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Research Article

Existence Results for General Mixed Quasivariational Inequalities

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We consider and study a new class of variational inequality, which is called the general mixed quasivariational inequality. We use the auxiliary principle technique to study the existence of a solution of the general mixed quasivariational inequality. Several special cases are also discussed. Results proved in this paper may stimulate further research in this area.

1. Introduction

Variational inequalities theory, which was introduced and studied in the 1960's has seen a dramatic increase in its application in various branches of pure and applied sciences. Variational inequalities have been extended and generalized in various directions using novel and innovative ideas. A useful and important generalization is called the mixed quasi variational inequality involving the bifunction. It has been shown that a wide class of problems which arise in the elasticity with nonlocal friction laws, fluid flow thorough porous media and structural analysis can be studied in the unified framework of the mixed quasi variational inequalities, see [1–12].

In recent years, Noor [13] has shown that the optimality conditions of the differentiable nonconvex functions involving one arbitrary functions can be characterized by a class of variational inequalities, which is called the general variational inequalities. We would like to mention that one can show that the minimum of the sum of differentiable nonconvex (g-convex) function and a nondifferentiable g-convex bifunction can be characterized by a class of variational inequality. Motivated by this result, we introduce a new class of mixed variational inequalities, which is called *general mixed quasi variational inequality* involving the bifunction and three different operators. Due to the presence of the bifunction, projection and resolvent operator techniques, and their variation forms cannot be extended for solving the general mixed quasi variational inequalities. Thanks to the auxiliary principle technique, one can overcome this drawback. This technique is mainly due to Glowinski et al. [3]. This technique is more flexible and has been used to develop several numerical methods for solving the variational inequalities and the equilibrium problems. Noor [10, 14] has used this technique to study the existence of the general mixed quasi variational inequalities. This technique deals with considering an auxiliary problem and proving that the solution of the auxiliary problem is the solution of the original problem using the fixed point theory. This technique does not involve projection and resolvent of the operator. We again use the auxiliary principle technique to study the existence of a solution of the general mixed quasi variational inequalities, which is the main result (Theorem 3.1). We use this technique to suggest and analyze an iterative method for solving the general mixed quasi variational inequalities. Since the general mixed quasi variational inequalities include various classes of variational inequalities and complementarity problems as special cases, results proved in this paper continue to hold for these problems. Results proved in this paper may be viewed as important and significant improvement of the previously known results. It is interesting to explore the applications of these general variational inequalities in mathematical and engineering sciences with new and novel aspects. This may lead to new research in this field.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed and convex set in H. Let $\varphi(\cdot, \cdot) : H \times \to R \cup \{\infty\}$ be a continuous bifunction.

For given nonlinear operators $T, g, h : H \to H$, consider the problem of finding $u \in H$ such that

$$\langle Tu, g(v) - u \rangle + \varphi(g(v), u) - \varphi(u, u) \geq 0, \quad \forall v \in H : g(v) \in H. \tag{2.1}$$

Inequality of type (2.1) is called the *general mixed quasi variational inequality.*

One can show that the minimum of sum of differentiable nonconvex (g-convex) function and a class of nondifferentiable nonconvex (g-convex) function on the g-convex set K in H can be characterized by general mixed quasi variational inequality (2.1). For this purpose, we recall the following well-known concepts, see [10, 13, 15–18].

Definition 2.1. Let K be any set in H. The set K is said to be g-convex if there exists a function $g : H \to H$ such that

$$u + t(g(v) - u) \in K, \quad \forall u, v \in H : u,\ g(v) \in K,\ t \in [0,1]. \tag{2.2}$$

Note that every convex set is g-convex, but the converse is not true, see [15].

Definition 2.2. The function $F : K \to H$ is said to be g-convex on the g-convex set K if there exists a function g such that

$$F(u + t(g(v) - u)) \leq (1 - t)F(u) + tF(g(v)), \quad \forall u, v \in H : u,\ g(v) \in K,\ t \in [0,1]. \tag{2.3}$$

Clearly every convex function is g-convex, but the converse is not true

$$I[v] = F(v) + \varphi(v, v), \qquad \forall v \in H. \tag{2.4}$$

Using the technique of Noor [12, 18, 19], one can easily show the minimum of a differentiable g-convex function, and nondifferentiable nonconvex bifunction on a g-convex set K in H can be characterized by the general mixed quasi variational inequality (2.1).

Lemma 2.3. *Let $F : K \to H$ be a differentiable g-convex function on the g-convex set K. Then $u \in K$ is the minimum of the functional $I[v]$ defined by* (2.4) *on the g-convex set K if and only if $u \in K$ satisfies the inequality*

$$\langle F'(u), g(v) - u\rangle + \varphi(g(v), u) - \varphi(u, u) \geq 0, \qquad \forall v \in H : g(v) \in K, \tag{2.5}$$

where $F'(u)$ is the differential of F at $u \in K$.

Lemma 2.3 implies that g-convex programming problem can be studied via the general mixed variational inequality (2.1) with $Tu = F'(u)$.

We now list some special cases of the general mixed quasi variational inequality (2.1).

(I) For $g = I$, the identity operator, the general mixed quasi variational inequality (2.1) is equivalent to finding $u \in H$ such that

$$\langle Tu, v - u\rangle + \varphi(u, u) - \varphi(u, u) \geq 0, \qquad \forall v \in H : g(v) \in H, \tag{2.6}$$

which is also called the mixed quasi variational inequality, see [1–3, 8, 9, 11, 12].

(II) If the bifunction $\varphi(\cdot, \cdot)$ is the indicator function of a closed convex-valued set $K(u)$ in H, that is,

$$\varphi(u, u) = K_{(u)}(u) = \begin{cases} 0, & u \in K(u), \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.7}$$

then problem (2.1) is equivalent to finding $u \in K(u)$ such that

$$\langle Tu, g(v) - u\rangle \geq 0, \quad \forall v \in H : g(v) \in K(u). \tag{2.8}$$

Problems of type (2.8) are called general quasi variational inequalities.

(III) If $K(u) \equiv K$, the convex set, then problem (2.8) is equivalent to finding $u \in K$ such that

$$\langle Tu, g(v) - u\rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.9}$$

which is called the general variational inequality, introduced and studied by Noor [13].

(IV) If $g = I$, the identity operator, then, problem (2.5) is equivalent to finding $u \in K(u)$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K(u), \tag{2.10}$$

which is called the quasi variational inequality.

(V) If $K(u) = K$, the convex set, then problem (2.10) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.11}$$

which is called the classical variational inequality, introduced and studied by Stampacchia [20]. For the applications, formulations, generalizations, numerical method and other aspects of the variational inequalities, see [1–30] and the references therein.

We would like to mention that one can obtain several known and new classes of variational inequalities as special cases of the problem (2.1). From the above discussion, it is clear that the general mixed quasi variational inequalities (2.1) is most general and includes several previously known classes of variational inequalities and related optimization problems as special cases. These variational inequalities have important applications in mathematical programming and engineering sciences.

We also need the following concepts and results.

Definition 2.4. For all $u, v \in H$, an operator $T : H \to H$ is said to be

(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|u - v\|^2, \tag{2.12}$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|u - v\|. \tag{2.13}$$

From (i) and (ii), it follows that $\alpha \leq \beta$.

Definition 2.5. The bifunction $\varphi(\cdot,\cdot)$ is said to be *skew symmetric,* if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0, \quad \forall u, v \in H. \tag{2.14}$$

Clearly, if the bifunction $\varphi(\cdot,\cdot)$ is linear in both arguments, then

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) = \varphi(u - v, u - v) \geq 0, \quad \forall u, v \in H, \tag{2.15}$$

which shows that the bifunction $\varphi(\cdot,\cdot)$ is nonnegative.

Remark 2.6. It is worth mentioning that the points $(u,u),(u,v),(v,u)$, and (v,v) make up a set of the four vertices of the square. In fact, the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ can be written in the form

$$\frac{1}{2}\varphi(u,u)+\frac{1}{2}\varphi(v,v)\geq\frac{1}{2}\varphi(u,v)+\frac{1}{2}\varphi(v,u),\quad \forall u,v\in H. \tag{2.16}$$

This shows that the arithmetic average value of the skew-symmetric bifunction calculated at the north-east and south-west vertices of the square is greater than or equal to the arithmetic average value of the skew-symmetric bifunction computed at the north-west and south-west vertices of the same square. The skew-symmetric bifunction has the properties which can be considered an analogs of monotonicity of gradient and nonnegativity of a second derivative for the convex functions.

3. Main Results

In this Section, we use the auxiliary principle technique of Glowinski et al. [3] to study the existence of a solution of the general mixed quasi variational inequality (2.1).

Theorem 3.1. *Let T be a strongly monotone with constant $\alpha>0$ and Lipschitz continuous with constant $\beta>0$. Let g be a strongly monotone and Lipschitz continuous operator with constants $\sigma>0$ and $\delta>0$, respectively. Let the bifunction $\varphi(\cdot,\cdot)$ be skew symmetric. If there exists a constant $\rho>0$ such that*

$$\left|\rho-\frac{\alpha}{\beta^2}\right|<\frac{\sqrt{\alpha^2-\beta^2k(2-k)}}{\beta^2},\quad \alpha>\beta\sqrt{k(2-k)},\ k<1, \tag{3.1}$$

where

$$\theta=k+\sqrt{1-2\rho\alpha+\rho^2\beta^2}, \tag{3.2}$$

$$k=\sqrt{1-2\sigma+\delta^2}. \tag{3.3}$$

then the general mixed quasi variational inequality (2.1) *has a unique solution.*

Proof. We use the auxiliary principle technique to prove the existence of a solution of (2.1). For a given $u\in K$ satisfying the general mixed quasi variational inequality (2.1), we consider the problem of finding a solution $w\in K$ such that

$$\langle\rho Tu+w-g(u),g(v)-w\rangle+\rho\varphi(g(v),w)-\rho\varphi(w,w)\geq0,\quad \forall v\in H:g(v)\in K, \tag{3.4}$$

where $\rho>0$ is a constant. The inequality of type (3.4) is called the auxiliary general mixed quasi variational inequality associated with the problem (2.1). It is clear that the relation (3.4) defines a mapping $u\rightarrow w$. It is enough to show that the mapping $u\rightarrow w$ defined by the relation (3.4) has a unique fixed point belonging to H satisfying the general mixed quasi variational inequality (2.1). Let $w_1 \neq w_2$ be two solutions of (3.4) related to $u_1, u_2 \in H$, respectively. It is sufficient to show that for a well chosen $\rho > 0$,

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|, \tag{3.5}$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $g(v) = w_2$ (respectively w_1) in (3.4) related to u_1 (respectively u_2), adding the resultant and using the skew-symmetry of the bifunction $\varphi(\cdot,\cdot)$, we have

$$\langle w_1 - w_2, w_1 - w_2 \rangle \leq \langle g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2),\ w_1 - w_2 \rangle, \tag{3.6}$$

from which we have

$$\begin{aligned} \|w_1 - w_2\| &\leq \left\| g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2) \right\| \\ &\leq \left\| u_1 - u_2 - \left(g(u_1) - g(u_2)\right) \right\| + \left\| u_1 - u_2 - \rho(Tu_1 - Tu_2) \right\|. \end{aligned} \tag{3.7}$$

Since T is both strongly monotone and Lipschitz continuous operator with constants $\alpha > 0$ and $\beta > 0$ respectively, it follows that

$$\begin{aligned} \left\| u_1 - u_2 - \rho(Tu_1 - Tu_2) \right\|^2 &\leq \|u_2 - u_2\|^2 - 2\rho\langle u_1 - u_2, Tu_1 - Tu_2 \rangle + \rho^2 \|Tu_1 - Tu_2\|^2 \\ &\leq \left(1 - 2\rho\alpha + \rho^2\beta^2\right) \|u_1 - u_2\|^2. \end{aligned} \tag{3.8}$$

In a similar way, using the strongly monotonicity with constant $\sigma > 0$ and Lipschitz continuity with constant $\delta > 0$, we have

$$\left\| u_1 - u_2 - \left(g(u_1) - g(u_2)\right) \right\| \leq \sqrt{1 - 2\sigma + \delta^2} \|u_1 - u_2\|. \tag{3.9}$$

From (3.7), (3.8), and (3.9) and using the fact that the operator h is firmly expanding, we have

$$\begin{aligned} \|w_1 - w_2\| &\leq \left\{ k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \right\} \|u_1 - u_2\| \\ &= \theta \|u_1 - u_2\|. \end{aligned} \tag{3.10}$$

From (3.1) and (3.2), it follows that $\theta < 1$ showing that the mapping defined by (3.4) has a fixed point belonging to K, which is the solution of (2.1), the required result. □

We note that if $w = u$, then w is a solution of the general mixed quasi variational inequality (2.1). This observation enables us to suggest and analyze the following iterative method for solving the general mixed quasi variational inequality (2.1), and this is one of the main motivation of this paper.

Algorithm 3.2. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_n + u_{n+1} - g(u_n), g(v) - u_{n+1} \rangle + \rho\varphi(g(v), u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \forall v \in H, \tag{3.11}$$

where $\rho > 0$ is a constant. Algorithm 3.2 is called the explicit iterative method. For different and suitable choice of the operators and spaces, one can obtain various iterative methods for solving the quasi variational inequalities and its variant forms. One can consider the convergence analysis of Algorithm 3.2 using the technique of Noor [24]. We leave this to the interested readers.

4. Conclusion

In this paper, we have introduced and studied a new class of variational inequalities, which is called the general mixed quasi variational inequality. We have shown that this class is related the optimality conditions of the nonconvex differentiable functions. One can easily obtain various classes of variational inequalities as special cases of this new class. We have used the auxiliary principle technique to study the existence of a solution of the general mixed quasi variational inequalities under some suitable conditions. Our technique does not involve the projection or resolvent operator. We have also suggested an iterative method for solving the general mixed quasi variational inequality. We expect that the results proved in this paper may stimulate further research in this field. The interested readers are encouraged to find the novel and new applications of the general mixed quasi variational inequalities in various branches of pure and applied sciences.

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