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Research Article

Global Existence of Classical Solutions to a Three-Species Predator-Prey Model with Two Prey-Taxes

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We are concerned with three-species predator-prey model including two prey-taxes and Holling type II functional response under no flux boundary condition. By applying the contraction mapping principle, the parabolic Schauder estimates, and parabolic L^p estimates, we prove that there exists a unique global classical solution of this system.

1. Introduction

In addition to random diffusion of the predator and the prey, the spatial-temporal variations of the predators' velocity are directed by prey gradient. Several field studies measuring characteristics of individual movement confirm the basis of the hypothesis about the dependence of acceleration on a stimulus [1]. Understanding spatial and temporal behaviors of interacting species in ecological system is a central problem in population ecology. Various types of mathematical models have been proposed to study problem of predator-prey. Recently, the appearance of prey-taxis in relation to ecological interactions of species was studied by many scholars, ecologists, and mathematicians [2–5].

In [2] the authors proved the existence and uniqueness of weak solutions to the two-species predator-prey model with one prey-taxis. In [3], the author extended the results of [2] to an $n \times m$ reaction-diffusion-taxis system. In [4], the author proved the existence and uniqueness of classical solutions to this model. In this paper, we deal with three-species predator-prey model with two prey-taxes including Holling type II functional response as follows:

$$\begin{gathered}
\frac{\partial u_1}{\partial t}-d_1\Delta u_1+\nabla\cdot(\beta_1 u_1\nabla u_2)+\nabla\cdot(\beta_2 u_1\nabla u_3)=-au_1+\frac{e_2c_2u_1u_2}{m_2+b_2u_2}+\frac{e_3c_3u_1u_3}{m_3+b_3u_3}\quad \text{in } (0,T)\times\Omega,\\
\frac{\partial u_2}{\partial t}-d_2\Delta u_2=r_2\left(1-\frac{u_2}{K_2}\right)u_2-\frac{c_2u_1u_2}{m_2+b_2u_2}\quad \text{in } (0,T)\times\Omega,\\
\frac{\partial u_3}{\partial t}-d_3\Delta u_3=r_3\left(1-\frac{u_3}{K_3}\right)u_2-\frac{c_3u_1u_3}{m_3+b_3u_3}\quad \text{in } (0,T)\times\Omega,\\
\frac{\partial u_1}{\partial \nu}=\frac{\partial u_2}{\partial \nu}=\frac{\partial u_3}{\partial \nu}=0\quad \text{on } (0,T)\times\partial\Omega,\\
(u_1(0,x),u_2(0,x),u_3(0,x))=(u_{10}(x),u_{20}(x),u_{30}(x))\geq(0,0)\quad \text{in } \Omega,
\end{gathered}\tag{1.1}$$

where Ω is a bounded domain in $R^N(N\geq 1$ is an integer) with a smooth boundary $\partial\Omega$; u_1 and $u_i\,(i=2,3)$ represent the densities of the predator and prey, respectively; the positive constants d_1, d_2, and d_3 are the diffusion coefficient of the corresponding species; the positive constants $a, K_i, r_i, m_i, e_i, m_i/c_i, b_i/c_i, m_i/b_i\ (i=2,3)$ represent the death rate of the predator, the carrying capacity of prey, the prey intrinsic growth rate, the half-saturation constant, the conversion rate, the time spent by a predator to catch a prey, the manipulation time which is a saturation effect for large densities of prey, the density of prey necessary to achieve one-half the rate, respectively; the predators are attracted by the preys, and the positive constant $\beta_i\,(i=1,2)$ denotes their prey-tactic sensitivity. The parts $\beta_1u_1\nabla u_2$ and $\beta_2u_1\nabla u_3$ of the flux are directed toward the increasing population density of u_2 and u_3, respectively. In this way, the predators move in the direction of higher concentration of the prey species.

The aim of this paper is to prove that there is a unique classical solution to the model (1.1). It is difficult to deal with the two prey-taxes terms. To get our goal we employ the techniques developed by [6, 7] to investigate.

Throughout this paper we assume that

$$\beta_1=0,\qquad \beta_2=0,\quad \text{for } u_1\geq u_{1m}.\tag{1.2}$$

The assumptions that $\beta_1=0$ for $u_1\geq u_{1m}$ and $\beta_2=0$ for $u_1\geq u_{1m}$ have a clear biological interpretation [2]: the predators stop to accumulate at given point of Ω after their density attains a certain threshold value u_{1m} and the prey-tactic sensitivity β_1 and β_2 vanishes identically when $u_1\geq u_{1m}$.

Throughout this paper we also assume that

$$\begin{gathered}
u_{20},u_{30}\leq K,\quad \partial\Omega\in C^{2+\alpha},\quad u_{10}(x),u_{20}(x),u_{30}(x)\in C^{2+\alpha}\left(\overline{\Omega}\right),\quad \text{where } 0<\alpha<1,\\
\frac{\partial u_{10}}{\partial\nu}=\frac{\partial u_{20}}{\partial\nu}=\frac{\partial u_{30}}{\partial\nu}=0,\quad \text{on } \partial\Omega.
\end{gathered}\tag{1.3}$$

Denote by $C_{x,t}^{m+\alpha,\beta}(Q_T)$ ($m \geq 0$ is integer, $0 < \alpha < 1$, $0 < \beta < 1$) the space of function $u(x,t)$ with finite norm [8]:

$$\|u\|_{C_{x,t}^{m+\alpha,\beta}(Q_T)} = \sum_{|l|=0}^{m} \left[\sup_{Q_T} \left| D_x^l u \right| + \left\langle D_x^l u \right\rangle_{x,Q_T}^{(\alpha)} + \left\langle D_x^l u \right\rangle_{t,Q_T}^{(\beta)} \right], \tag{1.4}$$

where

$$\begin{aligned} \langle w \rangle_{x,Q_T}^{(\alpha)} &= \sum_{(x,t),(y,t)\in Q_T} \frac{\left| w(x,t) - w(y,t) \right|}{\left| x-y \right|^{\alpha}}, \\ \langle w \rangle_{t,Q_T}^{(\beta)} &= \sum_{(x,t),(x,\tau)\in Q_T} \frac{\left| w(x,t) - w(x,\tau) \right|}{\left| t-\tau \right|^{\beta}}. \end{aligned} \tag{1.5}$$

We denote by $C_{x,t}^{2+\alpha,1+\beta}(Q_T)$ the space of functions $u(x,t)$ with norm

$$\|u\|_{C_{x,t}^{2+\alpha,\beta}(Q_T)} + \|u_t\|_{C_{x,t}^{\alpha,\beta}(Q_T)}. \tag{1.6}$$

The main result of this paper is as follows.

Theorem 1.1. *Under assumptions* (1.2) *and* (1.3), *for any given* $T > 0$ *there exists a unique solution* $\mathbf{U} = (u_1, u_2, u_3) \in C^{2+\alpha,1+(\alpha/2)}(Q_T)$ *of the system* (1.1), *where* $Q_T = (0,T) \times \Omega$. *Moreover,*

$$u_1(x,t) \geq 0, \quad 0 \leq u_2(x,t) \leq K_2, \quad 0 \leq u_3(x,t) \leq K_3, \tag{1.7}$$

for any $x \in \Omega$ *and* $t > 0$.

This paper is organized as follows. In Section 2, we present some preliminary lemmas that will be used in proving later theorem. In Section 3, we prove local existence and uniqueness to system (1.1). In Section 4, we prove global existence to system (1.1).

2. Some Preliminaries

For the convenience of notations, in what follows we denote various constants which depend on T by N, while we denote various constants which are independent of T by N_0.

Lemma 2.1. *Let* $(u,x) \in C^{2+\alpha,\ 1+(\alpha/2)}(Q_T)$. *Then*

$$\|u(x,t) - u(x,0)\|_{C^{1+\alpha,\alpha/2}(Q_T)} \leq N_0 \eta(T) \|u\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)}, \tag{2.1}$$

where $\eta(T) = \max\{T^{\alpha/2},\ T^{(1-\alpha)/2}\}$.

Proof. Using the definition of Hölder norm, we have

$$
\begin{aligned}
\frac{|u(x,t)-u(x,0)|}{|t|^{(1+\alpha)/2}} &\leq |D_t u|\cdot|T|^{(1-\alpha)/2},\\
[u(x,t)-u(x,0)]_{C^{1+\alpha,0}(Q_T)} &\leq N_0\left\|D_x^2u(x,t)\ \ -D_x^2u(x,0)\right\|_{L^\infty(Q_T)}\\
&\leq N_0\left[D_x^2u\right]_{C^{0,\alpha/2}(Q_T)}\cdot|T|^{\alpha/2}\\
&\leq N_0[u]_{C^{2,\alpha/2}(Q_T)}\cdot|T|^{\alpha/2},
\end{aligned}
\tag{2.2}
$$

which yields that

$$
\|u(x,t)-u(x,0)\|_{C^{1+\alpha,(1+\alpha)/2}(Q_T)} \leq N_0\eta\|u\|_{C^{2+\alpha,(1+\alpha)/2}(Q_T)}. \tag{2.3}
$$

Therefore,

$$
\|u(x,t)-u(x,0)\|_{C^{1+\alpha,\alpha/2}(Q_T)} \leq N_0\eta\|u\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)}. \tag{2.4}
$$

□

We now consider the following nonlinear parabolic problem:

$$
\begin{gathered}
\frac{\partial u_1}{\partial t}-d_1\Delta u_1+\nabla\cdot\left(\beta_1u_1\nabla u_2\right)+\nabla\cdot\left(\beta_2u_1\nabla u_3\right)=u_1f \quad \text{in } (0,T)\times\Omega,\\
\frac{\partial u_1}{\partial \nu}=0 \quad \text{on } (0,T)\times\partial\Omega,\\
u_1(0,x)=u_{10}(x) \quad \text{in } \Omega.
\end{gathered}
\tag{2.5}
$$

By the parabolic maximum principle, we have $u_1(x,t)\geq 0$.

Lemma 2.2. *Let*

$$
\begin{gathered}
u_2(x,t),u_3(x,t)\in C^{2+\alpha,\ 1+(\alpha/2)}(Q_T)\ , \qquad f(x,t)\in C^{\alpha,\alpha/2}(Q_T),\\
\|u_2\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)},\|u_3\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)},\|f\|_{C^{\alpha,\alpha/2}(Q_T)}\leq N_0.
\end{gathered}
\tag{2.6}
$$

Then, under assumptions (1.2) *and* (1.3), *there exists a unique nonnegative solution* $u_1(x,t)\in C^{2+\alpha,1+(\alpha/2)}(Q_T)$ *of the nonlinear problem* (2.5) *for small* $T>0$ *which depends on* $\|u_{10}(x)\|_{C^{2+\alpha}(\Omega)}$.

Proof. This proof is similar to that of Lemma 2.1 in [4]. For reader's convenience we include the proof here. We will prove by a fixed point argument. Let us introduce the Banach space X

of function u_1 with norm $\|u_1\|_{C^{1+\alpha,\alpha/2}(Q_T)}(0 < T < 1)$ and a subset $X_A = \{u_1 \in X : u_1(x,0) = u_{10}(x)$ and $\|u_1\|_{C^{1+\alpha,\alpha/2}(Q_T)} \le A\}$, where $A = \|u_{10}(x)\|_{C^{2+\alpha}(\Omega)} + \|u_{20}(x)\|_{C^{2+\alpha}(\Omega)} + \|u_{30}(x)\|_{C^{2+\alpha}(\Omega)} + 1$. For any $u_1 \in X_A$, we define a corresponding function $\tilde{u}_1 = Fu_1$, where $\tilde{u}_1$ satisfies the equations

$$\begin{aligned} &\frac{\partial \tilde{u}_1}{\partial t} - d_1 \Delta \tilde{u}_1 - \tilde{u}_1 f = -\beta_1 \nabla u_1 \cdot \nabla u_2 - \beta_2 \nabla u_1 \cdot \nabla u_3 - \beta_1 u_1 \Delta u_2 - \beta_2 u_1 \Delta u_3 \quad \text{in } (0,T)\times\Omega, \\ &\frac{\partial \tilde{u}_1}{\partial \nu} = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ &\tilde{u}_1(0,x) = u_{10}(x) \quad \text{in } \Omega. \end{aligned} \tag{2.7}$$

By $u_1 \in X_A$, we have

$$h_1 \triangleq -\beta_1 \nabla u_1 \cdot \nabla u_2 - \beta_2 \nabla u_1 \cdot \nabla u_3 - \beta_1 u_1 \Delta u_2 - \beta_2 u_1 \Delta u_3 \in C^{\alpha,\alpha/2}(Q_T). \tag{2.8}$$

By the parabolic Schauder theory, this yields that there exists a unique solution $\tilde{u}$ to (2.7) and

$$\begin{aligned} \|\tilde{u}_1\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} &\le \|\tilde{u}_1|_{t=0}\|_{C^{2+\alpha}(\Omega)} + M_1(A) \\ &\le A + M_1(A) \triangleq M_2(A), \end{aligned} \tag{2.9}$$

where M_2(A) is some constant which depends only on A. For any function $\tilde{u}_1(x,t)$, by Lemma 2.1 and combining (2.9), if T is sufficiently small (T depends only on A), then we have

$$\begin{aligned} \|\tilde{u}_1(x,t)\|_{C^{1+\alpha,\alpha/2}(Q_T)} &\le \|\tilde{u}_1(x,0)\|_{C^{1+\alpha,\alpha/2}(Q_T)} + N_0 \eta(T) \|\tilde{u}_1\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \\ &\le \|\tilde{u}_{10}(x)\|_{C^{2+\alpha}} + 1 \le A. \end{aligned} \tag{2.10}$$

Therefore, $\tilde{u}_1(x,t) \in X_A$ and F maps X_A into itself. We now prove that F is contractive. Take u_{11}, u_{12} in X_A, and set $\tilde{u}_{11} = Fu_{11}$, $\tilde{u}_{12} = Fu_{12}$, $\tilde{v} = \tilde{u}_{11} - \tilde{u}_{12}$. Then, it follows from (2.7) that $\tilde{v}$ solves the following systems:

$$\begin{aligned} &\frac{\partial \tilde{v}}{\partial t} - d_1 \Delta \tilde{v} - \tilde{v} f = h_2 \quad \text{in } (0,T)\times\Omega, \\ &\frac{\partial \tilde{v}}{\partial \nu} = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ &\tilde{v}(0,x) = 0 \quad \text{in } \Omega, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} h_2 \triangleq & -\beta_1(\nabla u_{11} - \nabla u_{12}) \cdot \nabla u_2 - \beta_2(\nabla u_{11} - \nabla u_{12}) \cdot \nabla u_3 \\ & - \beta_1(u_{11} - u_{12})\Delta u_2 - \beta_2(u_{11} - u_{12})\Delta u_3. \end{aligned} \tag{2.12}$$

By $u_{11}, u_{12} \in X_A$ and conditions of Lemma 2.2, it is easy to check that

$$\begin{aligned}\|h_2\|_{L^\infty(Q_T)} &\le \left\|\beta_1(\nabla u_{11} - \nabla u_{12}) \cdot \nabla u_2\right\|_{C^0(Q_T)} + \left\|\beta_2(\nabla u_{11} - \nabla u_{12}) \cdot \nabla u_3\right\|_{C^0(Q_T)} \\ &\quad + \left\|\beta_1(u_{11} - u_{12})\Delta u_2\right\|_{C^0(Q_T)} + \left\|\beta_2(u_{11} - u_{12})\Delta u_3\right\|_{C^0(Q_T)} \\ &\le N_0\|u_{11} - u_{12}\|_{C^{1,0}} + N_0\|u_{11} - u_{12}\|_{C^0(Q_T)} \\ &\le N_0\|u_{11} - u_{12}\|_{C^{1,0}(Q_T)}.\end{aligned} \tag{2.13}$$

Using the assumption $\|f\|_{\alpha,\alpha/2} \le N_0$ and the L^p-estimate, we have

$$\|f\|_{L^\infty(Q_T)} \le \|f\|_{C^{\alpha,\alpha/2}} \le N_0, \qquad \|\tilde{v}\|_{w_p^{2,1}} \le N_0\|h_2\|_{L^\infty(Q_T)}. \tag{2.14}$$

For any $p \ge 1$, by using Sobolev embedding $W_p^{2,1}(Q_T) \hookrightarrow C^{1+\gamma,(1+\gamma)/2}(Q_T)$ ($\gamma = 1 - (5/p) > \alpha$ if we take p sufficiently large), we have

$$\|\tilde{v}\|_{C^{1+\gamma,(1+\gamma)/2}(Q_T)} \le N_0\|h_2\|_{L^\infty(Q_T)} \le N_0\|u_{11} - u_{12}\|_{C^{1,0}(Q_T)}. \tag{2.15}$$

Then, noting $\gamma > \alpha$, we can easily check that [4]

$$\|\tilde{v}\|_{C^{1+\alpha,\alpha/2}(Q_T)} \le N_0 T^{\alpha/2}\|u_{11} - u_{12}\|_{C^{1+\alpha,\alpha/2}(Q_T)}. \tag{2.16}$$

Taking T small such that $N_0 T^{\alpha/2} < 1/2$, we conclude from (2.16) that F is contractive in X_A. Therefore F has a unique fixed point u_1, which is the unique solution to (2.5). Moreover, we can raise the regularity of u_1 to $C^{2+\alpha,1+(\alpha/2)}(Q_T)$ by using the parabolic Schauder estimates. □

3. Local Existence and Uniqueness of Solutions

In this section, we will prove Theorem 3.1 which show that system (1.1) has a unique solution $\mathbf{U}(x,t) = (u_1, u_2, u_3) \in C^{2+\alpha,1+(\alpha/2)}(Q_T)$ as done in [6, 7].

Theorem 3.1. *Assume that* (1.2) *and* (1.3) *hold, then there exists a unique solution* $\mathbf{U}(x,t) = (u_1, u_2, u_3) \in C^{2+\alpha,1+(\alpha/2)}(Q_T)$ *of the system* (1.2) *for small* $T > 0$ *which depends on*

$$\|\mathbf{U}_0(x)\|_{C^{2+\alpha}(\Omega)} \triangleq \|u_{10}(x)\|_{C^{2+\alpha}(\Omega)} + \|u_{20}(x)\|_{C^{2+\alpha}(\Omega)} + \|u_{30}(x)\|_{C^{2+\alpha}(\Omega)}. \tag{3.1}$$

Furthermore, $u_1(x,t) \ge 0,\ u_2(x,t) \ge 0,\ u_3(x,t) \ge 0$.

Proof. We will prove the local existence by a fixed point argument again. Introducing the Banach space X of the function $\mathbf{U}$, we define the norm

$$\|\mathbf{U}\|_{C^{\alpha,\alpha/2}(Q_T)} = \|u_1\|_{C^{\alpha,\alpha/2}(Q_T)} + \|u_2\|_{C^{\alpha,\alpha/2}(Q_T)} + \|u_3\|_{C^{\alpha,\alpha/2}(Q_T)} \quad (0 < T < 1), \tag{3.2}$$

and a subset

$$X_A = \left\{ \mathbf{U} \in X : u_1, u_2, u_3 \geq 0,\ \|\mathbf{U}\|_{C^{\alpha,\alpha/2}(Q_T)} \leq A \right\}, \tag{3.3}$$

where

$$\begin{gathered} \mathbf{U}(x,0) = (u_{10}(x), u_{20}(x), u_{30}(x)), \\ A = \|u_{10}(x)\|_{C^{2+\alpha}(\Omega)} + \|u_{20}(x)\|_{C^{2+\alpha}(\Omega)} + \|u_{30}(x)\|_{C^{2+\alpha}(\Omega)} + 1. \end{gathered} \tag{3.4}$$

For any $\mathbf{U} \in X_A$, we define correspondingly function $\overline{\mathbf{U}} = \mathbf{H}\mathbf{U}$ by $\overline{\mathbf{U}} = (\overline{u}_1, \overline{u}_2, \overline{u}_3)$, where $\overline{\mathbf{U}}$ satisfies the equations

$$\begin{gathered} \frac{\partial \overline{u}_2}{\partial t} - d_2 \Delta \overline{u}_2 = \left[r_2 \left(1 - \frac{u_2}{K_2} \right) - \frac{c_2 u_1}{m_2 + b_2 u_2} \right] \overline{u}_2 \quad \text{in } (0,T) \times \Omega, \\ \frac{\partial \overline{u}_3}{\partial t} - d_3 \Delta \overline{u}_3 = \left[r_3 \left(1 - \frac{u_3}{K_3} \right) - \frac{c_3 u_1}{m_3 + b_3 u_3} \right] \overline{u}_3 \quad \text{in } (0,T) \times \Omega, \\ \frac{\partial \overline{u}_2}{\partial \nu} = \frac{\partial \overline{u}_3}{\partial \nu} = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ \overline{u}_2(x,0) = u_{20}(x), \quad \overline{u}_3(x,0) = u_{30}(x), \quad x \in \Omega, \end{gathered} \tag{3.5}$$

$$\begin{gathered} \frac{\partial \overline{u}_1}{\partial t} - d_1 \Delta \overline{u}_1 + \nabla \cdot (\beta_1 \overline{u}_1 \nabla \overline{u}_2) + \nabla \cdot (\beta_2 \overline{u}_1 \nabla \overline{u}_3) = -a\overline{u}_1 + \frac{e_2 c_2 \overline{u}_1 u_2}{m_2 + b_2 u_2} + \frac{e_3 c_3 \overline{u}_1 u_3}{m_3 + b_3 u_3} \quad \text{in } (0,\mathrm{T}) \times \Omega, \\ \frac{\partial \overline{u}_1}{\partial \nu} = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ \overline{u}_1(x,0) = u_{10}(x), \quad x \in \Omega. \end{gathered} \tag{3.6}$$

By (3.5), $(u_1, u_2, u_3) \in X_A$, assumption (1.3), and the parabolic Schauder theory, we have that there exists a unique solution $\overline{u}_2, \overline{u}_3$ to (3.5) and

$$\|\overline{u}_2\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \leq \|\overline{u}_2|_{t=0}\|_{C^{2+\alpha}} + M_3(A) \leq A + M_3(A) \triangleq M_4(A). \tag{3.7}$$

Similarly,

$$\|\overline{u}_3\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \leq \|\overline{u}_3|_{t=0}\|_{C^{2+\alpha}} + M_5(A) \leq A + M_5(A) \triangleq M_6(A). \tag{3.8}$$

Moreover, by parabolic maximum principle, we have

$$\overline{u}_2(x,t) \geq 0 \quad \text{in } Q_T, \qquad \overline{u}_3(x,t) \geq 0 \quad \text{in } Q_T. \tag{3.9}$$

Similarly, by using Lemma 2.2, from (3.6) we can conclude that there exists a unique solution $\overline{u}_1$ satisfying

$$\|\overline{u}_1\|_{C^{2+\alpha,1+(\alpha/2)}Q_T} \leq M_7(A), \tag{3.10}$$

and by parabolic maximum principle we have $\overline{u}_1(x,t) \geq 0$ in Q_T. From (3.7), (3.8), and (3.10), we have

$$\left\|\overline{\mathbf{U}}\right\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \leq M_8(A). \tag{3.11}$$

For any function $\overline{\mathbf{U}}(x,t)$, using Lemma 2.1 we get

$$\left\|\overline{\mathbf{U}}(x,t) - \overline{\mathbf{U}}(x,0)\right\|_{C^{\alpha,\alpha/2}(Q_T)} \leq N_0\eta(T)\|\mathbf{U}\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)}. \tag{3.12}$$

From (3.11) and (3.12), if T is sufficiently small we have

$$\begin{aligned}\left\|\overline{\mathbf{U}}(x,t)\right\|_{C^{\alpha,\alpha/2}(Q_T)} &\leq \left\|\overline{\mathbf{U}}(x,0)\right\|_{C^{\alpha,\alpha/2}} + N_0\eta(T)M_8(A)\\ &\leq \|\mathbf{U}_0(x)\|_{C^{2+\alpha}(\Omega)} + 1 \equiv A,\end{aligned} \tag{3.13}$$

which yields $\overline{\mathbf{U}} \in X_A$. Therefore, H maps X_A into itself.

Next, we can prove that H is contractive as done in the proof of Lemma 2.2 in X_A if we take T sufficiently small. By the contraction mapping theorem H has a unique fixed point $\mathbf{U}$, which is the unique solution of (1.1). Moreover, we can raise the regularity of $\mathbf{U}$ to $C^{2+\alpha,1+(\alpha/2)}(Q_T)$ by using the parabolic Schauder estimates. □

4. Global Existence

First we establish some a priori estimates to (1.1).

Lemma 4.1. *Suppose that* $\mathbf{U} = (u_1, u_2, u_3) \in C^{2,1}(Q_T)$ *is a solution to the system* (1.1), *then there holds*

$$u_1 \geq 0, \quad 0 \leq u_2 \leq K_2, \quad 0 \leq u_3 \leq K_3. \tag{4.1}$$

Proof. It follows from (1.1) that

$$\begin{gathered}
\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 + (\beta_1 \nabla u_2 + \beta_2 \nabla u_3) \cdot \nabla u_1 \\
+ \left(\beta_1 \Delta u_2 + \beta_2 \Delta u_3 + a - \frac{e_2 c_2 u_2}{m_2 + b_2 u_2} - \frac{e_3 c_3 u_3}{m_3 + b_3 u_3} \right) u_1 = 0 \quad \text{in } (0,T) \times \Omega, \\
\frac{\partial u_1}{\partial \nu} = 0 \quad \text{on } (0,T) \times \partial\Omega, \\
u_1(0,x) = u_{10}(x) \geq 0 \quad \text{in } \Omega.
\end{gathered} \tag{4.2}$$

Obviously, $\underline{u}_1 \equiv 0$ is a subsolution to (4.2). Using the maximum principle, we get $u_1 \geq 0$. Similarly, we have $u_2 \geq 0$ and $u_3 \geq 0$.

On the other hand, it follows from model (1.1) that

$$\begin{gathered}
\frac{\partial u_2}{\partial t} - d_2 \Delta u_2 - r_2 \left(1 - \frac{u_2}{K_2} \right) u_2 + \frac{c_2 u_1 u_2}{m_2 + b_2 u_2} = 0 \leq \frac{c_2 K_2 u_1}{m_2 + b_2 K_2} \quad \text{in } (0,T) \times \Omega, \\
\frac{\partial u_2}{\partial \nu} = 0 \quad \text{on } (0,T) \times \partial\Omega, \\
u_2(0,x) = u_{20}(x) \quad \text{in } \Omega,
\end{gathered} \tag{4.3}$$

which implies that K_2 is a subsolution to problem (4.3). Hence we have $0 \leq u_2(x,t) \leq K_2$. Similarly, we get $0 \leq u_3(x,t) \leq K_3$. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Suppose that* $\mathbf{U} = (u_1, u_2, u_3) \in C^{2,1}(Q_T)$ *is a solution to the system* (1.1), *then for any* $p > 1$ *there holds*

$$\|u_1\|_{L^p(Q_T)} \leq N, \qquad \|u_2\|_{L^p(Q_T)} \leq N, \qquad \|u_2\|_{L^p(Q_T)} \leq N. \tag{4.4}$$

Proof. Multiplying the first equation of (1.1) by u_1^{p-1}, integrating over Q_T, using the no-flux boundary condition, and noting $u_1 \geq 0$, we get

$$\begin{aligned}
&\frac{1}{p} \int_\Omega u_1^p(x,t) dx - \frac{1}{p} \int_\Omega u_0^p(x,t) dx + (p-1) d_1 \int_0^t \int_\Omega u_1^{p-2} |\nabla u_1|^2 dx\, dt \\
&\quad \leq (p-1) \int_0^t \int_\Omega \beta_1 u_1^{p-1} \nabla u_1 \cdot \nabla u_2 dx\, dt + (p-1) \int_0^t \int_\Omega \beta_2 u_1^{p-1} \nabla u_1 \cdot \nabla u_3 dx\, dt \\
&\qquad + \frac{e_2 c_2}{b_2} \int_0^t \int_\Omega u_1^p dx\, dt + \frac{e_3 c_3}{b_3} \int_0^t \int_\Omega u_1^p dx\, dt.
\end{aligned} \tag{4.5}$$

For $u_1 \geq u_{1m}$, we get

$$\begin{aligned}
&\frac{1}{p}\int_\Omega u_1^p(x,t)dx - \frac{1}{p}\int_\Omega u_0^p(x,t)dx + (p-1)d_1\int_0^t\int_\Omega u_1^{p-2}|\nabla u_1|^2 dx\,dt \\
&\quad \leq \left(\frac{e_2c_2}{b_2} + \frac{e_3c_3}{b_3}\right)\int_0^t\int_\Omega u_1^p dx\,dt.
\end{aligned} \tag{4.6}$$

Therefore

$$\int_\Omega u_1^p(x,t)dt \leq N_0 + N_0\int_0^t\int_\Omega u_1^p dx\,dt. \tag{4.7}$$

Using Gronwall's Lemma, we have

$$\int_0^t\int_\Omega u_1^p(x,t)dt \leq N. \tag{4.8}$$

Therefore, for $u_1 < u_{1m}$, we have

$$\int_0^t\int_\Omega u_1^p(x,t)dt \leq \int_0^t\int_\Omega u_{1m}^p(x,t)dt \leq N. \tag{4.9}$$

Obviously, we have

$$\begin{aligned}
&\int_0^t\int_\Omega u_2^p(x,t)dt \leq \int_0^t\int_\Omega K_2^p(x,t)dt \leq N, \\
&\int_0^t\int_\Omega u_3^p(x,t)dt \leq \int_0^t\int_\Omega K_3^p(x,t)dt \leq N.
\end{aligned} \tag{4.10}$$

This completes the proof of Lemma 4.2. □

Lemma 4.3. *Suppose that* $\mathbf{U} = (u_1, u_2, u_3) \in C^{2,1}(Q_T)$ *is a solution to the system* (1.1)*, then for any* $p > 5$ *there holds*

$$\|u_1\|_{w_p^{2,1}(Q_T)} \leq N, \qquad \|u_2\|_{w_p^{2,1}(Q_T)} \leq N, \qquad \|u_3\|_{w_p^{2,1}(Q_T)} \leq N. \tag{4.11}$$

Proof. Note that the second equation of (1.1) can be rewritten as follows:

$$\frac{\partial u_2}{\partial t} - d_2\Delta u_2 - \left(r_2 - \frac{r_2}{K_2}u_2 - \frac{c_2u_1}{m_2 + b_2u_2}\right)u_2 = 0, \tag{4.12}$$

where $\|r_2 - (r_2/K_2)u_2 - (c_2u_1/(m_2 + b_2u_2))\|_{L^p(Q_T)} \leq N$.

By the parabolic L^p-estimate, we have

$$\|u_2\|_{w_p^{2,1}(Q_T)} \leq N. \tag{4.13}$$

Using the Sobolev embedding theorem (taking $p > 5$), we get

$$\|\nabla u_2\|_{L^\infty(Q_T)} \leq N. \tag{4.14}$$

Similarly, we can obtain

$$\begin{aligned} &\|u_3\|_{w_p^{2,1}(Q_T)} \leq N, \\ &\|\nabla u_3\|_{L^\infty(Q_T)} \leq N. \end{aligned} \tag{4.15}$$

It follows from the first equation of (1.1) that

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 + (\beta_1 \nabla u_2 + \beta_2 \nabla u_3) \cdot \nabla u_1 \\ &\quad = -\left(\beta_1 \Delta u_2 + \beta_2 \Delta u_3 + a - \frac{e_2 c_2 u_2}{m_2 + b_2 u_2} - \frac{e_3 c_3 u_3}{m_3 + b_3 u_3}\right) u_1 \quad \text{in } (0,T) \times \Omega, \\ &\frac{\partial u_1}{\partial \nu} = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ &u_1(0,x) = u_{10}(x) \geq 0 \quad \text{in } \Omega, \end{aligned} \tag{4.16}$$

where

$$\left\| -\left(\beta_1 \Delta u_2 + \beta_2 \Delta u_3 + a - \frac{e_2 c_2 u_2}{m_2 + b_2 u_2} - \frac{e_3 c_3 u_3}{m_3 + b_3 u_3}\right) u_1 \right\|_{L^p(Q_T)} \leq N. \tag{4.17}$$

Using the parabolic L^p-estimates again, we have

$$\|u_1\|_{w_p^{2,1}(Q_T)} \leq N. \tag{4.18}$$

This completes the proof of Lemma 4.3. □

Lemma 4.4. *Suppose that* $\mathbf{U} = (u_1, u_2, u_3) \in C^{2,1}(Q_T)$ *is a solution to the system* (1.1), *then there holds*

$$\|u_1\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \leq N, \qquad \|u_2\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \leq N, \qquad \|u_3\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \leq N. \tag{4.19}$$

Proof. Using the Sobolev embedding theorem (taking $p > 5$) and Lemma 4.3, we have

$$\|u_1\|_{C^{\alpha,\alpha/2}(Q_T)} \leq N, \qquad \|u_2\|_{C^{\alpha,\alpha/2}(Q_T)} \leq N, \qquad \|u_3\|_{C^{\alpha,\alpha/2}(Q_T)} \leq N. \tag{4.20}$$

Using (4.20) and the Schauder estimates to the second and third equation of model (1.1), we have

$$\|u_2\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \le N, \quad \|u_3\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \le N. \tag{4.21}$$

Applying the parabolic Schauder estimate to (4.16) and using (4.21), we have

$$\|u_1\|_{C^{2+\alpha,1+(\alpha/2)}(Q_T)} \le N. \tag{4.22}$$

This completes the proof of Lemma 4.4. □

Therefore, we can extend the local solution established in Theorem 3.1 to all $t > 0$, as done in [6, 7]. Namely, we have the following.

Theorem 4.5. *Under assumptions* (1.2) *and* (1.3), *there exists a unique solution* $\mathbf{U} = (u_1, u_2, u_3) \in C^{2+\alpha,1+(\alpha/2)}(Q_T)$ *of the system* (1.2) *for any given* $T > 0$. *Moreover,*

$$u_1(x,t) \ge 0, \qquad 0 \le u_2 \le K_2, \qquad 0 \le u_3 \le K_3. \tag{4.23}$$

for any $x \in \Omega$ *and* $t > 0$.

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