sum_{|t| \le R} \left(\nabla H\left(t, u\left(Sx^{(k)}(t)\right)\right) - \nabla H(t, u(Sx(t))) \cdot u\left(Sy(t)\right)\right) \right| < \frac{\epsilon}{2}, \tag{4.30}$$

is true for all integer $k \ge k_0$ and all $y \in E$ with $\|y\| = 1$. Combining (4.28) and (4.30), it is easy to see

$$\left\| \nabla H\left(t, u\left(Sx^{(k)}\right)\right) - \nabla H(t, u(Sx)) \right\| < \epsilon, \quad k \ge k_0. \tag{4.31}$$

It follows from the arbitrariness of ϵ that $h(x) \in C^1(E, \mathbf{R})$.

Finally, let us complete the proof of Theorem 4.1 by verifying $F(x)$ satisfies the Palais-Smale condition. □

Lemma 4.7. *With assumptions of Theorem 4.1, $F(x)$ satisfies the* $(\mathrm{PS})^*$ *condition.*

Proof. Let $(x^{(k)})$ be a $(\mathrm{PS})^*$ sequence, that is, $x^{(k)} \in E_n$, for all $k \in \mathbf{N}$, and $F(x^{(k)}) \le C$, $F'(x^{(k)}) \to 0$, as $k \to \infty$. We claim that $(x^{(k)})$ is bounded. If not, passing to a subsequence if necessary, we may assume that $\|x^{(k)}\| \to \infty$ as $k \to \infty$.

Denote $I_1 = \{t \in \mathbf{Z} \mid |x^{(k)}(t)| \ge r/m_0\}$ and $I_1 = \{t \in \mathbf{Z} \mid |x^{(k)}(t)| < r/m_0\}$ for all $k \in \mathbf{N}$. By (H_4), (H_5), (4.5), and Lemma 4.6, we have

$$\begin{aligned} 2F\left(x^{(k)}\right) - F'\left(x^{(k)}\right) \cdot x^{(k)} &= \sum_{t \in \mathbf{Z}} \left[H'\left(t, u\left(Sx^{(k)}(t)\right)\right) \cdot u\left(Sx^{(k)}(t)\right) - 2H\left(t, u\left(Sx^{(k)}(t)\right)\right)\right] \\ &\ge \sum_{\{t \in \mathbf{Z} \mid u(Sx^{(k)}(t)) \mid \ge r\}} b\left|u\left(Sx^{(k)}x(t)\right)\right|^{\beta} \ge \sum_{t \in I_1} m_0^{\beta} \left|x^{(k)}\right|^{\beta}, \end{aligned} \tag{4.32}$$

which implies that

$$\frac{1}{\|x^{(k)}\|} \sum_{t \in I_1} \left|x^{(k)}\right|^{\beta} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{4.33}$$

Making use of (4.5) and (4.6), we obtain that there is a constant C_3 such that

$$\left| H'\left(t, u\left(Sx^{(k)}(t)\right)\right) \right| \le C_3 \left(\left|x^{(k)}\right| + \left|x^{(k)}\right|^{\alpha} \right), \quad \forall x \in E. \tag{4.34}$$

Hence from (4.34), there holds

$$\begin{aligned}
F_n'\left(x^{(k)}\right)\cdot x_{(k)}^{+} &= \frac{1}{2}\left\|x_{(k)}^{+}\right\|^2 - \sum_{t\in\mathbf{Z}} H\left(t,u\left(Sx^{(k)}(t)\right)\right)\cdot u\left(Sx_{(k)}^{+}(t)\right)\\
&\geq \frac{1}{2}\left\|x_{(k)}^{+}\right\|^2 - \sum_{t\in\mathbf{Z}}\left|H\left(t,u\left(Sx^{(k)}(t)\right)\right)\right|\cdot\left|u\left(Sx_{(k)}^{+}(t)\right)\right|\\
&\geq \frac{1}{2}\left\|x_{(k)}^{+}\right\|^2 - \sum_{t\in\mathbf{Z}} C_3\left(\left|x^{(k)}\right| + \left|x^{(k)}\right|^{\alpha}\right)\left|u\left(Sx_{(k)}^{+}(t)\right)\right|\\
&\geq \frac{1}{2}\left\|x_{(k)}^{+}\right\|^2 - \sum_{t\in\mathbf{Z}} C_3M_0\left|x^{(k)}\right|\left|x_{(k)}^{+}\right| - \sum_{t\in\mathbf{Z}} C_3M_0\left|x^{(k)}\right|^{\alpha}\left|x_{(k)}^{+}\right|.
\end{aligned} \tag{4.35}$$

By Hölder inequality and Proposition 3.2, we achieve

$$\begin{aligned}
\sum_{t\in\mathbf{Z}}\left|x_{(k)}^{+}\right|^{\alpha}\left|x_{(k)}^{+}\right| &= \sum_{t\in I_1}\left|x^{(k)}\right|^{\alpha}\left|x_{(k)}^{+}\right| + \sum_{t\in I_2}\left|x^{(k)}\right|^{\alpha}\left|x_{(k)}^{+}\right|\\
&\leq \left(\frac{r}{m_0}\right)^{\alpha}\sum_{t\in I_1}\left|x_{(k)}^{+}\right| + \left(\sum_{t\in I_2}\left(\left|x^{(k)}\right|^{\alpha}\right)^{\beta/\alpha}\right)^{\alpha/\beta}\cdot\left(\sum_{t\in I_2}\left|x_{(k)}^{+}\right|^{\beta/(\beta-\alpha)}\right)^{(\beta-\alpha)/\beta}\\
&\leq \left(\frac{r}{m_0}\right)^{\alpha}\sum_{t\in\mathbf{Z}}\left|x_{(k)}^{+}\right| + \left(\sum_{t\in I_2}\left|x^{(k)}\right|^{\beta}\right)^{\alpha/\beta}\cdot\left(\sum_{t\in\mathbf{Z}}\left|x_{(k)}^{+}\right|^{\beta/(\beta-\alpha)}\right)^{(\beta-\alpha)/\beta}\\
&= \left(\frac{r}{m_0}\right)^{\alpha}\left\|x_{(k)}^{+}\right\|_{l^1} + \left(\sum_{t\in I_2}\left|x^{(k)}\right|^{\beta}\right)^{\alpha/\beta}\left\|x_{(k)}^{+}\right\|_{l^{\beta/(\beta-\alpha)}}\\
&\leq \left(\frac{r}{m_0}\right)^{\alpha}\lambda_1\left\|x_{(k)}^{+}\right\| + \left(\sum_{t\in I_2}\left|x^{(k)}\right|^{\beta}\right)^{\alpha/\beta}\lambda_{\beta/(\beta-\alpha)}\left\|x_{(k)}^{+}\right\|\\
&= \left[\left(\frac{r}{m_0}\right)^{\alpha}\lambda_1 + \lambda_{\beta/(\beta-\alpha)}\left(\sum_{t\in I_2}\left|x^{(k)}\right|^{\beta}\right)^{\alpha/\beta}\right]\left\|x_{(k)}^{+}\right\|.
\end{aligned} \tag{4.36}$$

Similarly,

$$\left|x^{(k)}\right|\left|x_{(k)}^{+}\right| \leq \left[\frac{r}{m_0}\lambda_1 + \lambda_{\beta/(\beta-1)}\left(\sum_{t\in I_2}\left|x^{(k)}\right|^{\beta}\right)^{1/\beta}\right]\left\|x_{(k)}^{+}\right\|. \tag{4.37}$$

Combing (4.35)–(4.37), yields

$$F_n'\left(x^{(k)}\right)\cdot x_{(k)}^{+} \geq \frac{1}{2}\left\|x_{(k)}^{+}\right\|^2 - \lambda_1 C_3 M_0\left[\left(\frac{r}{m_0}\right)^{\alpha} + \frac{r}{m_0}\right]\left\|x_{(k)}^{+}\right\|$$
$$- C_3 M_0 \lambda_{\beta/(\beta-\alpha)}\left(\sum_{t\in I_2}\left|x^{(k)}\right|^{\beta}\right)^{\alpha/\beta}\left\|x_{(k)}^{+}\right\| \tag{4.38}$$
$$- C_3 M_0 \lambda_{\beta/(\beta-1)}\left(\sum_{t\in I_2}\left|x^{(k)}\right|^{\beta}\right)^{1/\beta}\left\|x_{(k)}^{+}\right\|,$$

Since $1 < \alpha < \beta$, we deduce from (4.33) and (4.38) that

$$\frac{\left\|x_{(k)}^{+}\right\|}{\left\|x^{(k)}\right\|} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \tag{4.39}$$

similarly,

$$\frac{\left\|x_{(k)}^{-}\right\|}{\left\|x^{(k)}\right\|} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{4.40}$$

Now, let

$$y^{(k)}(t) = \begin{cases} x^{(k)}(t), & \text{if } \left|x^{(k)}(t)\right| \leq \dfrac{r}{m_0} \\ 0, & \text{if } \left|x^{(k)}(t)\right| > \dfrac{r}{m_0}, \end{cases} \tag{4.41}$$

$$z^{(k)}(t) = x^{(k)}(t) - y^{(k)}(t), \quad \forall k \in \mathbf{N},\ t \in \mathbf{Z}. \tag{4.42}$$

Then

$$\sum_{t\in\mathbf{Z}}\left|z^{(k)}\right|^{\beta} = \sum_{t\in\mathbf{Z}}\left|x^{(k)} - y^{(k)}\right|^{\beta} = \sum_{t\in I_1}\left|x^{(k)} - y^{(k)}\right|^{\beta} + \sum_{t\in I_2}\left|x^{(k)} - y^{(k)}\right|^{\beta}$$
$$= \sum_{t\in I_1}\left|x^{(k)}\right|^{\beta} \leq \sum_{t\in\mathbf{Z}}\left|x^{(k)}\right|^{\beta}, \tag{4.43}$$

that is

$$\left\|z^{(k)}\right\|_{l^{\beta}}^{\beta} \leq \left\|x^{(k)}\right\|_{l^{\beta}}^{\beta}. \tag{4.44}$$

For $\beta > 1$, by (4.32), (4.41) and Proposition 3.2, there exists a constant $C_4 > 0$ such that

$$C_4\left(1 + \left\|x^{(k)}\right\|\right) \geq \left\|z^{(k)}\right\|_{l^\beta}^{\beta}, \quad \forall k \in \mathbf{N}. \tag{4.45}$$

Since E^0 is of finite dimension, using Hölder inequality and (4.45), for any $x_{(k)}^0 \in E^0$, we have

$$\begin{aligned}
\left\|x_{(k)}^0\right\|_{l^2}^2 &= \left(x_{(k)}^0, x^{(k)}\right)_{l^2} = \left(x_{(k)}^0, z^{(k)} + y^{(k)}\right)_{l^2} \\
&= \left(x_{(k)}^0, z^{(k)}\right)_{l^2} + \left(x_{(k)}^0, y^{(k)}\right)_{l^2} \\
&\leq \sum_{t\in\mathbf{Z}} \left|x_{(k)}^0\right| \left|z^{(k)}\right| + \sum_{t\in\mathbf{Z}} \left|x_{(k)}^0\right| \left|y^{(k)}\right| \\
&\leq \sum_{t\in\mathbf{Z}} \left|x_{(k)}^0\right| \frac{r}{m_0} + \left(\sum_{t\in\mathbf{Z}} \left|x_{(k)}^0\right|^{\beta/(\beta-1)}\right)^{(\beta-1)/\beta} \left(\sum_{t\in\mathbf{Z}} \left|z^{(k)}\right|^{\beta}\right)^{1/\beta} \\
&= \frac{r}{m_0} \left\|x^{0(k)}\right\|_{l^1} + \left\|x_{(k)}^0\right\|_{l^{\beta/(\beta-1)}} \left\|z^{(k)}\right\|_{l^\beta} \\
&\leq C_5 \left\|x_{(k)}^0\right\|_{l^2} \left(1 + \left\|z^{(k)}\right\|_{l^\beta}\right),
\end{aligned} \tag{4.46}$$

where $C_5 > 0$ is a constant.

Hence by (4.45) and (4.46), there exist positive constants C_6, C_7 such that

$$\left\|x_{(k)}^0\right\| \leq C_6 C_5 \left(1 + \left\|z^{(k)}\right\|_{l^\beta}\right) \leq C_7 \left(1 + \left\|z^{(k)}\right\|^{1/\beta}\right), \tag{4.47}$$

which implies

$$\frac{\left\|x_{(k)}^0\right\|}{\left\|x^{(k)}\right\|} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \tag{4.48}$$

when $\beta > 1$.

By (4.39), (4.40), and (4.48), it follows

$$1 = \frac{\left\|x^{(k)}\right\|}{\left\|x^{(k)}\right\|} = \frac{\left\|x_{(k)}^+\right\| + \left\|x_{(k)}^-\right\| + \left\|x_{(k)}^0\right\|}{\left\|x^{(k)}\right\|} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \tag{4.49}$$

which is a contradiction. Therefore, $(x^{(k)})$ must be bounded. That is, $F(x)$ satisfies the $(\mathrm{PS})^*$ condition. □

By Lemma 3.1, $F(x)$ possesses a critical point $x \in E$ such that $F(x) \geq \delta > 0$ and (1.1) has a nontrivial homoclinic orbit.

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