Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 705197, 11 pages
doi:10.1155/2012/705197

Research Article

Existence of Traveling Fronts in a Food-Limited Population Model with Spatiotemporal Delay

Hai-Qin Zhao[1] and San-Yang Liu[2]

[1] *College of Mathematics and Information Science, Xianyang Normal University, Xianyang, Shaanxi 712000, China*
[2] *Department of Mathematics, Xidian University, Xi'an, Shaanxi 710071, China*

Correspondence should be addressed to Hai-Qin Zhao, xinyi98_01@126.com

Received 25 August 2012; Accepted 23 October 2012

Academic Editor: Wan-Tong Li

This paper is concerned with the traveling fronts of a diffusive food-limited population model with spatiotemporal delay. Sufficient conditions are established for the existence of traveling wave fronts by choosing different kinds of delay kernels. The approach used here is the upper-lower solution method and monotone iteration technique. Our work extends and/or covers some previous results.

1. Introduction

This paper is concerned with the traveling fronts for the following food-limited model:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\frac{1 - au(x,t) - b(g*u)(x,t)}{1 + adu(x,t) + bd(g*u)(x,t)}, \quad x \in \mathbb{R},\ t \geq 0, \tag{1.1}$$

where a, b, and d are nonnegative constants, $a+b>0$, and the kernel $g(x,t)$ is any integrable nonnegative function satisfying $g(-x,t) = g(x,t)$,

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} g(y,s)\,dy\,ds = 1, \qquad (g*u)(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{+\infty} g(x-y,t-s)u(y,s)\,dy\,ds, \tag{1.2}$$

which was first proposed and analyzed by Gourley and So [1] on a finite domain Ω.

In the case $a = 0$, $b = 1$, $d = \beta$, (1.1) becomes

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\frac{1-(g*u)(x,t)}{1+\beta(g*u)(x,t)}. \tag{1.3}$$

Recently, many researchers studied the existence of traveling fronts of (1.3) with some specific $g(x,t)$. For the case

$$g(x,t) = \delta(t-\tau)\delta(x), \tag{1.4}$$

where $\delta(\cdot)$ is the Dirac delta function, Gourley [2] showed that, for any $c > 2$, there exists $\tau^*(c) > 0$ such that, for any $\tau < \tau^*(c)$, (1.3) has a traveling front connecting the equilibria 0 and 1, by using the approach developed by Wu and Zou [3]. For the case

$$g(x,t) = \frac{1}{\tau}e^{-t/\tau}\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}, \tag{1.5}$$

Gourley and Chaplain [4] proved the existence of traveling fronts for any $c \geq 2$ and sufficient small $\tau > 0$, by employing linear chain techniques to recast the traveling wave equations as a finite-dimensional system of ODEs and using Fenichel's geometric singular perturbation theory [5] and the Fredholm alternative. For the case

$$g(x,t) = \delta(t-\tau)\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}, \tag{1.6}$$

Gourley and Chaplain [4], by using the method of Canosa [6], obtained some information on the monotonicity of traveling fronts for sufficiently large c. Furthermore, for these cases

$$\begin{gathered} g(x,t) = \frac{t}{\tau^2}e^{-t/\tau}\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}, \qquad g(x,t) = \delta(t-\tau)\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}, \\ g(x,t) = \delta(t)\frac{1}{2\rho}e^{-|x|/\rho}, \quad \rho > 0, \qquad g(x,t) = \frac{1}{\tau^2}e^{-t/\tau}\delta(x), \quad \tau > 0, \end{gathered} \tag{1.7}$$

Wang and Li [7] showed that, for any $c > 2$, there exists $\tau^*(c) > 0$ (or $\rho^*(c) > 0$) such that for any $\tau < \tau^*(c)$ (or $\rho < \rho^*(c)$), (1.3) has a traveling front connecting the equilibria 0 and 1.

In this paper, based on the monotone iteration technique as well as the upper and lower solution method developed by Wang et al. [8], we will establish the existence of traveling fronts of (1.1) with the kernel functions (1.4)–(1.7). More precisely, we shall show that for any $c > 2$, there exists $\tau^*(c) > 0$ (or $\rho^*(c) > 0$) such that, for any $\tau < \tau^*(c)$ (or $\rho < \rho^*(c)$), (1.1) has a traveling front connecting the equilibria 0 and $K = 1/(a+b)$ (see Theorems 2.5 and 2.9 and Remark 2.10), which includes, improves, and/or complements a number of existing results in [2–4, 7, 9, 10].

The rest of the paper is organized as follows. In Section 2, we establish the existence of traveling wave fronts of (1.1) with the kernel functions (1.4)–(1.7). For the sake of convenience, we present in the Appendix some results developed by Wang et al. [8].

2. Existence of Traveling Fronts

In this section, we will use Theorem A.2 to establish the existence of traveling fronts of (1.1) by choosing different kernel function g, such as (1.4)–(1.7). It is easy to see that (1.1) has two uniform steady states $K_0 = 0$ and $K = 1/(a+b)$.

Let $u(x,t) = \phi(\xi)$, $\xi = x + ct$. Then a traveling front $\phi(\xi)$ of (1.1) satisfies the boundary conditions $\phi(-\infty) = K_0$ and $\phi(+\infty) = K$, and the following equation:

$$\phi''(\xi) - c\phi'(\xi) + \phi(\xi)\frac{1 - a\phi(\xi) - b(g*\phi)(\xi)}{1 + ad\phi(\xi) + bd(g*\phi)(\xi)} = 0, \quad \xi \in \mathbb{R}. \tag{2.1}$$

For $c > 2$, let $\Delta_c(\mu) = \mu^2 - c\mu + 1$ and $\lambda = (c - \sqrt{c^2-4})/2$. Then $\Delta_c(\lambda) = 0$. Let $0 < \varepsilon < \lambda$, $\alpha > 0$, $M > 1$ and $\gamma > \lambda$ such that

$$\lambda + \varepsilon < \gamma, \qquad \lambda + \varepsilon < \frac{c + \sqrt{c^2-4}}{2}, \qquad \alpha < \frac{\lambda}{2(\gamma+\lambda)}, \qquad \frac{1}{2} \le M\alpha \le M - 1. \tag{2.2}$$

Clearly, $\Delta_c(\lambda+\varepsilon) < 0$. Define $\phi_+(\xi) = K/(1 + \alpha e^{-\lambda\xi})$ and $\phi_-(\xi) = \max\{Ke^{\lambda\xi}(1 - Me^{\varepsilon\xi}), 0\}$. Then we have the following observations.

Lemma 2.1. (i) *$\phi_+(\xi)$ is increasing in $\xi \in \mathbb{R}$ and satisfies $\phi_+(-\infty) = K_0$ and $\phi_+(+\infty) = K$;*

(ii) *$\phi_+(\xi) \ge \phi_-(\xi)$ for all $\xi \in \mathbb{R}$;*

(iii) *$e^{\gamma\xi}[\phi_+(\xi) - \phi_-(\xi)]$ is increasing and $e^{-\gamma\xi}[\phi_+(\xi) - \phi_-(\xi)]$ is decreasing in $\xi \in \mathbb{R}$;*

(iv) *$e^{\gamma\xi}[\phi_+(\xi+\eta) - \phi_+(\xi)]$ is increasing and $e^{-\gamma\xi}[\phi_+(\xi+\eta) - \phi_+(\xi)]$ is decreasing in $\xi \in \mathbb{R}$ for every $\eta > 0$.*

Clearly, Lemma 2.1 implies that, for $\gamma > \lambda + \varepsilon$, $\phi_+(\xi) \in \Gamma^*$, $\phi_+(\xi) \in \Gamma^{**}$ and $\sup_{\xi\in\mathbb{R}}\phi_-(\xi) > 0$. Now, we show that $\phi_+(\xi)$ and $\phi_-(\xi)$ are lower and upper solutions of (2.1) by choosing different kernel functions g, respectively.

For the sake of convenience, throughout this section, we let

$$f(\phi(\xi), (g*\phi)(\xi)) = \phi(\xi)\frac{1 - a\phi(\xi) - b(g*\phi)(\xi)}{1 + ad\phi(\xi) + bd(g*\phi)(\xi)}, \quad \xi \in \mathbb{R}. \tag{2.3}$$

2.1. The Case $g(x,t) = \delta(t)(1/2\rho)e^{-|x|/\rho}$, $\rho > 0$

Clearly, $g(x,t) = \delta(t)(1/2\rho)e^{-|x|/\rho}$ satisfies (H_0) and in this case

$$(g*\phi)(\xi) = \int_{-\infty}^{+\infty} \frac{1}{2\rho}e^{-|y|/\rho}\phi(\xi - y)\,dy. \tag{2.4}$$

Lemma 2.2. *For sufficient small $\rho > 0$, $f(\phi(\xi), (g*\phi)(\xi))$ satisfies (H_1^{**}).*

Proof. Let $A = (a^2 + 2abd + a^2d + ab)/(a+b)^2$ and $B = b(d+1)/(a+b) + abd/(a+b)^2$. Fix $\gamma > A + 2B$. Let $\phi_1, \phi_2 \in C(\mathbb{R},\mathbb{R})$ with $0 \le \phi_1(\xi) \le \phi_2(\xi) \le K$ so that $e^{\gamma\xi}[\phi_2(\xi) - \phi_1(\xi)]$ is increasing and $e^{-\gamma\xi}[\phi_2(\xi) - \phi_1(\xi)]$ is decreasing in $\xi \in \mathbb{R}$. It is easy to see that for any $\eta \in \mathbb{R}$,

$e^{\gamma\xi}[\phi_2(\xi+\eta)-\phi_1(\xi+\eta)]$ is increasing and $e^{-\gamma\xi}[\phi_2(\xi+\eta)-\phi_1(\xi+\eta)]$ is decreasing in $\xi\in\mathbb{R}$. For sufficiently small $\rho>0$ satisfying $1-\rho\gamma>1/2$, there is

$$\begin{aligned}
&(g*\phi_2)(\xi)-(g*\phi_1)(\xi)\\
&\quad=\int_{-\infty}^{+\infty}\frac{1}{2\rho}e^{-|y|/\rho}[\phi_2(\xi-y)-\phi_1(\xi-y)]dy\\
&\quad=\int_{0}^{+\infty}\frac{1}{2\rho}e^{-y/\rho}[\phi_2(\xi-y)-\phi_1(\xi-y)]dy+\int_{0}^{+\infty}\frac{1}{2\rho}e^{-y/\rho}[\phi_2(\xi+y)-\phi_1(\xi+y)]dy\\
&\quad=\int_{0}^{+\infty}\frac{1}{2\rho}e^{-y/\rho}e^{\gamma y}\Big\{e^{-\gamma y}[\phi_2(\xi-y)-\phi_1(\xi-y)]+e^{-\gamma y}[\phi_2(\xi+y)-\phi_1(\xi+y)]\Big\}dy\\
&\quad\le 2[\phi_2(\xi)-\phi_1(\xi)]\int_{0}^{+\infty}\frac{1}{2\rho}e^{-y/\rho}e^{\gamma y}dy\\
&\quad=\frac{1}{1-\rho\gamma}[\phi_2(\xi)-\phi_1(\xi)]\le 2[\phi_2(\xi)-\phi_1(\xi)].
\end{aligned}\tag{2.5}$$

Hence,

$$\begin{aligned}
&\phi_2(1-a\phi_2-bg*\phi_2)(1+ad\phi_1+bdg*\phi_1)-\phi_1(1-a\phi_1-bg*\phi_1)(1+ad\phi_2+bdg*\phi_2)\\
&\quad=(\phi_2-\phi_1)\Big[1+bdg*\phi_1-(a+abdg*\phi_1)(\phi_2+\phi_1)-a^2d\phi_1\phi_2-bg*\phi_2-b^2dg*\phi_1g*\phi_2\Big]\\
&\qquad+\Big(abd\phi_1^2-bd\phi_1-b\phi_1-abd\phi_1\phi_2\Big)(g*\phi_2-g*\phi_1)\\
&\quad\ge -A(\phi_2-\phi_1)-B(g*\phi_2-g*\phi_1)\\
&\quad\ge -(A+2B)(\phi_2-\phi_1)>-\gamma(\phi_2-\phi_1).
\end{aligned}\tag{2.6}$$

Therefore,

$$\begin{aligned}
&f(\phi_2(\xi),(g*\phi_2)(\xi))-f(\phi_1(\xi),(g*\phi_1)(\xi))\\
&\quad=\phi_2(\xi)\frac{1-a\phi_2(\xi)-b(g*\phi_2)(\xi)}{1+ad\phi_2(\xi)+bd(g*\phi_2)(\xi)}-\phi_1(\xi)\frac{1-a\phi_1(\xi)-b(g*\phi_1)(\xi)}{1+ad\phi_1(\xi)+bd(g*\phi_1)(\xi)}\\
&\quad\ge\frac{-\gamma[\phi_2(\xi)-\phi_1(\xi)]}{[1+ad\phi_2(\xi)+bd(g*\phi_2)(\xi)][1+ad\phi_1(\xi)+bd(g*\phi_1)(\xi)]}\\
&\quad>-\gamma[\phi_2(\xi)-\phi_1(\xi)].
\end{aligned}\tag{2.7}$$

This completes the proof. □

Lemma 2.3. *Assume that $1-\lambda\rho>0$. Then for sufficiently large $M>1$, $\phi_-(\xi)$ is a lower solution of* (2.1).

Proof. For $\xi \geq \xi_0 = (1/\varepsilon)\ln(1/M)$, $\phi_-(\xi) = 0$, then

$$\phi_-''(\xi) - c\phi_-'(\xi) + \phi_-(\xi)\frac{1 - a\phi_-(\xi) - b(g * \phi_-)(\xi)}{1 + ad\phi_-(\xi) + bd(g * \phi_-)(\xi)} = 0. \tag{2.8}$$

Let

$$M \geq -\frac{(d+1)aK}{\Delta_c(\lambda+\varepsilon)} - \frac{(d+1)bK}{(1-\rho\lambda)(1+\rho\lambda)\Delta_c(\lambda+\varepsilon)}. \tag{2.9}$$

For $\xi < \xi_0 < 0$, $\phi_-(\xi) = Ke^{\lambda\xi}(1 - Me^{\varepsilon\xi})$, since

$$\begin{aligned}(g * \phi_-)(\xi) &= \int_{-\infty}^{+\infty} \frac{1}{2\rho} e^{-|y|/\rho} \phi_-(\xi - y)\,dy \\ &= \int_{\xi-\xi_0}^{+\infty} \frac{1}{2\rho} e^{-|y|/\rho} e^{\lambda(\xi-y)} K\left(1 - Me^{\varepsilon(\xi-y)}\right) dy \\ &\leq K\int_{-\infty}^{+\infty} \frac{1}{2\rho} e^{-|y|/\rho} e^{\lambda(\xi-y)}\,dy = \frac{Ke^{\lambda\xi}}{(1-\rho\lambda)(1+\rho\lambda)},\end{aligned} \tag{2.10}$$

and $h(z) = (1-z)/(1+dz) \geq 1 - (d+1)z$ for all $z > 0$, then

$$\begin{aligned}&\phi_-''(\xi) - c\phi_-'(\xi) + \phi_-(\xi)\frac{1 - a\phi_-(\xi) - b(g * \phi_-)(\xi)}{1 + ad\phi_-(\xi) + bd(g * \phi_-)(\xi)} \\ &\quad \geq \phi_-''(\xi) - c\phi_-'(\xi) + \phi_-(\xi)\{1 - (d+1)[a\phi_-(\xi) + b(g * \phi_-)(\xi)]\} \\ &\quad \geq K\left[\lambda^2 - M(\lambda+\varepsilon)^2 e^{\varepsilon\xi}\right]e^{\lambda\xi} - Kc\left[\lambda - M(\lambda+\varepsilon)e^{\varepsilon\xi}\right]e^{\lambda\xi} + Ke^{\lambda\xi}\left(1 - Me^{\varepsilon\xi}\right) \\ &\qquad - (d+1)aK^2e^{2\lambda\xi}\left(1 - Me^{\varepsilon\xi}\right)^2 - \frac{(d+1)bK^2e^{2\lambda\xi}}{(1-\rho\lambda)(1+\rho\lambda)} \\ &\quad \geq Ke^{(\lambda+\varepsilon)\xi}\left[-M\Delta_c(\lambda+\varepsilon) - a(d+1)K - \frac{(d+1)bK}{(1-\rho\lambda)(1+\rho\lambda)}\right] \geq 0.\end{aligned} \tag{2.11}$$

Thus, we showed that $\phi_-(\xi)$ is a lower solution of (2.1). This completes the proof. □

Lemma 2.4. *For sufficiently small $\rho > 0$, $\phi_+(\xi)$ is an upper solution of* (2.1).

Proof. Note that

$$\phi_+'(\xi) = \frac{K\alpha\lambda e^{-\lambda\xi}}{(1+\alpha e^{-\lambda\xi})^2}, \qquad \phi_+''(\xi) = \frac{-K\alpha\lambda^2 e^{-\lambda\xi} + K\alpha^2\lambda^2 e^{-2\lambda\xi}}{(1+\alpha e^{-\lambda\xi})^3}. \tag{2.12}$$

By an argument similar to [7, Lemma 3.5], for $\rho > 0$ such that $1 - 2\rho\lambda > 0$, we have

$$(g * \phi_+)(\xi) \geq \frac{K}{1 + \alpha e^{-\lambda\xi}} - \frac{K\alpha\lambda^2\rho^2}{1 - \lambda^2\rho^2} \cdot \frac{e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^2}. \tag{2.13}$$

Then for sufficiently small $\rho > 0$ with $2\lambda^2 - Kb\lambda^2\rho^2/(1 - \lambda^2\rho^2) > 0$,

$$\begin{aligned}
&\phi_+''(\xi) - c\phi_+'(\xi) + \phi_+(\xi)\frac{1 - a\phi_+(\xi) - b(g * \phi_+)(\xi)}{1 + ad\phi_+(\xi) + bd(g * \phi_+)(\xi)} \\
&\quad \leq \phi_+''(\xi) - c\phi_+'(\xi) + \phi_+(\xi)\left[1 - a\phi_+(\xi) - b(g * \phi_+)(\xi)\right] \\
&\quad \leq \frac{-K\alpha\lambda^2 e^{-\lambda\xi} + K\alpha^2\lambda^2 e^{-2\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^3} - \frac{Kc\alpha\lambda e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^2} + \frac{K\alpha e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^2} \\
&\qquad + \frac{K^2 b\alpha\lambda^2\rho^2}{1 - \lambda^2\rho^2} \cdot \frac{e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^3} \\
&\quad = \frac{K\alpha^2(\lambda^2 - c\lambda + 1)e^{-2\lambda\xi} - K\alpha(\lambda^2 + c\lambda - 1 - Kb\lambda^2\rho^2/(1 - \lambda^2\rho^2))e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^3} \\
&\quad = \frac{-K\alpha(2\lambda^2 - Kb\lambda^2\rho^2/(1 - \lambda^2\rho^2))e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^3} < 0.
\end{aligned} \tag{2.14}$$

This completes the proof. □

Therefore, by Theorem A.2(ii), we have the following result.

Theorem 2.5. *For any $c > 2$, there exists $\rho^*(c) > 0$ such that, for any $\rho < \rho^*(c)$,* (1.1) *has an increasing traveling wave front $\phi(\xi)$ that satisfies $\phi(-\infty) = 0$, $\phi(+\infty) = K$ and $\lim_{\xi\to-\infty}\phi(\xi)e^{-\lambda\xi} = 1$.*

2.2. *The Case* $g(x,t) = (t/\tau^2)e^{-t/\tau}\delta(x)$, $\tau > 0$

It is easy to see that $g(x,t) = (t/\tau^2)e^{-t/\tau}\delta(x)$ satisfies (H_0) and in this case

$$(g * \phi)(\xi) = \int_0^{+\infty} \frac{s}{\tau^2} e^{-s/\tau}\phi(\xi - cs)ds. \tag{2.15}$$

The following two lemmas are similar to Lemmas 2.1 and 2.3, and their proofs are omitted.

Lemma 2.6. *For sufficient small $\tau > 0$, $f(\phi(\xi), (g * \phi)(\xi))$ satisfies (H_1^*).*

Lemma 2.7. *For sufficiently large $M \geq 1$, $\phi_-(\xi)$ is a lower solution of* (2.1).

Lemma 2.8. *For sufficiently small $\tau > 0$, $\phi_+(\xi)$ is an upper solution of* (2.1).

Proof. Note that, for $\tau > 0$ such that $1 - 2\lambda c\tau > 0$,

$$
\begin{aligned}
&(g * \phi_+)(\xi) \\
&\quad = \int_0^{+\infty} \frac{s}{\tau^2} e^{-s/\tau} \frac{K}{1 + \alpha e^{-\lambda(\xi - cs)}} ds \\
&\quad = \int_0^{+\infty} \frac{1}{\tau} e^{-s/\tau} \frac{K}{1 + \alpha e^{-\lambda(\xi - cs)}} ds - \alpha\lambda c e^{-\lambda\xi} \int_0^{+\infty} \frac{s}{\tau} e^{-s/\tau} \frac{K e^{(\lambda c - 1/\tau)s}}{\left[1 + \alpha e^{-\lambda(\xi - cs)}\right]^2} ds \\
&\quad \geq \frac{K}{1 + \alpha e^{-\lambda\xi}} - \frac{K\alpha\lambda c\tau e^{-\lambda\xi}}{(1 - \lambda c\tau)\left(1 + \alpha e^{-\lambda\xi}\right)^2} - \frac{K\alpha\lambda c\tau e^{-\lambda\xi}}{(1 - \lambda c\tau)^2\left(1 + \alpha e^{-\lambda\xi}\right)^2} \\
&\quad = \frac{K}{1 + \alpha e^{-\lambda\xi}} - \frac{K\alpha\lambda c\tau e^{-\lambda\xi}(2 - \lambda c\tau)}{(1 - \lambda c\tau)^2\left(1 + \alpha e^{-\lambda\xi}\right)^2}.
\end{aligned}
\tag{2.16}
$$

Then for sufficiently small $\tau > 0$ with $2\lambda^2 - bK\lambda c\tau(2 - \lambda c\tau)/(1 - \lambda c\tau)^2 > 0$,

$$
\begin{aligned}
&\phi_+''(\xi) - c\phi_+'(\xi) + \phi_+(\xi) \frac{1 - a\phi_+(\xi) - b(g * \phi_+)(\xi)}{1 + ad\phi_+(\xi) + bd(g * \phi_+)(\xi)} \\
&\quad \leq \phi_+''(\xi) - c\phi_+'(\xi) + \phi_+(\xi)\left[1 - a\phi_+(\xi) - b(g * \phi_+)(\xi)\right] \\
&\quad \leq \frac{-K\alpha\lambda^2 e^{-\lambda\xi} + K\alpha^2\lambda^2 e^{-2\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^3} - \frac{Kc\alpha\lambda e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^2} + \frac{K\alpha e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^2} \\
&\qquad + \frac{K^2 b\alpha\lambda c\tau e^{-\lambda\xi}(2 - \lambda c\tau)}{(1 - \lambda c\tau)^2\left(1 + \alpha e^{-\lambda\xi}\right)^3} \\
&\quad = \frac{K\alpha^2\left(\lambda^2 - c\lambda + 1\right)e^{-2\lambda\xi} - K\alpha\left(\lambda^2 + c\lambda - 1 - bK\lambda c\tau(2 - \lambda c\tau)/(1 - \lambda c\tau)^2\right)e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^3} \\
&\quad = \frac{-K\alpha\left(2\lambda^2 - bK\lambda c\tau(2 - \lambda c\tau)/(1 - \lambda c\tau)^2\right)e^{-\lambda\xi}}{\left(1 + \alpha e^{-\lambda\xi}\right)^3} < 0.
\end{aligned}
\tag{2.17}
$$

This completes the proof. □

Now, by Theorem A.2(i), we have the following result.

Theorem 2.9. *For any $c > 2$, there exists $\tau^*(c) > 0$ such that, for any $\tau < \tau^*(c)$, (1.1) has an increasing traveling wave front $\phi(\xi)$ that satisfies $\phi(-\infty) = 0$, $\phi(+\infty) = K$ and* $\lim_{\xi \to -\infty}\phi(\xi)e^{-\lambda\xi} = 1$.

Remark 2.10. Being a careful observation, for these cases where

$$g(x,t) = \delta(t-\tau)\delta(x), \qquad g(x,t) = \frac{1}{\tau}e^{-t/\tau}\delta(x), \qquad g(x,t) = \frac{1}{\tau}e^{-t/\tau}\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t},$$
$$g(x,t) = \frac{t}{\tau^2}e^{-t/\tau}\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}, \qquad g(x,t) = \delta(t-\tau)\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t} \tag{2.18}$$

by using the above method, we can get similar results, respectively.

Remark 2.11. In the case $a = 0$, $b = 1$, $d = 0$, (1.1) reduces to

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\left[1 - (g * u)(x,t)\right], \tag{2.19}$$

which has been studied by many researchers, for example, Ashwin et al. [9], Gourley [10], and Wu and Zou [3] and references therein. It is easy to see that our results include and complement those of Ashwin et al. [9], Gourley [10], and Wu and Zou [3].

Remark 2.12. We mention that Ou and Wu [11] obtained the persistence of traveling fronts of delayed nonlocal reaction-diffusion equations. Their abstract results could be applied to the model (1.1) to obtain the existence of traveling fronts. But, their results cannot prove the precise asymptotic behavior of the traveling fronts.

Appendix

In this appendix, we present some general results developed by Wang et al. [8]. Consider the following reaction-diffusion system with spatiotemporal delays:

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f\left(u(x,t), (g_1 * u)(x,t), \dots, (g_m * u)(x,t)\right), \tag{A.1}$$

where $x \in \mathbb{R}$, $t \geq 0$, $D = \operatorname{diag}(d_1, \dots, d_n)$, $d_i > 0$, $i = 1, \dots, n$, $n \in \mathbb{N}$; $f \in C((\mathbb{R}^{m+1})^n, \mathbb{R}^n)$, $u(x,t) = (u_1(x,t), \dots, u_n(x,t))^T$, and

$$(g_j * u)(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{+\infty} g_j(x-y, t-s)u(y,s)\,dy\,ds, \quad j = 1, \dots, m,\ m \in \mathbb{N}, \tag{A.2}$$

and the kernel $g_j(x,t)$ is any integrable nonnegative function satisfying $g_j(-x,t) = g_j(x,t)$, $\int_0^{+\infty}\int_{-\infty}^{+\infty} g_j(y,s)\,dy\,ds = 1$, and the following assumption:

(H_0) $\int_{-\infty}^{+\infty} g_j(x,t)dx$ is uniformly convergent for $t \in [0,a]$, $a > 0$, $j = 1, \dots, m$. In other words, if given $\varepsilon > 0$, then there exists $M > 0$ such that $\int_M^{+\infty} g_j(x,t)dx < \varepsilon$ for any $t \in [0,a]$.

Assume $u(x,t) = \phi(\xi)$ and $\xi = x + ct$, and then we can write (A.1) in the following form:

$$-D\phi''(\xi) + c\phi'(\xi) = f\left(\phi(\xi), (g_1 * \phi)(\xi), \dots, (g_m * \phi)(\xi)\right), \quad \xi \in \mathbb{R}. \tag{A.3}$$

A traveling wave front with a wave speed $c > 0$ to (A.1) is a function $\phi \in BC^2(\mathbb{R}, \mathbb{R}^n)$ and a number $c > 0$ which satisfy (A.3) and the following boundary condition:

$$\phi(-\infty) = \mathbf{0}, \quad \phi(+\infty) = \mathbf{K} = (K_1, \dots, K_n)^T \quad \text{with } K_i > 0,\ i = 1, \dots, n. \tag{A.4}$$

In order to tackle the existence of traveling fronts, we need the following monotonicity conditions and assumptions.

(H_1^*) There exists a matrix $\gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$ with $\gamma_i > 0$, $i = 1, \dots, n$, such that

$$\begin{aligned} &f\left(\psi(\xi), (g_1 * \psi)(\xi), \dots, (g_m * \psi)(\xi)\right) + \gamma\psi(\xi) \\ &\quad \geq f\left(\phi(\xi), (g_1 * \phi)(\xi), \dots, (g_m * \phi)(\xi)\right) + \gamma\phi(\xi), \end{aligned} \tag{A.5}$$

where $\phi, \psi \in C(\mathbb{R}, \mathbb{R}^n)$ satisfy $\mathbf{0} \leq \phi(\xi) \leq \psi(\xi) \leq \mathbf{K}$ in $\xi \in \mathbb{R}$ and $e^{\gamma t}[\psi(\xi) - \phi(\xi)]$ is increasing in $\xi \in \mathbb{R}$.

(H_1^{**}) There exists a matrix $\gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$ with $\gamma_i > 0$, $i = 1, \dots, n$, such that

$$\begin{aligned} &f\left(\psi(\xi), (g_1 * \psi)(\xi), \dots, (g_m * \psi)(\xi)\right) + \gamma\psi(\xi) \\ &\quad \geq f\left(\phi(\xi), (g_1 * \phi)(\xi), \dots, (g_m * \phi)(\xi)\right) + \gamma\phi(\xi), \end{aligned} \tag{A.6}$$

where $\phi, \psi \in C(\mathbb{R}, \mathbb{R}^n)$ satisfy $\mathbf{0} \leq \phi(\xi) \leq \psi(\xi) \leq \mathbf{K}$ in $\xi \in \mathbb{R}$, $e^{\gamma\xi}[\psi(\xi) - \phi(\xi)]$ is increasing in $\xi \in \mathbb{R}$, and $e^{-\gamma\xi}[\psi(\xi) - \phi(\xi)]$ is decreasing in $\xi \in \mathbb{R}$.

(H_2) $f(\mu_1, \dots, \mu_n) \neq 0$ for $\mathbf{0} < \mu < \mathbf{K}$.

(H_3) $f(\mu_1, \dots, \mu_n) = \mathbf{0}$ when $\mu = \mathbf{0}$ or $\mu = \mathbf{K}$.

Let $BC[\mathbf{0}, \mathbf{K}] = \{x \in BC(\mathbb{R}, \mathbb{R}^n) : \mathbf{0} \leq x(t) \leq \mathbf{K},\ t \in \mathbb{R}\}$, $Y = \{\phi \in BC(\mathbb{R}, \mathbb{R}^n) : \phi', \phi'' \in L^\infty(\mathbb{R}, \mathbb{R}^n)\}$ and

$$\Gamma^* = \left\{ \begin{array}{l} \phi \in Y\colon \ \text{(i) } \phi(\xi) \text{ is nondecreasing in } \xi \in \mathbb{R}; \\ \qquad\quad \text{(ii) } \mathbf{0} \leq \lim\limits_{\xi \to -\infty} \phi(\xi) < \mathbf{K}, \qquad \lim\limits_{\xi \to +\infty} \phi(\xi) = \mathbf{K}; \\ \qquad\quad \text{(iii) } e^{\gamma\xi}\left[\phi(\xi + \eta) - \phi(\xi)\right] \text{is increasing in } \xi \in \mathbb{R} \text{ for every } \eta > 0 \end{array} \right\}, \tag{A.7}$$

$$\Gamma^{**} = \left\{ \begin{array}{l} \phi \in Y\colon \ \text{(i) } \phi(\xi) \text{ is nondecreasing in } \xi \in \mathbb{R}; \\ \qquad\quad \text{(ii) } \mathbf{0} \leq \lim\limits_{\xi \to -\infty} \phi(\xi) < \mathbf{K}, \qquad \lim\limits_{\xi \to +\infty} \phi(\xi) = \mathbf{K}; \\ \qquad\quad \text{(iii) } e^{\gamma\xi}\left[\phi(\xi + \eta) - \phi(\xi)\right] \text{ is increasing in } \xi \in \mathbb{R} \text{ and} \\ \qquad\quad e^{-\gamma\xi}\left[\phi(\xi + \eta) - \phi(\xi)\right] \text{ is decreasing in } \xi \in \mathbb{R} \text{ for every } \eta > 0 \end{array} \right\}. \tag{A.8}$$

Define an operator $F : BC[\mathbf{0}, \mathbf{K}] \to BC(\mathbb{R}, \mathbb{R}^n)$ by

$$F(\phi)(\xi) = f(\phi(\xi), (g_1 * \phi)(\xi), \dots, (g_m * \phi)(\xi)), \quad \xi \in \mathbb{R}. \tag{A.9}$$

Now we give definitions of the lower and upper solutions of (A.3) as follows.

Definition A.1. A continuous function $\phi : \mathbb{R} \to \mathbb{R}$ is called an upper solution of (A.3) if ϕ' and ϕ'' exist almost everywhere in $\mathbb{R}$ and are essentially bounded on $\mathbb{R}$, and if ϕ satisfies,

$$-D\phi''(\xi) + c\phi'(\xi) \geq f(\phi(\xi), (g_1 * \phi)(\xi), \dots, (g_m * \phi)(\xi)), \text{ a.e. in } \mathbb{R}. \tag{A.10}$$

A lower solution of (A.3) is defined in a similar way by reversing the inequality in (A.10).

Theorem A.2. *Assume that* (H_2), (H_3), *and* (H_0) *hold. Also assume that* ϕ *and* ψ, *where* $\phi \in BC[\mathbf{0}, \mathbf{K}] \cap \Upsilon$ *with* $\phi \neq \widehat{\mathbf{0}}$, $\lim_{\xi \to -\infty} \phi(\xi) = \mathbf{0}$ *and* $\phi \leq \psi$, *are lower and upper solutions of* (A.3), *respectively. Then*

(i) *if* (H_1^*) *holds,* $\psi \in \Gamma^*$ *and* $e^{\gamma\xi}[\psi(\xi) - \phi(\xi)]$ *is increasing in* $\xi \in \mathbb{R}$, *then for* $c > 1 - \min\{\gamma_i d_i, i = 1, \dots n\}$, (A.1) *has a traveling wave front* ϕ^* *such that* (A.4) *holds with* $\phi \leq \phi^* \leq \psi$ *and for* $a, b \in \mathbb{R}$ *with* $a < b$,

$$\|\psi^m - \phi^*\|_{C([a,b],\mathbb{R}^n)} \longrightarrow 0, \tag{A.11}$$

where

$$\begin{gathered} -D(\psi^m)'' + c(\psi^m)' + \gamma\psi^m = F\psi^{m-1} + \gamma\psi^{m-1} \quad (m \in N), \\ \phi \leq \phi^* \leq \cdots \leq \psi^m \leq \cdots \leq \psi^1 \leq \psi^0 = \psi, \end{gathered} \tag{A.12}$$

(ii) *if* (H_1^{**}) *holds,* $\psi \in \Gamma^{**}$, $e^{\gamma\xi}[\psi(\xi) - \phi(\xi)]$ *is increasing in* $\xi \in \mathbb{R}$ *and* $e^{-\gamma\xi}[\psi(\xi) - \phi(\xi)]$ *is decreasing in* $\xi \in \mathbb{R}$, *where* $\min\{\gamma_i d_i, i = 1, \dots, n\} - 1 > 0$, *then for* $0 < c < \min\{\gamma_i d_i, i = 1, \dots, n\} - 1$, (A.1) *has a traveling wave front* ϕ^* *such that* (A.4) *holds with* $\phi \leq \phi^* \leq \psi$ *and for* $a, b \in \mathbb{R}$ *with* $a < b$, *and* (A.11) *and* (A.12) *hold.*

In particular, if $\lim_{\xi \to -\infty} \psi(\xi) = \mathbf{0}$, *then* $\|\psi^m - \phi^*\| \to 0$.

Acknowledgments

The authors are very grateful to the anonymous referees for careful reading and helpful suggestions. H.-Q. Zhao is supported by the Scientific Research Program Funded by Shaanxi Provincial Education Department (no. 12JK0860) and the Specialized Research Fund of Xianyang Normal University (11XSYK202), and S.-Y. Liu is supported by the NSF of China (60974082).

References

[1] S. A. Gourley and J. W.-H. So, "Dynamics of a food-limited population model incorporating nonlocal delays on a finite domain," *Journal of Mathematical Biology*, vol. 44, no. 1, pp. 49–78, 2002.

[2] S. A. Gourley, “Wave front solutions of a diffusive delay model for populations of Daphnia magna,” *Computers & Mathematics with Applications*, vol. 42, no. 10-11, pp. 1421–1430, 2001.

[3] J. Wu and X. Zou, “Traveling wave fronts of reaction-diffusion systems with delay,” *Journal of Dynamics and Differential Equations*, vol. 13, no. 3, pp. 651–687, 2001.

[4] S. A. Gourley and M. A. J. Chaplain, “Travelling fronts in a food-limited population model with time delay,” *Proceedings of the Royal Society of Edinburgh A*, vol. 132, no. 1, pp. 75–89, 2002.

[5] N. Fenichel, “Geometric singular perturbation theory for ordinary differential equations,” *Journal of Differential Equations*, vol. 31, no. 1, pp. 53–98, 1979.

[6] J. Canosa, “On a nonlinear diffusion equation describing population growth,” *International Business Machines Corporation*, vol. 17, pp. 307–313, 1973.

[7] Z.-C. Wang and W.-T. Li, “Monotone travelling fronts of a food-limited population model with nonlocal delay,” *Nonlinear Analysis. Real World Applications*, vol. 8, no. 2, pp. 699–712, 2007.

[8] Z.-C. Wang, W.-T. Li, and S. Ruan, “Travelling wave fronts in reaction-diffusion systems with spatio-temporal delays,” *Journal of Differential Equations*, vol. 222, no. 1, pp. 185–232, 2006.

[9] P. Ashwin, M. V. Bartuccelli, T. J. Bridges, and S. A. Gourley, “Travelling fronts for the KPP equation with spatio-temporal delay,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 53, no. 1, pp. 103–122, 2002.

[10] S. A. Gourley, “Travelling front solutions of a nonlocal Fisher equation,” *Journal of Mathematical Biology*, vol. 41, no. 3, pp. 272–284, 2000.

[11] C. Ou and J. Wu, “Persistence of wavefronts in delayed nonlocal reaction-diffusion equations,” *Journal of Differential Equations*, vol. 235, no. 1, pp. 219–261, 2007.