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Research Article

The Point Zoro Symmetric Single-Step Procedure for Simultaneous Estimation of Polynomial Zeros

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The point symmetric single step procedure PSS1 has R-order of convergence at least 3. This procedure is modified by adding another single-step, which is the third step in PSS1. This modified procedure is called the point zoro symmetric single-step PZSS1. It is proven that the R-order of convergence of PZSS1 is at least 4 which is higher than the R-order of convergence of PT1, PS1, and PSS1. Hence, computational time is reduced since this procedure is more efficient for bounding simple zeros simultaneously.

1. Introduction

The iterative procedures for estimating simultaneously the zeros of a polynomial of degree n were discussed, for example, in Ehrlich [1], Aberth [2], Alefeld and Herzberger [3], Farmer and Loizou [4], Milovanović and Petković [5] and Petković and Stefanović [6]. In this paper, we refer to the methods established by Kerner [7], Alefeld and Herzberger [3], Monsi, and Wolfe [8], Monsi [9] and Rusli et al. [10] to increase the rate of convergence of the point zoro symmetric single-step method PZSS1. The convergence analysis of this procedure is given in Section 3. This procedure needs some preconditions for initial points $x_i^{(0)}$ $(i = 1, \dots, n)$ to converge to the zeros x_i^* $(i = 1, \dots, n)$, respectively, as shown subsequently in the sequel. We also give attractive features of PZSS1 in Section 3.

2. Methods of Estimating polynomial zeros

Let $p : C \to C$ be a polynomial of degree n defined by

$$p(x) = \sum_{i=0}^{n} a_i x^i, \tag{2.1}$$

$a_i \in C$ $(i = 1, \ldots, n)$ and $a_n \neq 0$. Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ be the distinct zeros of $p(x) = 0$, expressed in the form:

$$p(x) = \prod_{i=1}^{n} (x - x_i^*) = 0, \tag{2.2}$$

with $a_n = 1$. Suppose that, for $j = 1, \ldots, n$, x_j is an estimate of x_j^*, and let $q : C \to C$ be defined by

$$q(x) = \prod_{j=1}^{n} (x - x_j). \tag{2.3}$$

Then,

$$q'(x_i) = \prod_{j \neq i}^{n} (x_i - x_j), \quad (i = 1, \ldots, n). \tag{2.4}$$

By (2.2), if for $i = 1, \ldots, n$, $x_i \neq x_j$ $(j = 1, \ldots, n; j \neq i)$, then

$$x_i^* = x_i - \frac{p(x_i)}{\prod_{j \neq i} \left(x_i - x_j^* \right)}. \tag{2.5}$$

Now, $x_j \approx x_j^*$ $(j = 1, \ldots, n)$ so by (2.5),

$$x_i^* \approx x_i - \frac{p(x_i)}{\prod_{j \neq i} (x_i - x_j)} \quad (i = 1, \ldots, n). \tag{2.6}$$

An iteration procedure PT1 of (2.6) is defined by

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p\left(x_i^{(k)} \right)}{\prod_{j \neq i} \left(x_i^{(k)} - x_j^{(k)} \right)} \quad (i = 1, \ldots, n) \ (k \geq 0), \tag{2.7}$$

which has been studied by Kerner [7]. Furthermore, the following procedure PS1

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k+1)}\right)\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k)}\right)} \tag{2.8}$$

has been studied by Alefeled and Herzberger [3].

The symmetric single-step idea of Aitken [11] and the procedure PS1 of Alefeld and Herzberger [3] are used to derive the point symmetric single-step procedure PSS1(Monsi [9]). The procedure PSS1 is defined by

$$\begin{gathered}
x_i^{(k,0)} = x_i^{(k)} \quad (i = 1, \ldots, n), \\
x_i^{(k,1)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k,1)}\right)\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k,0)}\right)} \quad (i = 1, \ldots, n), \\
x_i^{(k,2)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k,1)}\right)\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k,2)}\right)} \quad (i = 1, \ldots, n), \\
x_i^{(k+1)} = x_i^{(k,2)} \quad (i = 1, \ldots, n).
\end{gathered} \tag{2.9}$$

The following definitions and theorem (Alefeld and Herzberger [12], Ortega and Rheinboldt [13]) are very useful for evaluation of R-order of convergence of an iterative procedure I.

Definition 2.1. If there exists a $p \geq 1$ such that for any null sequence $\{w^{(k)}\}$ generated from $\{x^{(k)}\}$, then the R-factor of the sequence $\{w^{(k)}\}$ is defined to be

$$R_p\left(w^{(k)}\right) = \begin{cases} \lim\limits_{k \to \infty} \sup \left\| w^{(k)} \right\|^{1/k}, & p = 1, \\ \lim\limits_{k \to \infty} \sup \left\| w^{(k)} \right\|^{1/p^k}, & p > 1, \end{cases} \tag{2.10}$$

where R_p is independent of the norm $\| \cdot \|$.

Definition 2.2. We next define the R-order of the procedure I in terms of the R-factor as

$$O_R(I, x^*) = \begin{cases} +\infty \text{ if } R_p(I, x^*) = 0, & \text{for } p \geq 1, \\ \inf\{p \mid p \in [1, \infty), R_p(I, x^*) = 1\}, & \text{otherwise.} \end{cases} \tag{2.11}$$

Suppose that $R_p(w^{(k)}) < 1$, then it follows from Ortega and Rheinboldt [13] that the R-order of I satisfies the inequality $O_R(I, x^*) \geq p$.

Theorem 2.3. *Let I be an iterative procedure and let $\Omega(I, x^*)$ be the set of all sequences $\{x^{(k)}\}$ generated by I which converges to the limit x^*. Suppose that there exists a $p \geq 1$ and a constant γ such that for any $\{x^{(k)}\} \in \Omega(I, x^*)$,*

$$\left\| x^{(k+1)} - x^* \right\| \leq \gamma \left\| x^{(k)} - x^* \right\|^p, \quad k \geq k_0 = k_0\left(\left\{x^{(k)}\right\}\right). \tag{2.12}$$

Then, it follows that R-order of I satisfies the inequality $O_R(I, x^*) \geq p$.

We will use this result in order to calculate the R-order of convergence of PZSS1 in the subsequent section.

For comparison, the procedure (2.7) has R-order of convergence at least 2 or $O_R(\text{PT1}, x^*) \geq 2$, while the R-order of convergence of (2.8) is greater than 2 or $O_R(\text{PS1}, x^*) > 2$. However, the R-order of convergence of PSS1 is at least 3 or $O_R(\text{PSS1}, x^*) \geq 3$.

3. The Point Zoro Symmetric Single-Step Procedure PZSS1

The value of $x_i^{(k,2)}$ which is computed from (3.1c) requires $(n-i)$ multiplications, one division, and $(n-i+1)$ subtractions, increasing the lower bound on the R-order by unity compared with the R-order of PS1. Furthermore, the value of $x_i^{(k,3)}$ which is computed from (3.1d) requires $(n-i)$ multiplications, one division, and $(n-i+1)$ subtractions, increasing the lower bound on the R-order by unity compared with the R-order of PSS1. This observation gives rise to the idea that it might be advantageous to add another step in PSS1. This leads to what is so called the point zoro symmetric single-step procedure PZSS1 which consists of generating the sequences $\{x_i^{(k)}\}$ $(i = 1, \ldots, n)$ from

$$x_i^{(k,0)} = x_i^{(k)} \quad (i = 1, \ldots, n), \tag{3.1a}$$

$$x_i^{(k,1)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k,1)}\right)\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k,0)}\right)} \quad (i = 1, \ldots, n), \tag{3.1b}$$

$$x_i^{(k,2)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k,1)}\right)\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k,2)}\right)} \quad (i = 1, \ldots, n), \tag{3.1c}$$

$$x_i^{(k,3)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k,3)}\right)\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k,2)}\right)} \quad (i = 1, \ldots, n), \tag{3.1d}$$

$$x_i^{(k+1)} = x_i^{(k,3)} \quad (i = 1, \ldots, n)\ (k \geq 0). \tag{3.1e}$$

The procedure PZSS1 has the following attractive features.

From (3.1b), (3.1c), and (3.1d), it follows that for $k \geq 0$,

(i) the values $p(x_i^{(k)})$ $(i = 1, \ldots, n)$ which are computed for use in (3.1b) are reused in (3.1c) and (3.1d).

(ii) $x_n^{(k,2)} = x_n^{(k,1)}$ and $x_1^{(k,3)} = x_1^{(k,2)}$, so that $x_n^{(k,2)}$ and $x_1^{(k,3)}$ need not be computed.

(iii) The product

$$\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k,1)}\right) \quad (i = 2, \ldots, n), \tag{3.2}$$

which are computed for use in (3.1b) are reused in (3.1c).

(iv) The product

$$\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k,2)}\right) \quad (i = n-1, \ldots, 1) \tag{3.3}$$

which are computed for use in (3.1c) are reused in (3.1d).

The following lemmas (Monsi [9]) are required in the proof of Theorem 3.4.

Lemma 3.1. *If*

(i) $p : C \to C$ *is defined by* (2.1)*;*

(ii) $p_i : C \to C$ *is defined by*

$$p_i(x) = \prod_{m=1}^{i-1}(x - x_m^*) \prod_{m=i+1}^{n}(x - x_m^*) \quad (i = 1, \ldots, n); \tag{3.4}$$

(iii) $q_i : C \to C$ *is defined by*

$$q_i(x) = \prod_{m=1}^{i-1}(x - \overline{x}_m) \prod_{m=i+1}^{n}(x - \widehat{x}_m) \quad (i = 1, \ldots, n), \tag{3.5}$$

where $\overline{x}_j \neq \overline{x}_m$ *and* $\widehat{x}_j \neq \widehat{x}_m$ $(j, m = 1, \ldots, n; j \neq m)$*;*

(iv) $\phi_i : C \to C$ *is defined by*

$$\phi_i(x) = q_i(x) + \sum_{j=1}^{i-1} \frac{p_i(\overline{x}_j) q_i(x)}{q_i'(\overline{x}_j)(x - \overline{x}_j)} + \sum_{j=i+1}^{n} \frac{p_i(\widehat{x}_j) q_i(x)}{q_i'(\widehat{x}_j)(x - \widehat{x}_j)} \quad (i = 1, \ldots, n), \tag{3.6}$$

then

$$\phi_i(x) = p_i(x) \quad (\forall x \in C) \; (i = 1, \ldots, n). \tag{3.7}$$

Lemma 3.2. *If* (i)–(iv) of Lemma 3.1 are valid; (v) $\check{x}_i$ $(i = 1, \ldots, n)$ are such that $p(\check{x}_i) \neq 0$ $(i = 1, \ldots, n)$, $\check{x}_i \neq \overline{x}_m$ $(m = 1, \ldots, i-1)$, $\check{x}_i \neq \widehat{x}_m$ $(m = i+1, \ldots, n)$, and

$$\overline{x}_i = \check{x}_i - \frac{p(\check{x}_i)}{\prod_{m=1}^{i-1}(\check{x}_i - \overline{x}_m)\prod_{m=i+1}^{n}(\check{x}_i - \widehat{x}_m)} \quad (i = 1, \ldots, n); \tag{3.8}$$

$\check{w}_i = \check{x}_i - x_i^*$, $\widehat{w}_i = \widehat{x}_i - x_i^*$, and $\overline{w}_i = \overline{x}_i - x_i^*$ $(i = 1, \ldots, n)$, then

$$\overline{w}_i = \check{w}_i \left\{ \sum_{j=1}^{i-1} \overline{\gamma}_{ij} \overline{w}_j + \sum_{j=i+1}^{n} \widehat{\gamma}_{ij} \widehat{w}_j \right\} \quad (i = 1, \ldots, n), \tag{3.9}$$

where

$$\begin{aligned} \overline{\gamma}_{ij} &= \frac{\prod_{m \neq i,j} (\overline{x}_j - x_m^*)}{q_i'(\overline{x}_j)(\overline{x}_j - \check{x}_i)} \quad (j = 1, \ldots, i-1), \\ \widehat{\gamma}_{ij} &= \frac{\prod_{m \neq i,j} (\widehat{x}_j - x_m^*)}{q_i'(\widehat{x}_j)(\widehat{x}_j - \check{x}_i)} \quad (j = i+1, \ldots, n). \end{aligned} \tag{3.10}$$

Lemma 3.3. *If* (i)–(v) of Lemma 3.2 are valid; (vi) $|\check{x}_i - x_i^*| \leq \theta d/(2n-1)$ and $|\widehat{x}_i - x_i^*| \leq \theta d/(2n-1)$ $(i = 1, \ldots, n)$, where $d = \min\{|x_i^* - x_j^*| \mid i, j = 1, \ldots n;\ j \neq i\}$ and $0 < \theta < 1$, then $|\overline{w}_i| \leq \theta |\check{w}_i|$ $(i = 1, \ldots, n)$.

Theorem 3.4. *If* (i) $p : C \to C$ defined by (2.1) has n distinct zeros x_i^* $(i = 1, \ldots, n)$; (ii) $|x_i^{(0)} - x_i^*| \leq \theta d/(2n-1)$ $(i = 1, \ldots, n)$, where $0 < \theta < 1$ and $d = \min\{|x_i^* - x_j^*| \mid i, j = 1, \ldots, n : j \neq i\}$, and the sequences $\{x_i^{(k)}\}$ $(i = 1, \ldots, n)$ are generated from PZSS1 (i.e., from (3.1a)–(3.1e)), then $x_i^{(k)} \to x_i^*$ $(k \to \infty)$ $(i = 1, \ldots, n)$ and $O_R(\text{PZSS1}, x^*) \geq 4$.

Proof. For $i = 1, \ldots, n$, let

$$\begin{aligned} q_{1,i}(x) &= \prod_{m=1}^{i-1} \left(x - x_m^{(k,1)}\right) \prod_{m=i+1}^{n} \left(x - x_m^{(k,0)}\right), \\ q_{2,i}(x) &= \prod_{m=1}^{i-1} \left(x - x_m^{(k,1)}\right) \prod_{m=i+1}^{n} \left(x - x_m^{(k,2)}\right), \\ q_{3,i}(x) &= \prod_{m=1}^{i-1} \left(x - x_m^{(k,3)}\right) \prod_{m=i+1}^{n} \left(x - x_m^{(k,2)}\right). \end{aligned} \tag{3.11}$$

Then, by (3.5) and (3.6),

$$\begin{aligned} \phi_{1,i}(x) &= q_{1,i}(x) + \sum_{j=1}^{i-1} \frac{p_i\left(x_j^{(k,1)}\right) q_{1,i}(x)}{q_{1,i}'\left(x_j^{(k,1)}\right)\left(x - x_j^{(k,1)}\right)} + \sum_{j=i+1}^{n} \frac{p_i\left(x_j^{(k,0)}\right) q_{1,i}(x)}{q_{1,i}'\left(x_j^{(k,0)}\right)\left(x - x_j^{(k,0)}\right)}, \\ \phi_{2,i}(x) &= q_{2,i}(x) + \sum_{j=1}^{i-1} \frac{p_i\left(x_j^{(k,1)}\right) q_{2,i}(x)}{q_{2,i}'\left(x_j^{(k,1)}\right)\left(x - x_j^{(k,1)}\right)} + \sum_{j=i+1}^{n} \frac{p_i\left(x_j^{(k,2)}\right) q_{2,i}(x)}{q_{2,i}'\left(x_j^{(k,2)}\right)\left(x - x_j^{(k,2)}\right)}, \\ \phi_{3,i}(x) &= q_{3,i}(x) + \sum_{j=1}^{i-1} \frac{p_i\left(x_j^{(k,3)}\right) q_{3,i}(x)}{q_{3,i}'\left(x_j^{(k,3)}\right)\left(x - x_j^{(k,3)}\right)} + \sum_{j=i+1}^{n} \frac{p_i\left(x_j^{(k,2)}\right) q_{3,i}(x)}{q_{3,i}'\left(x_j^{(k,2)}\right)\left(x - x_j^{(k,2)}\right)}, \end{aligned} \tag{3.12}$$

where $p_i(x)$ is defined by (3.4).

By Lemmas 3.1 and 3.2 with $q_i = q_{1,i}$, $\check{x}_i = x_i^{(k)}$, $\hat{x}_i = x_i^{(k,0)}$, $\overline{x}_i = x_i^{(k,1)}$, $\phi_i = \phi_{1,i}$ $(i = 1, \ldots, n)$, it follows that, for $i = 1, \ldots, n$, $k \geq 0$,

$$w_i^{(k,1)} = w_i^{(k)} \left\{ \sum_{j=1}^{i-1} \alpha_{ij}^{(k,1)} w_j^{(k,1)} + \sum_{j=i+1}^{n} \alpha_{ij}^{(k,0)} w_j^{(k,0)} \right\}, \tag{3.13}$$

where

$$\begin{gathered} w_i^{(k,s)} = x_i^{(k,s)} - x_i^* \quad (s = 0, \ldots, 3), \\ \alpha_{ij}^{(k,1)} = \frac{\prod_{m \neq i,j} \left(x_j^{(k,1)} - x_m^* \right)}{q'_{1,i}\left(x_j^{(k,1)} \right) \left(x_j^{(k,1)} - x_i^{(k)} \right)} \quad (j = 1, \ldots, i-1), \\ \alpha_{ij}^{(k,0)} = \frac{\prod_{m \neq i,j} \left(x_j^{(k,0)} - x_m^* \right)}{q'_{1,i}\left(x_j^{(k,0)} \right) \left(x_j^{(k,0)} - x_i^{(k)} \right)} \quad (j = i+1, \ldots, n). \end{gathered} \tag{3.14}$$

Similarly, by Lemmas 3.1 and 3.2, with $q_i = q_{2,i}$, $\check{x}_i = x_i^{(k)}$, $\hat{x}_i = x_i^{(k,2)}$, $\overline{x}_i = x_i^{(k,1)}$, $\phi_i = \phi_{2,i}$ $(i = 1, \ldots, n)$, it follows that, for $i = 1, \ldots, n$, $k \geq 0$,

$$w_i^{(k,2)} = w_i^{(k)} \left\{ \sum_{j=1}^{i-1} \beta_{ij}^{(k,1)} w_j^{(k,1)} + \sum_{j=i+1}^{n} \beta_{ij}^{(k,2)} w_j^{(k,2)} \right\}, \tag{3.15}$$

where

$$\begin{gathered} \beta_{ij}^{(k,1)} = \frac{\prod_{m \neq i,j} \left(x_j^{(k,1)} - x_m^* \right)}{q'_{2,i}\left(x_j^{(k,1)} \right) \left(x_j^{(k,1)} - x_i^{(k)} \right)} \quad (j = 1, \ldots, i-1), \\ \beta_{ij}^{(k,2)} = \frac{\prod_{m \neq i,j} \left(x_j^{(k,2)} - x_m^* \right)}{q'_{2,i}\left(x_j^{(k,2)} \right) \left(x_j^{(k,2)} - x_i^{(k)} \right)} \quad (j = i+1, \ldots, n). \end{gathered} \tag{3.16}$$

Similarly, by Lemma 3.1 and Lemma 3.2, with $q_i = q_{3,i}$, $\check{x}_i = x_i^{(k)}$, $\hat{x}_i = x_i^{(k,2)}$, $\overline{x}_i = x_i^{(k,3)}$, $\phi_i = \phi_{3,i}$ $(i = 1, \ldots, n)$, it follows that, for $i = 1, \ldots, n$, $k \geq 0$,

$$w_i^{(k,3)} = w_i^{(k)} \left\{ \sum_{j=1}^{i-1} \gamma_{ij}^{(k,3)} w_j^{(k,3)} + \sum_{j=i+1}^{n} \gamma_{ij}^{(k,2)} w_j^{(k,2)} \right\}, \tag{3.17}$$

where

$$\gamma_{ij}^{(k,3)} = \frac{\prod_{m \neq i,j}\left(x_j^{(k,3)} - x_m^*\right)}{q'_{3,i}\left(x_j^{(k,3)}\right)\left(x_j^{(k,3)} - x_i^{(k)}\right)} \quad (j = 1, \dots, i-1),$$
$$\gamma_{ij}^{(k,2)} = \frac{\prod_{m \neq i,j}\left(x_j^{(k,2)} - x_m^*\right)}{q'_{3,i}\left(x_j^{(k,2)}\right)\left(x_j^{(k,2)} - x_i^{(k)}\right)} \quad (j = i+1, \dots, n). \tag{3.18}$$

It follows from (3.13)-(3.14) and Lemma 3.3 that $|w_i^{(0,1)}| \le \theta |w_i^{(0,0)}|$ $(i = 1, \dots, n)$, and it follows from (3.15)-(3.16) and Lemma 3.3 that $|w_i^{(0,2)}| \le \theta^2 |w_i^{(0,0)}|$ $(i = 1, \dots, n)$ follows from (3.1e). It follows from (3.17)-(3.18) and Lemma 3.3 that

$$\left|w_i^{(0,3)}\right| \le \theta^3 \left|w_i^{(0,0)}\right| \quad (i = 1, \dots, n), \tag{3.19}$$

whence $|w_i^{(1,0)}| \le \theta^3 |w_i^{(0,0)}|$ $(i = 1, \dots, n)$. It then follows by induction on k that, for all $k \ge 0$,

$$\left|w_i^{(k,0)}\right| \le \theta^{4^k - 1} \left|w_i^{(0,0)}\right| \quad (i = 1, \dots, n), \tag{3.20}$$

whence $x_i^{(k)} \to x_i^*$ $(k \to \infty)$, $(i = 1, \dots, n)$. Let

$$h_i^{(k,m)} = \frac{(2n-1)}{d}\left|w_i^{(k,m)}\right| \quad (i = 1, \dots, n)\ (m = 0, \dots, 3). \tag{3.21}$$

Then, by (3.13), (3.15), (3.17), and (3.21), for $i = 1, \dots, n$, (recall (3.1b), (3.1c), (3.1d))

$$h_i^{(k,1)} \le \frac{1}{(n-1)} h_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,1)} + \sum_{j=i+1}^{n} h_j^{(k,0)} \right\}, \tag{3.22}$$

for $i = n, \dots, 1$,

$$h_i^{(k,2)} \le \frac{1}{(n-1)} h_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,1)} + \sum_{j=i+1}^{n} h_j^{(k,2)} \right\}, \tag{3.23}$$

and for $i = 1, \dots, n$,

$$h_i^{(k,3)} \le \frac{1}{(n-1)} h_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,3)} + \sum_{j=i+1}^{n} h_j^{(k,2)} \right\}. \tag{3.24}$$

Let

$$u_i^{(1,1)} = \begin{cases} 2, & (i = 1, \dots, n-1), \\ 3, & (i = n), \end{cases}$$
$$u_i^{(1,2)} = \begin{cases} 4, & (i = 1), \\ 3, & (i = 2, \dots, n), \end{cases} \tag{3.25}$$
$$u_i^{(1,3)} = \begin{cases} 4, & (i = 1, \dots, n-1), \\ 5, & (i = n). \end{cases}$$

For $r = 1, 2, 3,$ let

$$u_i^{(k+1,r)} = \begin{cases} 4u_i^{(k,r)}, & (i = 1, \dots, n-1), \\ 4u_i^{(k,r)} + 1, & (i = n). \end{cases} \tag{3.26}$$

Then, by (3.25)–(3.26), for all $k \geq 1$,

$$u_i^{(k,1)} = \begin{cases} 2\left(4^{(k-1)}\right), & (i = 1, \dots, n-1), \\ \dfrac{10}{3}\left(4^{(k-1)}\right) - \dfrac{1}{3}, & (i = n), \end{cases}$$
$$u_i^{(k,2)} = \begin{cases} 4\left(4^{(k-1)}\right), & (i = 1), \\ 3\left(4^{(k-1)}\right), & (i = 2, \dots, n-1) \\ \dfrac{10}{3}\left(4^{(k-1)}\right) - \dfrac{1}{3}, & (i = n), \end{cases} \tag{3.27}$$

$$u_i^{(k,3)} = \begin{cases} 4\left(4^{(k-1)}\right), & (i = 1, \dots, n-1), \\ \dfrac{16}{3}\left(4^{(k-1)}\right) - \dfrac{1}{3}, & (i = n). \end{cases} \tag{3.28}$$

Suppose, without loss of generality, that

$$h_i^{(0,0)} \leq h < 1 \quad (i = 1, \dots, n). \tag{3.29}$$

Then, by a lengthy inductive argument, it follows from (3.21)–(3.29) that for $i = 1, \dots, n$, for all $k \geq 1$,

$$h_i^{(k,1)} \leq h^{u_i^{(k+1,1)}},$$
$$h_i^{(k,2)} \leq h^{u_i^{(k+1,2)}}, \tag{3.30}$$
$$h_i^{(k,3)} \leq h^{u_i^{(k+1,3)}},$$

whence, by (3.28) and (3.1e), for all $k \geq 1$,

$$h_i^{(k)} \leq h^{4^k} \quad (i = 1, \dots, n). \tag{3.31}$$

By (3.21) for $m = 3$,

$$\left| w_i^{(k,3)} \right| = \frac{d}{(2n-1)} h_i^{(k,3)} \quad (i = 1, \dots, n), \tag{3.32}$$

then by (3.1e),

$$\left| w_i^{(k+1)} \right| = \frac{d}{(2n-1)} h_i^{(k+1)} \quad (i = 1, \dots, n). \tag{3.33}$$

So,

$$\left| w_i^{(k)} \right| = \frac{d}{(2n-1)} h_i^{(k)} \quad (i = 1, \dots, n) \ (k \geq 0). \tag{3.34}$$

Let

$$\begin{aligned} w^{(k)} &= \max_{1 \leq i \leq n} \left\{ \left| w_i^{(k)} \right| \right\}, \\ h^{(k)} &= \max_{1 \leq i \leq n} \left\{ h_i{}^{(k)} \right\}. \end{aligned} \tag{3.35}$$

Then, by (3.22)–(3.35)

$$w^{(k)} \leq \frac{d}{(2n-1)} h^{4^k} \quad (\forall k \geq 0). \tag{3.36}$$

So,

$$R_4\left(w^{(k)}\right) = \lim_{k \to \infty} \sup \left\{ \left(w^{(k)}\right)^{1/4^k} \right\} \leq \lim_{k \to \infty} \sup \left\{ \left(\frac{d}{2n-1}\right)^{1/4^k} h \right\} = h < 1. \tag{3.37}$$

Therefore (Ortega and Rheindboldt [13]),

$$O_R\left(\mathrm{PZSS1}, x_i^*\right) \geq 4 \quad (i = 1, \dots, n). \tag{3.38}$$

□

4. Conclusion

The result above shows that the procedure PZSS1 has R-order of convergence at least 4 that is higher than does PT1, PS1, and PSS1. The attractive features given in Section 3 of this procedure will give less computational time. Our experiences in the implementation of the interval version of PZSS1, that is, the procedure IZSS1(Rusli et al. [10]) showed that this procedure is more efficient for bounding the zeros simultaneously.

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